Applied Linear Algebra
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Week 02
Column space, null space and rank of a matrix

All right, welcome to this lecture once again. We're going to now talk about column space. null space and rank of a matrix. Now this is sort of analogous to what we studied before in the context of linear transformations in the previous lecture where you looked at, you know, range, null and then there was this nice relationship which we call the fundamental theorem of linear map which related the dimension of V to the sum, I mean basically, dimension of V equals dimension of null(T) plus dimension of range(T). We know that linear maps are represented by matrices and all these properties for linear maps should also translate to some properties for matrices, and you will see that they translate to say, very nice, important properties that you might have studied earlier in other contexts, or maybe not. So that way, this lecture will sort of set up the nice connection between linear maps and matrices a little bit further, okay? So let us get started.

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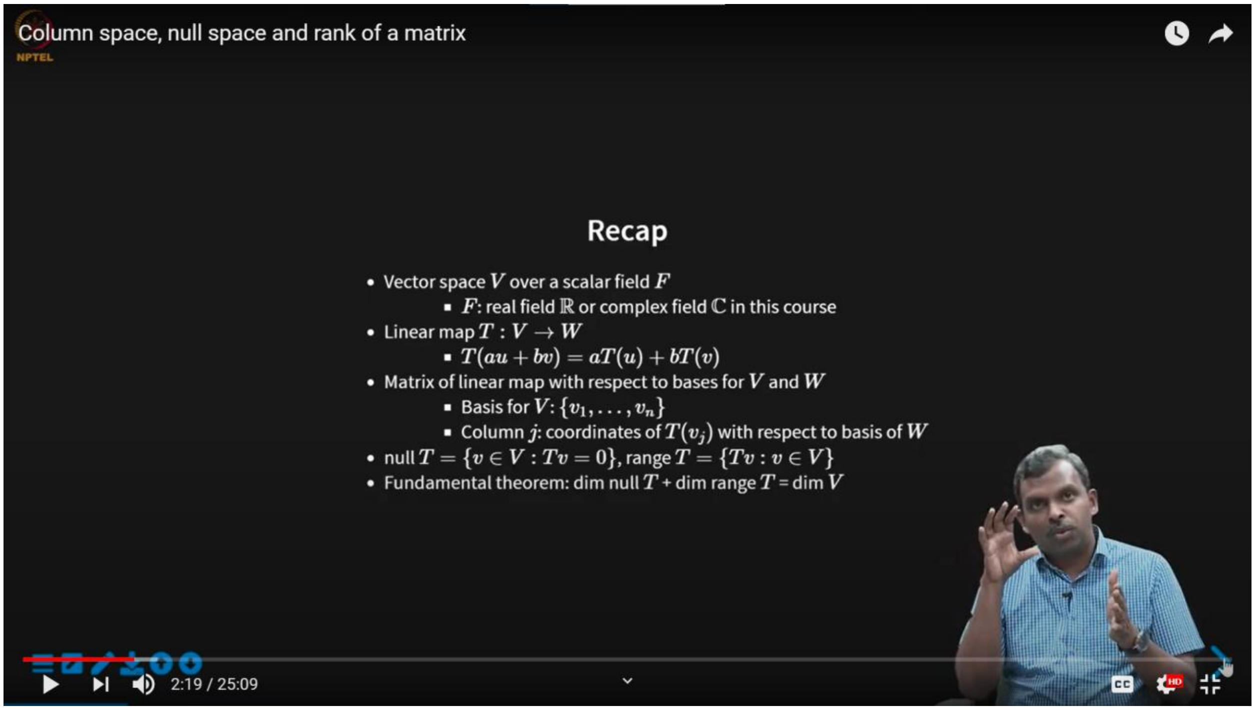

A quick recap. We saw linear maps. We saw the definition. We saw the matrix of a linear map, right? Once again, to remind you what it is, you pick a basis for V, pick a basis for W and then you look at $T(v_j)$, right? I mean the j^{th} element of the basis for V, you look at where it goes under the transformation T and then you express it in terms of the basis that you chose for W. So you'll get a column vector so to speak. So the j^th column of the matrix is simply $T(v_j)$, okay? So that's the important thing. And we saw definitions for null in terms of the transformation, in terms of the linear map T, and then range for the linear map which is also a simple definition. I drew these nice… I don't know if it's nice or not. But I drew these pictures with ellipses representing V and W, and then how linear transformations look in general. In particular, this fundamental theorem was really of great help in understanding what the linear map does. Well, the equation is one thing. To say the dimensions add up to give you dimension of V, but there was also this other picture that we got from the proof, right? You take a basis for the null and you extend it to a basis for the whole vector space and then how that sort of gives you a picture of the linear map which is very important to hold in your mind, okay? So that will help us going forward, okay? So let's proceed.

Okay. So let me remind you once again how one maps the, you know... Finds sort of the matrix corresponding to a linear map and how that works in practice. You know, how do you use that matrix to, you know, find... For instance, given an input, how do you find the output, okay? So how do you use that in the matrix is an important question. So this slide talks about that, okay? So we fix the basis for $\mathbb{F}^n$ and we fix the basis for $\mathbb{F}^m$, okay? v_1 to v_n and w_1 to w_m, and then you form this matrix for T, which is A. And then you know that the j^th column is nothing but the $T(v_j)$, right? And then expressed in the basis W, okay? So that's the matrix, that's given there. That's well and good. And then if you look at an input v. So now somebody gives you input v and one of your tasks is to find the output corresponding to this input, right? So how do you go about doing it in the matrix world? You know how to do it in the sort of the transform world when, the linear map world without referring to matrices. You first express v in terms of the bases $v_1, \dots, v_n$, okay? And you will get coordinates $a_1, a_2, \dots, a_n$ and usually we list that coordinates as a vector, okay? So $a_1, a_2, \dots, a_n$. Or you can put it as a column vector a_1 through a_n. And then what happens when you do T times v? It's simply $a_1 T(v_1) + \cdots + a_n T(v_n)$, right? So that is also easy to see. You can see, we have seen this before. Now $T(v_1)$ is nothing but the first column of the matrix A. So we can write a_1 times, and then replace $T(v_1)$ with the first column of the matrix A, plus...Like that you can do for a_2 also. a_2 is not shown here in the slide but you can see $a_2 T(v_2)$. $T(v_2)$ you replace with the second column from the matrix A, so on till a_n times the last column of the matrix A, and that, right, gives you the output corresponding to the input v, ok?

So you have the input v into the linear map $a_1, \dots, a_n$. How do you find the output vector corresponding to that? This is what you do. a_1 times this vector plus so on to a_n times this vector. You know how to do that right? So if you add it up, if you want in other words, this would be nothing but $a_1 A_{11} + \cdots + a_n A_{1n}, a_1 A_{21} + \dots + a_n A_{2n}$, so on till the last one which is $a_1 A_{m1} + \dots + a_n A_{mn}$, okay? So that is what you do. And then you know, so this... So you can notice what is going on here, right? So when you do $T(v)$, you are actually doing a linear combination of the columns of A, okay? So A is given to you, that's the matrix of the transform T. When you want to find the output corresponding to an input v, you first find the coefficients of v, the coordinates of

v in the basis and then simply use the same coordinates and do a linear combination of the columns of A, and then you get the vector corresponding to $T(v)$, okay? So this is what gives you, gives rise to this very popular matrix-vector product, right? So you take the matrix A, and multiply with the vector on the right, right? $a_1, a_2, \dots, a_n$. You see that, you know, a times that vector, this is how you evaluate it, right? So you take a_1 and multiply the first column, plus a_2 multiply the second column, like that. So when you do that you get the matrix-vector product.

And in fact, the notion, this sort of a notion is what gives you, gives rise to the definition of the matrix-vector product itself, right? So you might have studied that if you have a matrix multiplying a column vector, this is how you multiply. That would have been given as a rule to you, right? You take the first row, multiply etc. etc. And that rule sort of comes from this picture of a matrix as a linear transform. And this matrix-vector product is nothing but the output corresponding to the input which is represented by the column, okay? So this is one nice picture to remember, okay?

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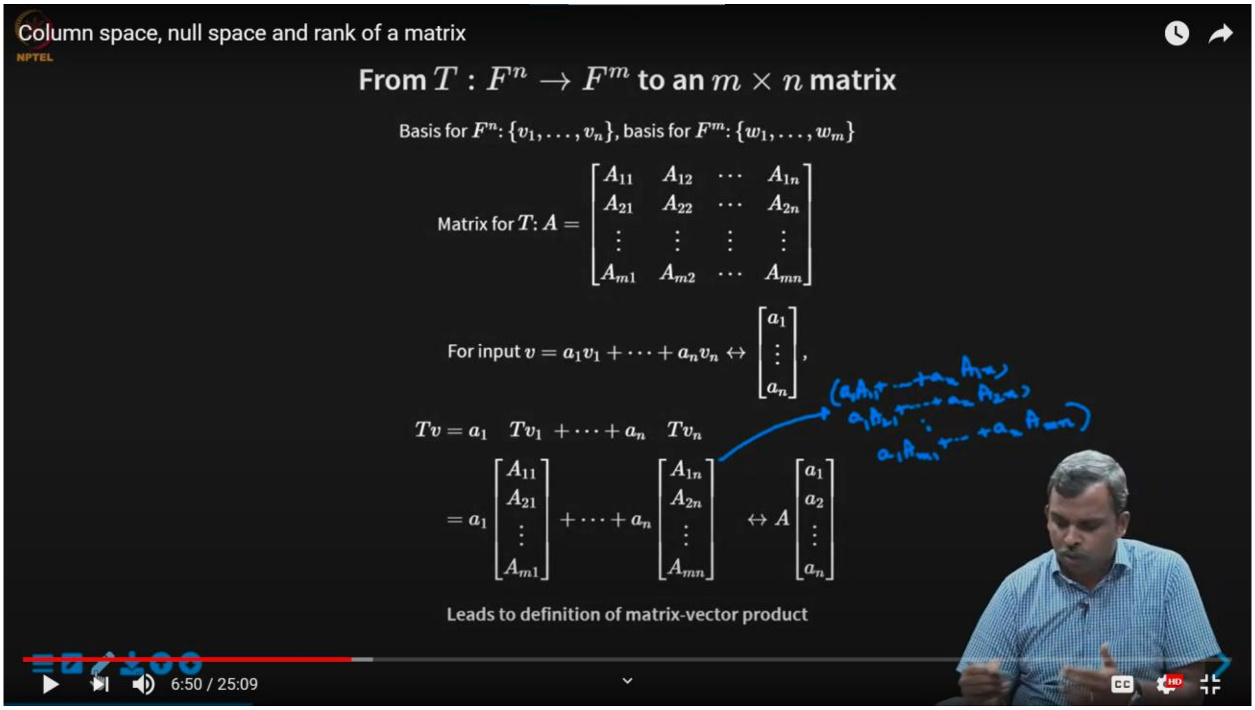

You can also sort of do the reverse and the next slide is about that, okay? So supposing I start with the matrix. So typically, you know, people just give you a matrix A, right? And then say this is the matrix. How do you associate a linear map with that matrix, what is the connection to that, okay? So of course you need a basis, and in most cases if somebody gives you a matrix they're not going to bother to give you the basis, okay? So if they don't specify the basis, the usual assumption is that you take it, take the basis to be the standard basis, okay? So usually when somebody gives a matrix they're supposed to talk about some basis, but in case they forget to do that it's usually assumed to be the standard basis. So once you fix the standard basis, you can now go from, you can now go from the matrix, $m \times n$ matrix to a linear map T from $\mathbb{F}^n$ to $\mathbb{F}^m$, okay? So that is possible. Again the same mapping goes. The j^{th} column is nothing but $T(v_j)$ expressed in the basis for W, okay? And the input vector v can be expressed as a column vector, and then the output $T(v)$ is the matrix vector product, okay? So this picture is important to keep in mind.

So quite often, even in this course, I will say a linear map and simply give you the matrix, okay? So I will not bother to define anything else. I'll say a linear map, this matrix. So what should you read? You should fill in all the remaining things, okay? If it's an $m \times n$ matrix, you should say I'm talking about a linear map $\mathbb{F}^n \to \mathbb{F}^m$, and then the basis is the standard basis, okay? So those are things that you should just assume as part of the other things. Sometimes I might say a linear map represented by this matrix, but even if that's not given, when somebody says linear map, or a matrix even, they're really referring to the same thing in some sense, okay? Hopefully that picture was clear to you, okay? So let us move on.

Now that we have identified matrices and linear maps as one and the same thing, what about properties like null, range, dimension of the range, dimension of the null? How do all those things correlate in the matrix world? What is the connection in the matrix world to that? And that is where this column space and null space of a matrix come from, okay? So like I said, we are going to think of an $m \times n$ matrix A as corresponding to a linear map $T: \mathbb{F}^n \to \mathbb{F}^m$. And now I can define null, right? So $null(T)$ is nothing but set of all vectors x in $\mathbb{F}^n$ such that $Ax = 0$, right? Ax I know is the output corresponding to an input x. What is null? The set of all inputs that give you the output 0. So what is null space now? It's simply set of all x such that $Ax = 0$, okay? So this is something you must have studied before as set of linear homogeneous equations. So the solution to the linear homogeneous equation $Ax = 0$ when A is a matrix is simply the null space of the linear map represented by that matrix, okay? So that's the first result. So that space of solutions to $Ax = 0$ is usually called the null space of the matrix itself, okay? So instead of saying null space with the linear map represented by the matrix, a lot of words, simply say null space of the matrix, okay? So and then there is the dimension of the null space. Some people refer to it as nullity. Nullity is called the dimension of the null space of a matrix, okay?

So the same thing with the… So there is a similar construct for range, okay? The range of the linear map is basically set of all Ax such that $x \in \mathbb{F}^n$. So all possible outputs, right? Ax where x ranges over all possible inputs gives you all possible outputs. And that is the range. So if you look at it carefully, you can see that that is also the span of the columns of A, right? Ax is nothing but... x specifies the combinations that you do to the columns of A, right? So when you do Ax, so if you go through all possible x, you're doing all possible linear combinations of the columns, okay? So the range of T is the span of the columns of A. And so that's why it's called the column space of A, okay? So you can define column space of A as the span of the columns of A and that would now correspond to the range of the linear map T represented by A, okay? So those are things to keep in mind, okay? Now the dimension of the column space is given a special name in the matrix world, it's called column rank, or simply rank, okay? So this reason why you can drop the column, you'll see it later on. So if you want to insist, you can call it the column rank but also you can call it rank, it's not wrong, okay?

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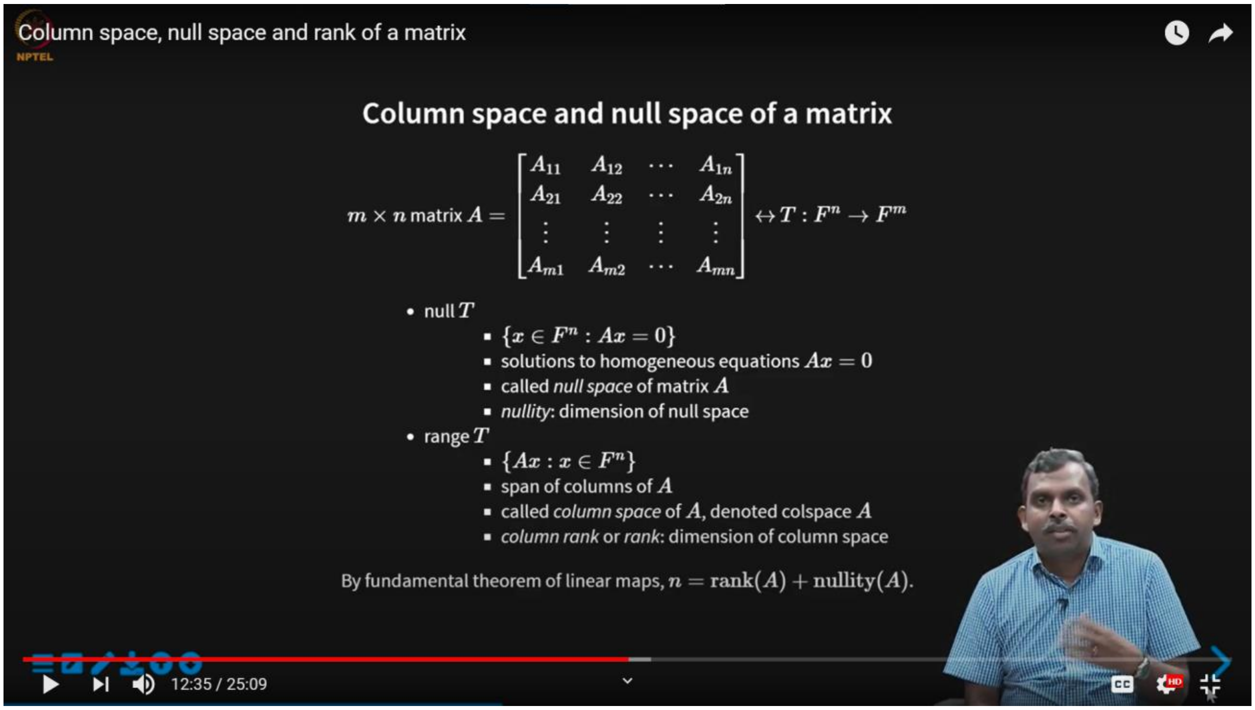

So hopefully this is clear to you. There is a column space which corresponds to range of the linear map represented by the matrix. There is null space which corresponds to the solution of the homogeneous equation defined by the matrix which is also the null space of the linear map represented by the matrix. And by the fundamental theorem you know that n, which is the number of columns, should be equal to rank of A plus nullity of A, okay? So this is also another result. There is no need to prove this separately, right? This is exactly the fundamental theorem of linear algebra, linear maps, okay? So that hopefully sort of completes the picture for you in the matrix world. So now this... Once you come to the matrix world, it’s very nice, right? So supposing you want to find the range of T, right? You know the columns are there. Supposing you want to find the dimension of the range of T or the rank of T. What do you do? You simply take the columns of A and do a Gaussian Elimination on it, right? So that will tell you what are all, how many linearly independent columns are actually there and that would be the rank. Once you find the rank, you can also find the nullity, right? Because you know the dimension n, so you use the fundamental theorem and you find, okay? So in the matrix world, you can also do these numerics to solve for whatever you want - rank, null space... The only thing you may be we have not studied in this class formally is how to solve the homogeneous equation, right? So how to solve Ax equals zero. You might have some experience from before, but even otherwise we will look at how to solve those things later on in this course, okay? So this gives you a complete picture of what all these things mean in the matrix world, okay?

So now we are going to see a whole bunch of examples. Very simple examples. I will do 2×2 matrices, you know, 2×3 matrices, 3×2 matrices just to give you an idea of what are the various simple situations where rank, and nullity and null space and column space can be easily looked at and figured out in simple terms. Towards the end, we'll also see a general method that I would present for column space, rank and nullity. Maybe not for the null space entirely at this point. We'll see later on how that works, okay? So here are some simple cases. A is just $[0\ 0; 0\ 0]$. Now this linear transformation is simply the zero linear map, right? Both the basis vectors on the input side get mapped to $(0,0), (0,0)$, right? Each column is the output, right? You should remember that. So the column space itself which is a span of two zero columns is zero. So rank becomes trivially zero. Nullity will become two. And once you say nullity is two and you are looking at $\mathbb{F}^2$, that needs to be the entire space $\mathbb{F}^2$, right? So that is the null space. So every vector goes to zero, okay? So that is the first thing.

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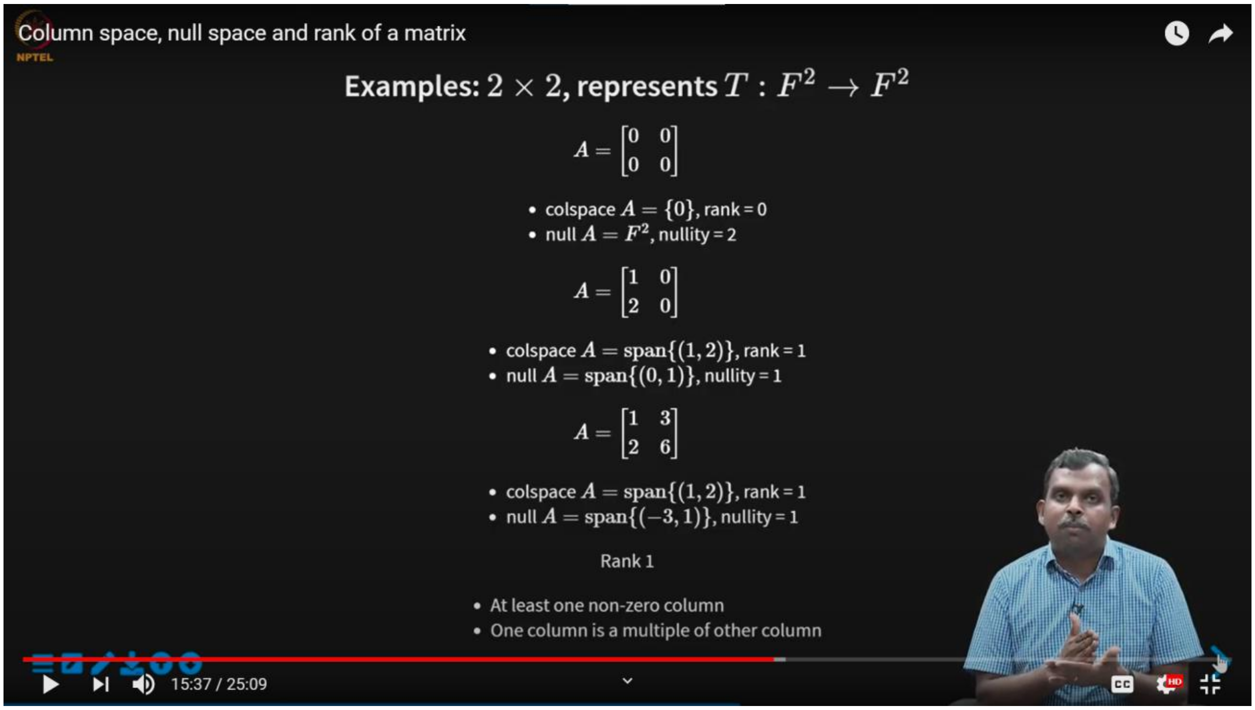

The next example is a rank 1 example. I will give you two rank 1 examples. So here you see that this is an example where the first basis vector gets mapped to $(1,2)$, the second one gets mapped to $(0,0)$, okay? So here, if you look at column space, it will be the span of $(1,2)$ and $(0,0)$ which is actually the same as span of $(1,2)$. So you clearly get rank to be 1, okay? So nullity becomes 1. Why does nullity become 1? Because rank plus nullity equals two, right? Number of columns. So nullity becomes 1. So I know that the null space is spanned by just one non-zero vector and that can be quite easily found. You can find one nonzero vector in the null space. $(0, 1)$ is in the null space. So the span of that should be equal to the entire null space A, okay? So these kind of arguments one can quickly make for small cases. They are not very non trivial.

So here is another example of a rank one column space matrix A, which is, you know, [(1 ; 2) (3 ; 6)]. You can quickly see that the second column is three times the first column. So the first basis vector gets mapped to $(1, 2)$. Second basis vector gets mapped to another vector. But it's linearly dependent on the first one. So you get only a rank of one. So if you do span, you just get the span of $(1, 2)$ and rank becomes one. So nullity becomes one and so all you have to do is find one vector in the null space and that completely describes the null space. And you can see $(-3, 1)$ would be a vector in the null space, okay? So that's a rank 1 example. So in general if you want to look at 2×2 matrices, or in fact any matrices, the rank 1 matrix should have at least one non-zero column, right? All the columns are zero, then the rank would be zero, right? So it should have at least one non-zero column, and all columns should be multiples, should be a multiple of that non-zero column, okay? So that's how it should work, right? So that's rank one for you.

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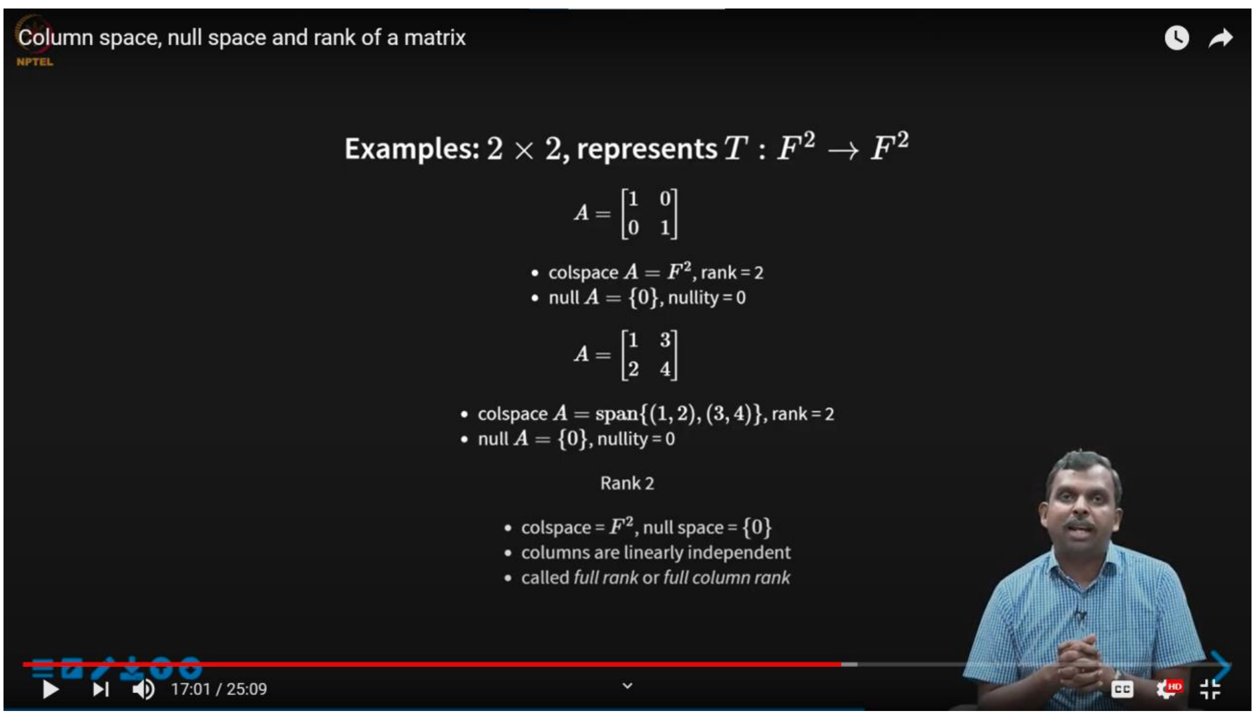

Here are the examples for rank two. Again 2×2. But rank 2, the simplest example is the identity map. [1 0; 0 1]. Here you can see that the column space is the entire $\mathbb{F}^2$, right? The span by $(1, 0)$ and $(0, 1)$. So rank becomes 2. Once rank becomes 2, nullity becomes 0 by the fundamental theorem. And once nullity is 0, you know, null space is just 0, okay? Same thing happens with any other set of columns which are linearly independent for this $\mathbb{F}^2$, okay? So like $[(1;2)\ (3;4)]$. If you have anything that is linearly independent, then you know this would happen. The rank for the matrix becomes 2. The column space is the entire $\mathbb{F}^n$, okay? So remember. This little guy here, this is actually equal to $\mathbb{F}^2$, okay? So that's something you can quickly see. So nullity becomes zero. So once nullity is zero, you know, the null space has nothing interesting in it, okay? So in general for rank two, the column space would be the entire $\mathbb{F}^2$. For the 2×2 case. I'm still dealing with 2×2 case by the way. And null space would be just 0. The columns will be linearly independent. Whenever the columns are linearly independent, the matrix is supposed to have full column rank. So this is the terminology you will hear quite often. Full rank or full column rank matrix. So in this case it's square also. So you can say full rank and full column rank, okay? Hopefully that was clean enough.

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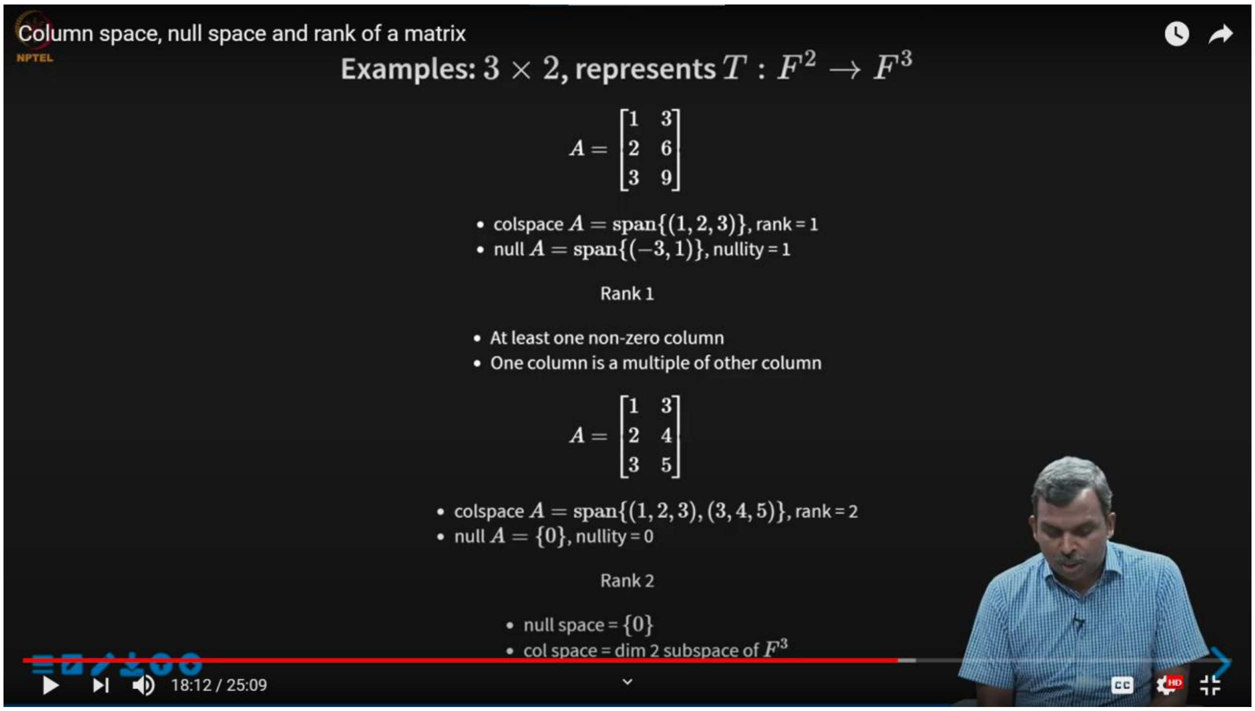

The next example we will see is a tall matrix 3×2, okay? Slightly bigger example than that we had. And once again I will skip the rank zero example and go to the rank one example where you have two columns. One is a multiple of the other, you can see clearly that the span is just the span of $(1,2,3)$ for the column space. So rank becomes one. So nullity becomes one. Remember two should be equal to rank plus nullity. Number of columns matter. So all I need to find is one vector in the null space and that's $(-3,1)$ again. So you see quite easily that this works, okay? So in general rank one, this is again true. So here is an example of a rank 2 case. $[(1;2;3)\ (3;4;5)]$. You can see clearly the it is not a multiple. So it is rank two. The span is dimension two and so nullity becomes zero. So null subspace is just the zero, okay? So in general, rank 2 for the 3×2

case will work like that. Null space will be zero. Column space will be a dimension two subspace of $\mathbb{F}^3$. I cannot say what it will be. There are multiple, so many of the dimension 2 subspaces are there, okay? So this is a picture for a tall matrix.

For a sort of a fat matrix one, you know, 2×3 sort of matrix. So here, if you want to look at a rank one case... So you see here there are three columns, right? And you see the first column is $(1, 2)$, second column is $(3, 6)$ which is three times the first column. Last column is $(5, 10)$. So again five times the first column. So it becomes a rank one matrix. The entire span is just $(1, 2)$. And now if you look at nullity. Nullity will become two, okay? Why is that? Because number of columns is three. Three equals one plus two. So you have nullity two. And you defined null space of A. You can find two basis vectors, then you will be done. And you can identify two vectors that are linearly independent, and that will definitely span the null space of A, okay? So that was a rank one example. So in general rank one will look like this.

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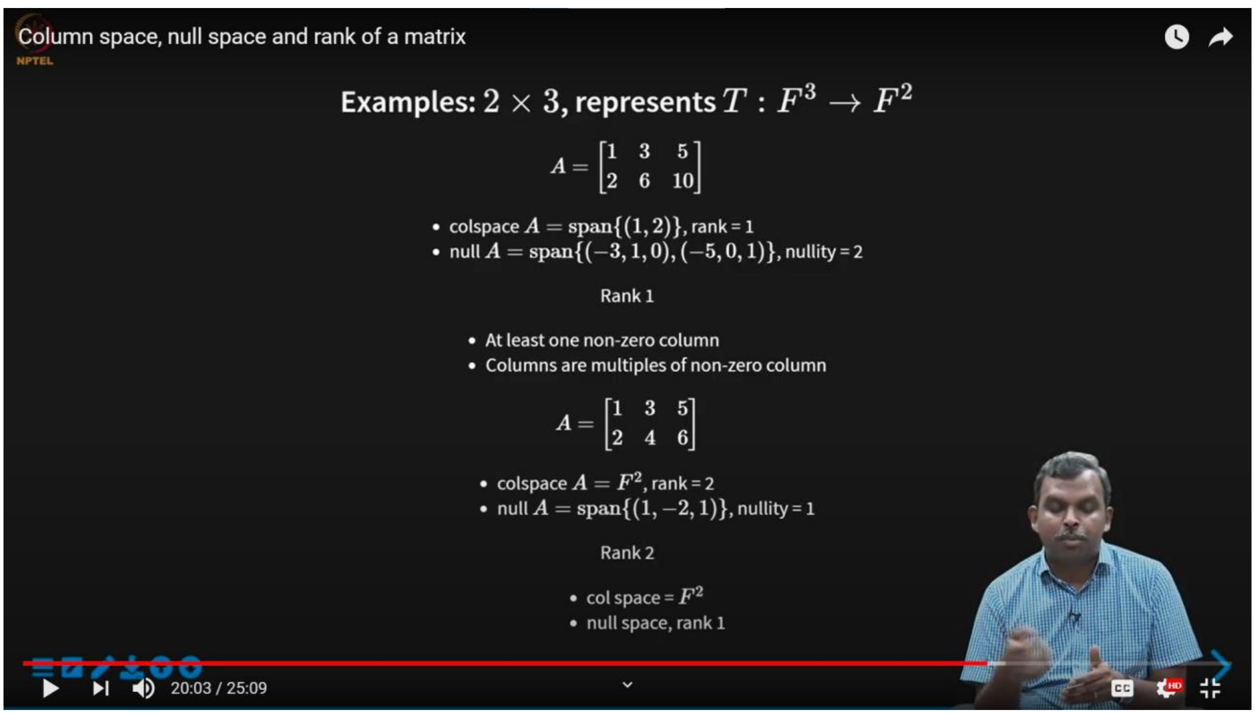

Rank two here is an example. $[(1; 2)\ (3; 4)\ (5; 6)]$. Column space is $\mathbb{F}^2$, rank is two. So once you have rank two, nullity will become one. So you have to just find one vector in the null space, and then you are done. In this case, by careful consideration, you can see $(1, -2, 1)$ is the null space. Later on we'll see a formal way to figure out a spanning set for the null space, okay? So I'll give you some... Using Gaussian Elimination and looking at it differently, you can do that also. But we will come back to that later on in the course, maybe in the next few lectures we will see that, okay? So in general, rank 2 will look like this. The column space will be $\mathbb{F}^2$ in the 2×3 case and the null space will have rank 1, okay? So that much you can say about 2×3. So hopefully these simple examples, you know, gave you some ideas on how to look at this. Null space, nullity, rank etc. hopefully gave you, I mean, some reasonable practice. Later on you'll do, you know, assignments and homework problems, you will get more practice with how to work with these things, okay? So in general, when you have an $m \times n$ matrix representing a linear map $T: \mathbb{F}^n \to \mathbb{F}^m$, the column space is the span of columns. So you start by looking at the columns. You know, even if m and n were very large, right? Say, let's say 100×1000 matrix or whatever, okay? Right? So you start by looking at the columns and then you use Gaussian Elimination to reduce, okay? When you reduce, you end up finding the dimension, right? So you know the dimension of the column space or the column rank, you can compute using Gaussian Elimination. Once you do that, you can use the Fundamental Theorem to find the dimension of the null space, okay? Once you find the dimension, you can go ahead, solve that homogeneous linear equation and find the basis for the null space. So that is the general method.

And there are a few observations you can make which are very important, okay? First observation is the dimension of the column space has to be less than or equal to m. Why is that? Because column space is a subspace of $\mathbb{F}^m$, right? So it can never be having dimension greater than m. So dimension has to be less than or equal to that m. Same thing with the null, dimension of the null space has to be less than or equal to n because null space is a subspace of $\mathbb{F}^n$, right? The input space, right? So its dimension has to be less than equal to n. There is, interesting, another result which says the dimension of the column space should also be less than or equal to n, okay? Where does that come from? If you think about it, it seems like the first two things come just because, you know, column space is a subset of Fm and null is a subspace of $\mathbb{F}^n$. So the first two results come because of this. Where does this come from? This comes because n equals dimension of column space plus dimension of null space, right? So dimension of column space has to be less than or equal to n also, right? Of course dimension of null space is also less than or equal to n, but that's nothing new. You knew that already, right? The fact that the dimension of column space is also less than or equal to n comes from the fundamental theorem of linear maps. It's not directly from just being the subset, okay? So that's something to keep in mind.

Now you can do further things which are interesting. Particularly when m is less than n, okay? So when the matrix is sort of fat, it's a bigger, wider matrix, then you can argue dimension of null will be strictly greater than zero, okay? How do I get that? Okay? So you know m is less than n, dimension of the column space is less than or equal to m, okay? So in this case, what ends up happening is because m is less than n and dimension of column space is less than or equal to m, dimension of column space is strictly less than n, okay? Now when you use that along with the fundamental theorem, that gives you this result, right? So n equals dimension of column space plus dimension of null space. And dimension of column space is strictly less than n. Which means dimension of null space better be non-zero because otherwise it won't add up to n, okay? So dimension of null space becomes strictly greater than zero. That means it's not injective. So if you have m, the number of rows being less than the number of columns, then that matrix cannot represent an injective linear map, okay? So it can never be injective.

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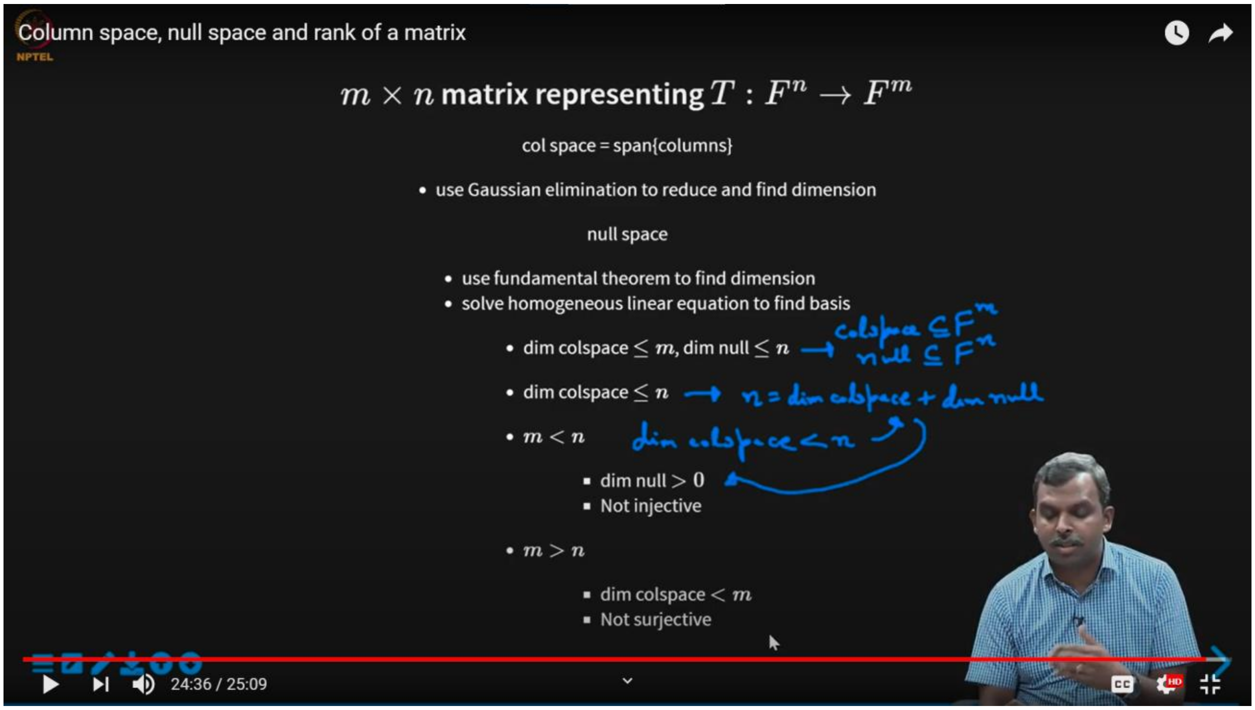

So the contrasting case is m greater than n, when the number of rows is greater in your tall matrix. In that case, again if you use this result, you know, that dimension of column space, right, is strictly less than or equal to, n you simply get dimension of column space is less than m, okay? So when dimension of column space becomes less than m, you can conclude that the linear map can never be surjective, okay? So when you go from smaller dimensions to larger dimension, you can never go and fill up the entire space there, right? So it will never be surjective. In the same way, when you go from larger dimension, you get compressed down to smaller dimension by a linear map, you should have some null space, okay? Otherwise you are not going to get that compression, okay? So that is the high level idea behind these two things.

But these are interesting results to remember and know, okay? So just by looking at the size of the matrix, you can make a lot of statements about how big is the null space and all that. So that's what's interesting about these results. And finally, I have a little quiz for you at the end of this lecture. Go ahead and fill it out, it'll give me some feedback on what you've understood. These are basic questions. And please feel free to fill it up and then submit so that I'll get a feel for how much of it is clear to you and how much of it is not, okay? Thank you very much. Let's move on to the next lecture and look at, you know, interesting things about algebra of linear maps, okay? Thank you.

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