Applied Linear Algebra
Prof. Andrew Thangaraj
Department of Electrical Engineering
Indian Institute of Technology, Madras

Week 11
Positive Operators

Hello and welcome to this lecture. We're going to talk about positive operators in this lecture. So there is a way in which you can think of classification of these operators. So if you remember, from the very first day, we have been looking at operators T which take you from the vector space to itself. Linear operators, how to classify them, how to study them, what are the various types, what are their properties, etc. One simple analogy is, you can think of, this is an analogy… Again, remember, what is an analogy? It is just something which is similar to, there is no hard proof or anything, is to think of operators also like, you know, you classify them in a similar manner as complex numbers are classified, right? So you think of complex numbers in the complex plane. Some of the complex numbers are real, right? So they are equal to their conjugate, okay? So what is the analogous picture in the operators? Operators have adjoints, and if the operator is equal to its adjoint, then you have these self-adjoint operators, which are like your, you know, real numbers in some sense, okay? And then some of the real numbers are positive, some of them are negative.

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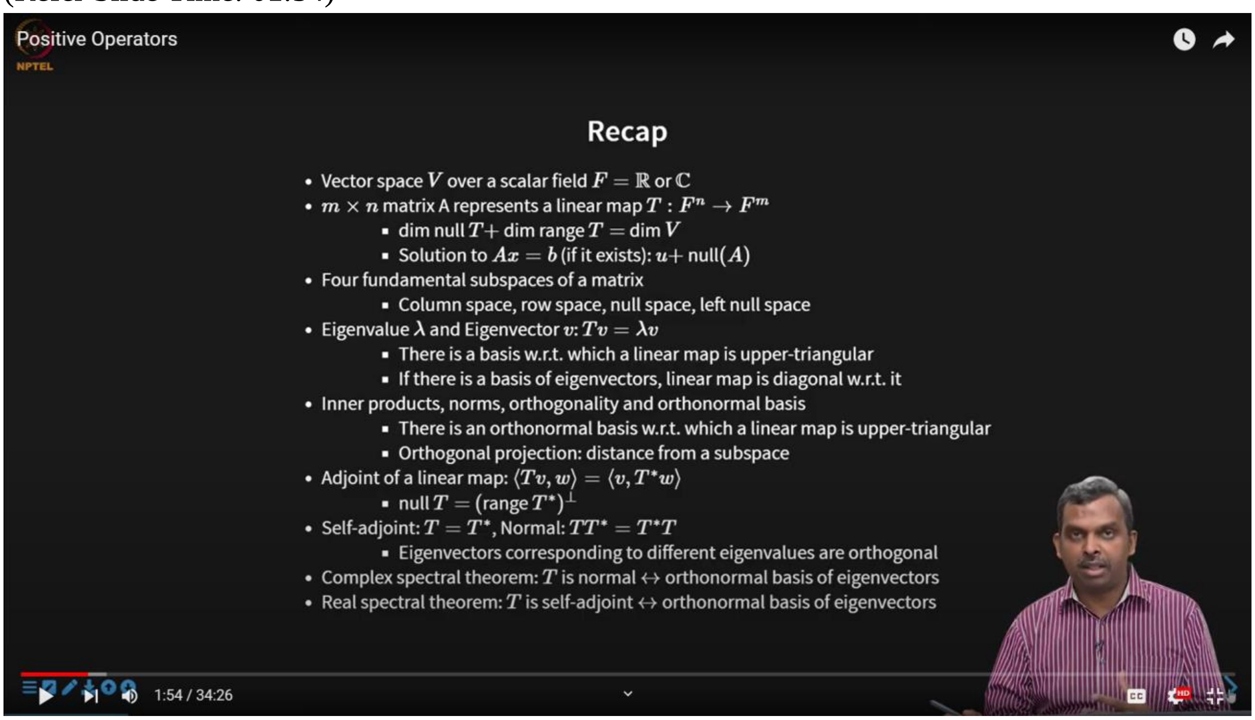

So we are now looking at positive operators. And then there are other things in the complex plane and we will slowly see that there is an analogy between how we classify and study operators and how we classify when we think of complex numbers, okay? So positive operators will have a lot of properties of the positive real numbers which when seen as a subset of the complex plane, okay? So let’s get started.

So here is a quick recap. We are looking at vector spaces over the real or complex field. We see that, you know, operators play an important role and there is a matrix representation, there is the fundamental theorem of algebra which talks about null space, range space and all that and the four fundamental spaces associated with the matrix. And eigenvectors, eigenvalues lead to a huge simplification in understanding how operators work. And upper triangularization is always possible. Some operators are diagonal, okay? And then there is this whole inner product and orthogonality which makes study so much more easier, which orthonormal basis, you see upper triangularization is possible. Then there is this idea of projection which solves a very nice optimization problem. And then we studied adjoint, what adjoint brings, this other picture into mind with respect to how inner product plays with a linear operator and in particular self-adjoint operators, normal operators. All of these are diagonalizable with respect to an orthonormal basis, right? Orthonormal basis of eigenvectors exist. We saw this complex spectral theorem, real spectral theorem. And now we are studying another type of operators called positive operators, which have huge applications once again. Particularly in the area of optimization, positive operators are used a lot. Maybe in a later lecture, we will look at applications separately.

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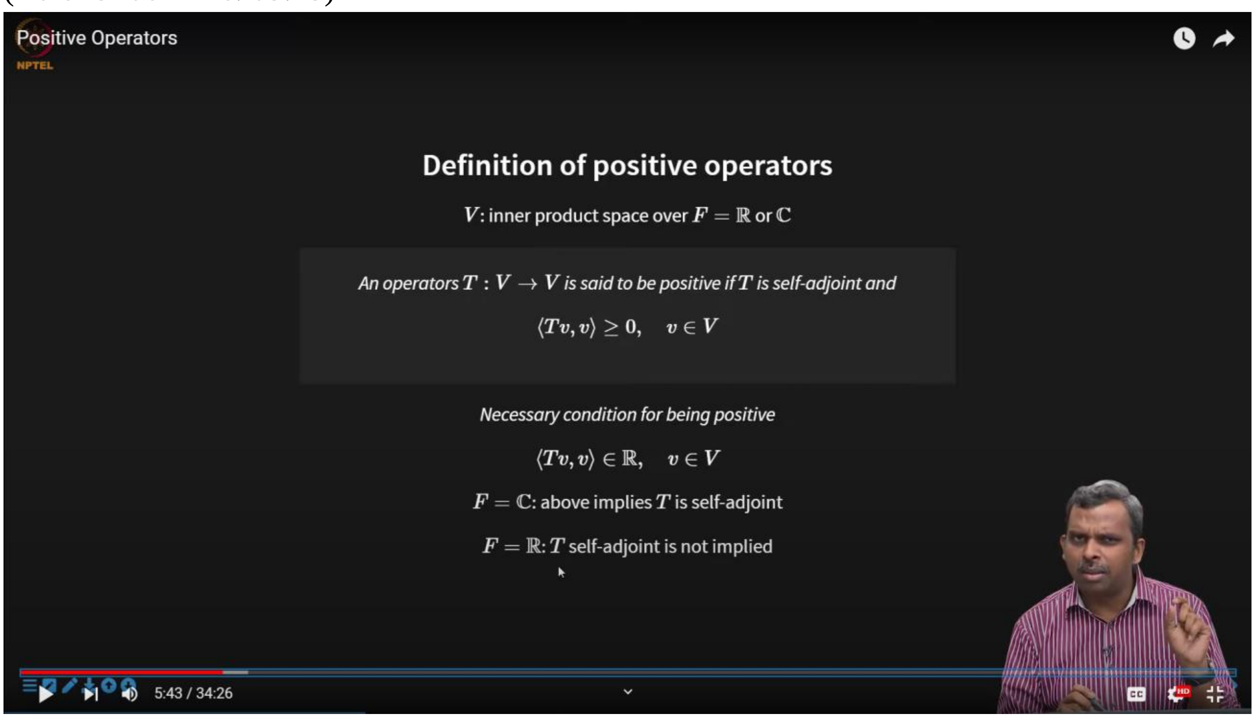

Okay. So here is the definition. It is actually a very simple and interesting definition. We have an inner product space and an operator T. It says operators T, it should be operator T. An operator T is said to be positive if T is self-adjoint. So first of all we will only associate, so even in the complex numbers, only real numbers can be positive or negative, right? Usually for complex numbers, we don't associate a sign, right? So there's no sign for complex numbers, only real numbers have either positive or negative. So only if T is self-adjoint we think about positive, negative. Otherwise we do not associate this notion of positive with it. And what is the condition for positivity? This is the condition, okay? The inner product $< Tv, v >$ should be non-negative, okay? Should be greater than or equal to zero, okay? So for all v this should happen, okay? So your operator T is self-adjoint. It takes v to Tv, but $< Tv, v >$ needs to be non-negative for that operator to be called positive, okay?

So immediately at the outset, I want to point out there are some terminology differences here. Some people would call it non-negative, right? Or positive semi-definite instead of positive. So all that is fine. In our course, we will just consider this as the definition for positive, okay? So be aware that there are some minor modifications in these definitions, whether or not you put equal to or not here, okay? Okay. So why is this self-adjoint a big deal? Because, you know, so we have this condition needed for being positive, right? See if $< Tv, v >$ is not real itself, then clearly it cannot be positive. I mean, that doesn't make sense, right? So $< Tv, v >$ should be real for all v, okay? Now that condition, if the field is complex, we've seen a result before, that immediately implies T is self-adjoint, right? So we have seen this definition of how if $< Tv, v >$ is real, you know, $\overline{< v, T^*v >}$ is actually $< T^*v, v >$. So if you subtract those two, you get $< (T - T^*)v, v >$. So if, and that goes real, so if it is real, then, you know, it has to be self adjoint. So we saw that very interesting little result there. So if $\mathbb{F}$ is $\mathbb{C}$, that is true. On the other hand, if $\mathbb{F}$ is $\mathbb{R}$, it looks like self adjoint is not really needed, but we have imposed that condition here, right? So it seems like the self-adjoint is an additional condition when $\mathbb{F}$ is $\mathbb{R}$ because when $\mathbb{F}$ is $\mathbb{R}$, $< Tv, v >$ is always real, so there is no problem with $< Tv, v >$ being real. But later on I will point out some particular case, particular argument to show that this condition that T be self-adjoint does not really limit you, okay? So this necessary condition is not a big deal. So for now just accept it, you'll see later on why even if when $\mathbb{F}$ is $\mathbb{R}$, the self adjoint is not a bad assumption, okay? So you'll see it's an okay assumption to make. Okay?

So this is the definition. So the definition is important to picture in your mind, so I am going to say this $< Tv, v > \geq 0$. And then I will say the operator T is self-adjoint. So some people like to say it's a property of this type of inner products. It's not really a property of the operator, but anyway… So it's, we will call it like that, okay? Okay. So what happens in terms of matrices? It is always good to just immediately think of matrices. Sometimes it gives you a clear picture of what is going on. If you think of an $n \times n$ matrix representing a transform T. A is positive if $A = A^H$. So this notation, hermitian, I will use to denote conjugate transpose. It's just a simple notation. Your book does not use it, but I'm going to use it to just cut short some of these notation, okay? So conjugate transpose I will call it as A^H, A hermitian, okay? So if $A = A^H$ and $x^H Ax \geq 0\ \ \forall x \in \mathbb{F}^n$, okay? So this is the definition for, in terms of matrices. The same thing I have just written it down in terms of, you know, $x^H Ax$ to point out that, you know, it is the inner product like this. This is a product of this form. So this $x^H Ax$ is something very interesting. It's, in fact it is called the quadratic form. Later on we will see some applications and all that. But that's what needs to be non-negative. So it's not like the operator itself is positive or an operator takes positive… Or what is the meaning of saying a matrix is positive, right? So that's not what we are worried about. We are thinking of this positive definite or positive in some sense, we associate with this quadratic form $x^H Ax$ should be non-negative for all x, okay?

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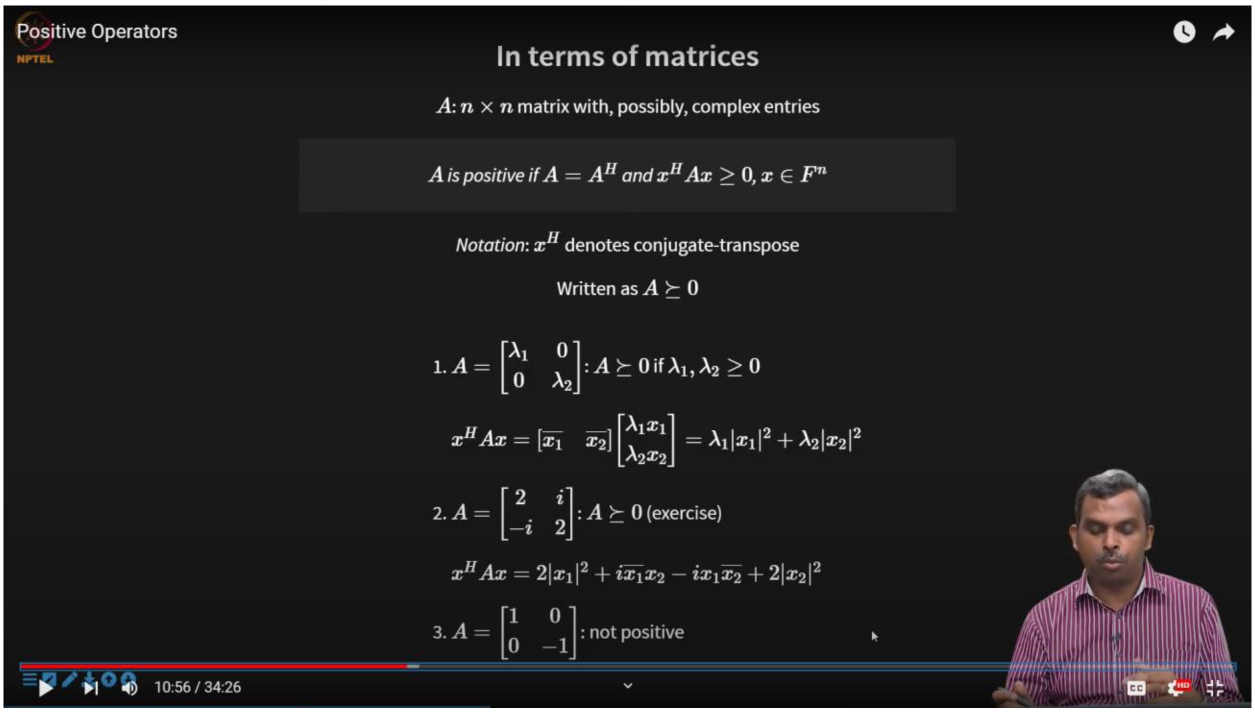

So why am I calling it quadratic form? We will see later on, but for now just accept that terminology, okay? So this is… And this is written as A… This, so it's not the usual positive, it's this sort of curly positive greater than or equal to, okay? So this is how one writes these things, okay? So let's see a few examples, okay? A few examples will clarify what is going on. Supposing I have a matrix $[\lambda_1\ 0;\ 0\ \lambda_2]$ and say $\lambda_1, \lambda_2 \in \mathbb{R}$, okay? They have to be real because I am saying is greater than equal to 0. So if you look at $x^H Ax$, you will see that it is, you know, $[\lambda_1 x_1; \lambda_2 x_2]$, that is Ax, right? Ax is that, okay? x^H is $[\overline{x_1}\ \ \overline{x_2}]$. You multiply this out, you get $\lambda_1 |x_1|^2 + \lambda_2 |x_2|^2$, okay? So when will a matrix like this, a diagonal matrix like this be positive? If the λ_1 and λ_2 are real and they are positive, okay? Non-negative, okay? So that case, A becomes a positive. represents a positive operator, okay? So this is a simple, nice enough example. But look at this guy, you know? So you have $[2\ i;\ -i\ 2]$, okay? This is A and it turns out this A is actually positive, okay? So it's an exercise, you can see that, you know, $x^H Ax$ has, you know, some crazy condition like this, you know, expansion like this and it, I mean I will leave it as an exercise to you. You can show it using some arguments that this is always positive whatever value of x_1 and x_2 you take. It's also real, I mean, real maybe you can quickly see, you know? There is this conjugation going on here. But positive needs a little bit of work. But it can be done, okay? So at this point I'll leave it as an exercise. Later we'll see a general result which will easily argue why, from which you can easily argue that this is positive. But for now there are very different looking matrices. Even with complex entries the matrix can be positive. So it's not obvious what is happening. You can see clearly that this is self-adjoint, right? So you can see that, you know, the conjugate transpose is itself, okay… So whenever we think of matrices representing operators, we are thinking standard basis, orthonormal basis. So conjugate transpose represents the conjugate of the operator, the adjoint of the operator and all, okay? So all that is true, okay? So two examples we saw. We will see more examples of positive operators. There are very very many interesting examples as well, okay?

So here is an example of an operator which is not positive, okay? So you should also see a simple example. So here is a simple example. So if you put 1, -1, I am going to get $|x_1|^2 - |x_2|^2$. So clearly it could be positive for some x, negative for some x, 0 for some x. So it's not really a positive operator. So you can have operators which are neither positive nor negative or anything like that, right? So this quadratic form need not have the same sign always, okay? So it can go all over the place. So there are operators like this. It's good to see an example clearly of what is not positive also. So hopefully this gives you an idea of how positive operators look like and, you know, what we can say about them. So later on we'll see a very nice characterization of positive operators. For now it seems like there is a variety of them, lots of different types of operators and they may be positive, non-positive. First of all, this last A is also self-adjoint, right? So it is symmetric but it's not positive, okay? So that's some examples for you.

Then let's talk about the self-adjoint property, right? For real spaces, if you want to think of positive, you know, quadratic forms, do you really need this self-adjoint? So let us take an example of an A which is not symmetric. Here is an example. $[1\ 1;\ -1\ 1]$. So it is not symmetric. But the quadratic form $x^T Ax$, you can do the multiplication, I am not showing you the multiplication. It's quite quick here, it will be $(x_1^2 + x_2^2)$ and that is clearly non-negative, okay? So maybe it looks like, you know, if you do not think about it for a little while, it seems like maybe you are losing something by saying A needs to be symmetric, right? So previously we took up only symmetric A and we said for non-symmetric A we won't even consider positive. But here is an example of a non-symmetric A which somehow has a positive quadratic form, right? The quadratic $x^T Ax$ ends up being non-negative. So in real spaces, some interesting things like this may happen. But notice this condition, okay? This is what's very interesting. If you look at $x^T Ax$, it is actually equal to $x^T \left(\frac{A + A^T}{2}\right) x$, okay?

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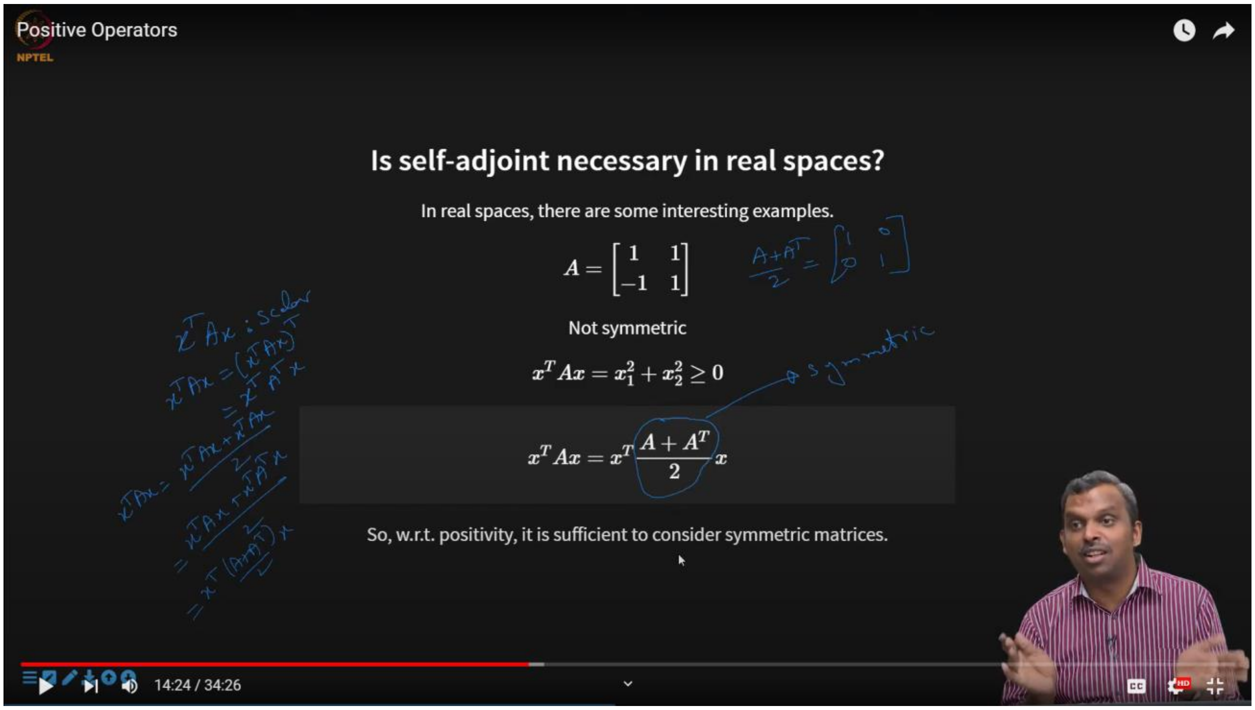

So how do I prove this? This is actually not very hard. So if you do $x^T Ax$, it is actually a scalar, right? So $x^T Ax$ is the same as $(x^T Ax)^T$, because it is a scalar, right? It's 1×1. So it's equal to its transpose and that is the same as $x^T A^T x$, okay? So how do you do it when you have a product of three matrices and you do transpose? You have to do the inverse, I mean the opposite direction, right? In reverse you have to come. So first x gets transposed, then A gets transposed then x^T gets transposed. You get this. So these two are equal, okay? So clearly $x^T Ax$ equals $(x^T Ax + x^T Ax)/2$ which is equal to $(x^T Ax + x^T \ldots)$, one of these things I will change to the transpose, okay? So that becomes the same as $x^T \frac{(A+A^T)}{2} x$, okay? So now what is so nice about this guy? $\frac{(A+A^T)}{2}$, this is symmetric, okay? So you can see that here, right? So what is $\frac{A+A^T}{2}$ here? This would just be $[1\ 0;\ 0\ 1]$, okay? So the quadratic form that you get by using A and the quadratic form that you get by getting, doing, using $(A + A^T)/2$ are exactly identical for all x, okay? So if you are worried only about quadratic forms, which you should be, right? Really, I mean, when you think of positive, only the quadratic form matters, the actual A inside does not matter, it is enough to restrict to symmetric forms, right? So it is enough to not worry about general A and only worry about symmetric A and then you would get, you know, positive operators, okay? So this is not a very big, bad assumption that we are making, this additional self-adjoint operators and only classifying them as positive or negative. It's not a big deal. In the complex case, anyway it's going to be self-adjoint. Once it's real, in the real case, even if you have something which is positive but not symmetric, you might as well move to the symmetry question and get the same quadratic form, okay? So this is something good to know.

Okay. Now here is another very interesting operator which we have studied very well, this orthogonal projection. And it turns out any orthogonal projection operator is positive, okay? So that's interesting, right? So we do not think of positive operators or even self adjoint operators when we think of projection. So now I am saying P_U for any orthogonal projection is positive. Which means what? It has to be self-adjoint. And then the quadratic form associated with it should be non-negative, okay? Any subspace that you project on to, orthogonal case, it is going to be positive, okay? So that's a, that's a very interesting result. And one can prove it using just the definition. It is not very hard. So I am going to write down the proof for this separately and later we will classify more things, okay?

Okay. So here is the proof. So what is the orthogonal projection? Supposing I say the orthogonal projection of v is u. Then I know v can be written as $u + w$, where $u \in U$ and $w \in U^{\perp}$, right? So this is the definition of orthogonal projection, isn't it? So now notice what happens when I look at $< v, u >$, okay? What is $< v, u >$? That is $< v, P_U(v) >$, right? $< v, P_U(v) >$, isn't it? That's what I am trying to find here. So $< v, u >$, v is $u + w$, so it's $< u, u > + < w, u >$. Now what is $< w, u >$? Because $w \in U^{\perp}$, $u \in U$, so $< w, u >$ goes away, right? So notice, you get just $< u, u >$. So this little quadratic form inner product $< v, P_U(v) >$ is actually the same as $< P_U(v), P_U(v) >$, so it is non-negative. All right? So this is non-negative, I have shown that, okay? So this has got to be greater than or equal to 0. So you have this interesting little result, okay?

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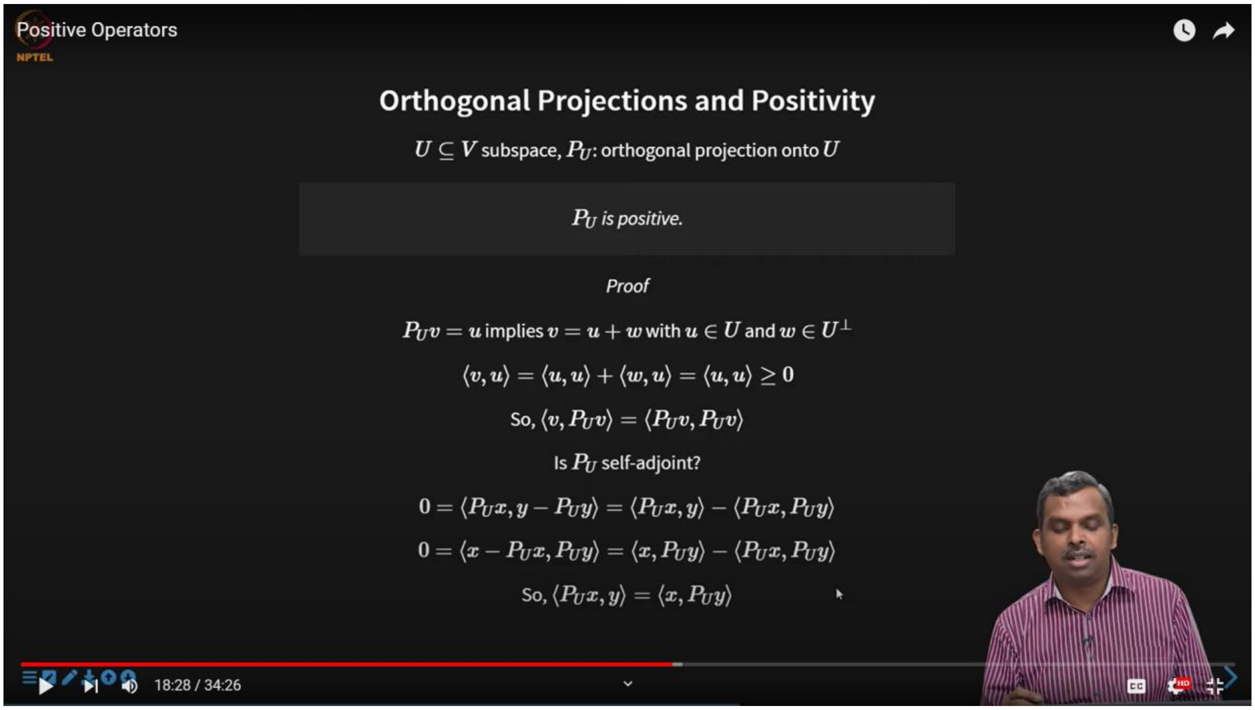

So that seems fine. But is P_U self adjoint? Okay? So that's also something worth checking, right? For the real case and all that it just clearly completes the picture, okay? So it turns out that's also true. You can show P_U is self adjoint. For any projection operator, is self adjoint. I have written down a proof argument here, you can think about how that works out. It's a bit of a, sometimes it is a bit confusing, but you can see. Finally I am able to show for any (x, y), $< P_U(x), y >$ is the same as $< x, P_U(y) >$, okay? So that means P_U its own adjoint, okay? P_U becomes its own adjoint and it's a, projection is a self adjoint operator, and it is also a positive operator, okay? So go through this proof and you can see the proof uses orthogonality in a very fundamental way, right? So this is the definition of orthogonality, isn't it? So if P_U is a projector, any $(y - P_U(y))$ is going to be orthogonal to any $P_U(x)$. So 0 is the same as, you just unravel this, you get this $< P_U(x), P_U(y) >$ here. And the $< P_U(x), P_U(y) >$ is common whether you do $(y - P_U(y))$ or $(x - P_U(x))$. So this guy has to be the same as well, okay? So it's a very interesting operator, right? So maybe you never thought of projection in such a, in such great detail. Look at what all it does for you. It does orthogonal projection, which finds the closest possible vector. But, you know, the way you write it, you see that you're looking at the orthogonal, orthonormal basis in U and simply taking dot product of that with v to find the projection, isn't it? So that's like a self-adjoint operator working on it. And then it's also positive, okay? So think about why all that is true. It's very interesting. We can maybe, maybe I'll give you more ways to think about it later on. So this is a very interesting result as to why self-adjoint, I mean why projections end up being positive, okay? So that's something to remember. In case you are worried about examples, if you want to come up with examples for, you know, positive operators, projections are your simplest and easiest examples you can think of.

Okay. The next notion is a square root, right? So one of the nice things about, nice or whatever things about positive numbers is: you can take square root, right? So you can take square root. So is there something like that for positive operators, right? So that's a natural question that people might ask. So here is a definition for square root of an operator, okay? The first is the definition. R is said to be a square root of T if $T = R^2$, okay? So if you apply R twice, you get a square root. Now it turns out, I mean you might think of square roots with operators, now you can have all types of square roots. In particular we will say a square root is positive if it is positive. R is a square root of T and if it is positive, then T has a positive square root, otherwise T may have square roots which are not positive. So let me give you examples, you know, square roots for operators is a bit tricky, you can have so many square roots and all sorts of crazy things can happen with square roots and operators, okay? So here is the first example. A is some operator like this, matrix representing an operator like this. Look at B. B is like this and B^2 is A, so B becomes a square root of A, okay? B becomes a square root and clearly B is not positive or anything, right? It's not even self-adjoint. So we don't bother about positive and all here. So you can have square roots which are not positive. Here's another example, very interesting example by the way. So you have this $[0\ 0; \alpha\ 1]$ and if you do P^2, you get P, okay? So P is its own square root. Anything which does $P^2 = P$ you can think of it as a projection, but it turns out in this case if α is not 0, this is not an orthogonal projection, okay? So it is a different type of projection. So that's an interesting little aside and exercise. But still, you know, anything like this for any alpha is a square root of itself, okay? So it doesn't really change.

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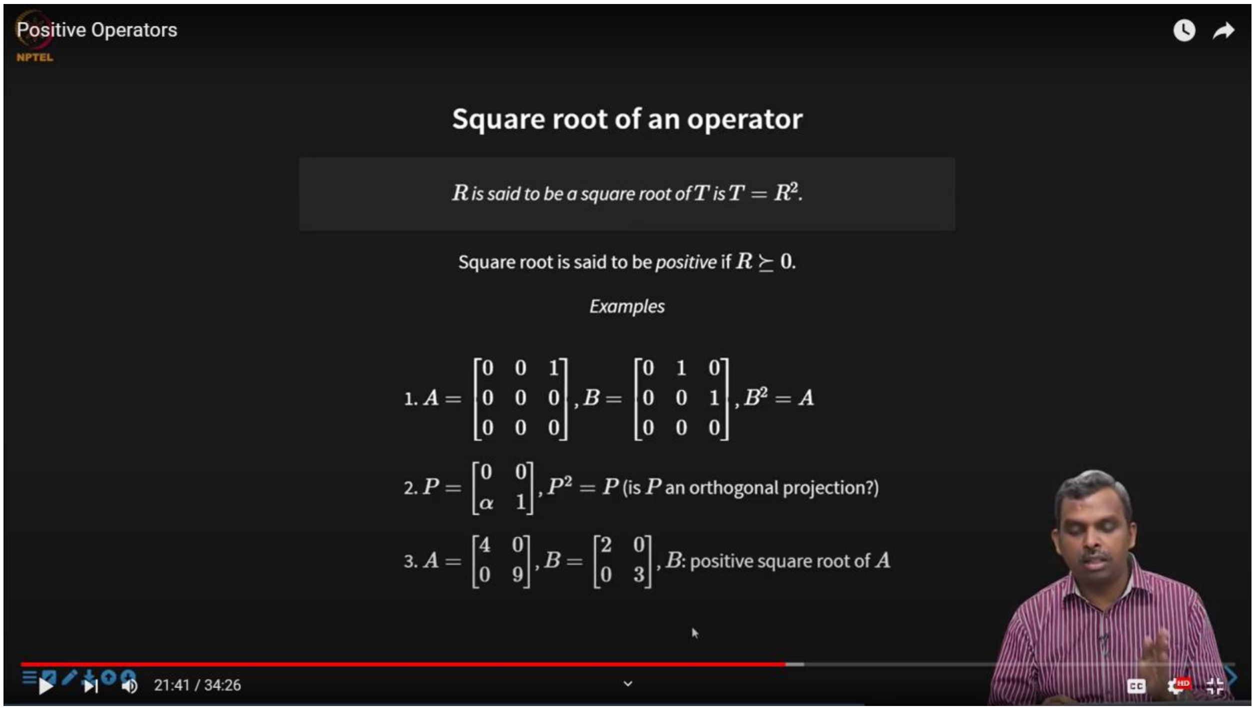

But here is a very, much more typical example. So here is A which is 4, 9 on the diagonal and B is 2, 3 on the diagonal. And it's a positive square root. So in this lecture at least, this is a good example for us. So here is a positive matrix, positive operator which has a positive square root. So that's nice. So in the previous two cases, we did not have a positive square root, particularly when alpha is not zero, right? So clearly this is not positive, no? You can easily come up with other examples. So here is, I mean, hopefully these examples give you a picture of, you know… There are various types of square roots with respect to operators. In fact you can have multiple numbers of square roots and all that so all sorts of interesting properties are possible with respect to, you know, operator square roots. But it looks like there are also good cases. As in, you can have a positive square root sometimes when you have a positive operator and it seems to work out in some sense, okay?

All right. So this is a whole bunch of definitions we have done. I think this gives us all the ingredients that we need to try and characterize positive operators, okay? So I'll give you one main theorem or result which provides all the possible characterizations for positive operators and we'll do a quick proof and you will also see how it is very interesting, okay? Okay. So before we go there, just one more idea, okay? One more type of operator we have studied before which ends up being positive which we may not have thought of as positive, but it ends up being positive, is this operator-adjoint product, okay? So I think this notation came out a bit badly, so hopefully you can see what I mean here, okay? So it's T^*... TT^* and T^*T, okay? This product of operator and its adjoint both ways, right? So these are also operators. For any T that you take from V to W, V and W can be different here, and then T^* you can take from W to V, you can define a TT^* which will go from W to W and T^*T which will go from V to V. Both of these are in fact positive operators, any product of operator-adjoint is positive, okay? So that is a very nice result. And the proof is actually very, very clear. T^*T and TT^* are clearly self-adjoint, there is no problem here. And to show that they are positive, you just have to look at the product here. Push this T^* to this side, you get $||Tv||$. Push this T to this side you get $||T^*\ v||$^2. So both of these are clearly positive operators. So we see projection is positive. We see that this operator-adjoint product is positive. All sorts of interesting properties are there for positive operators. Is there a very clean nice characterization?

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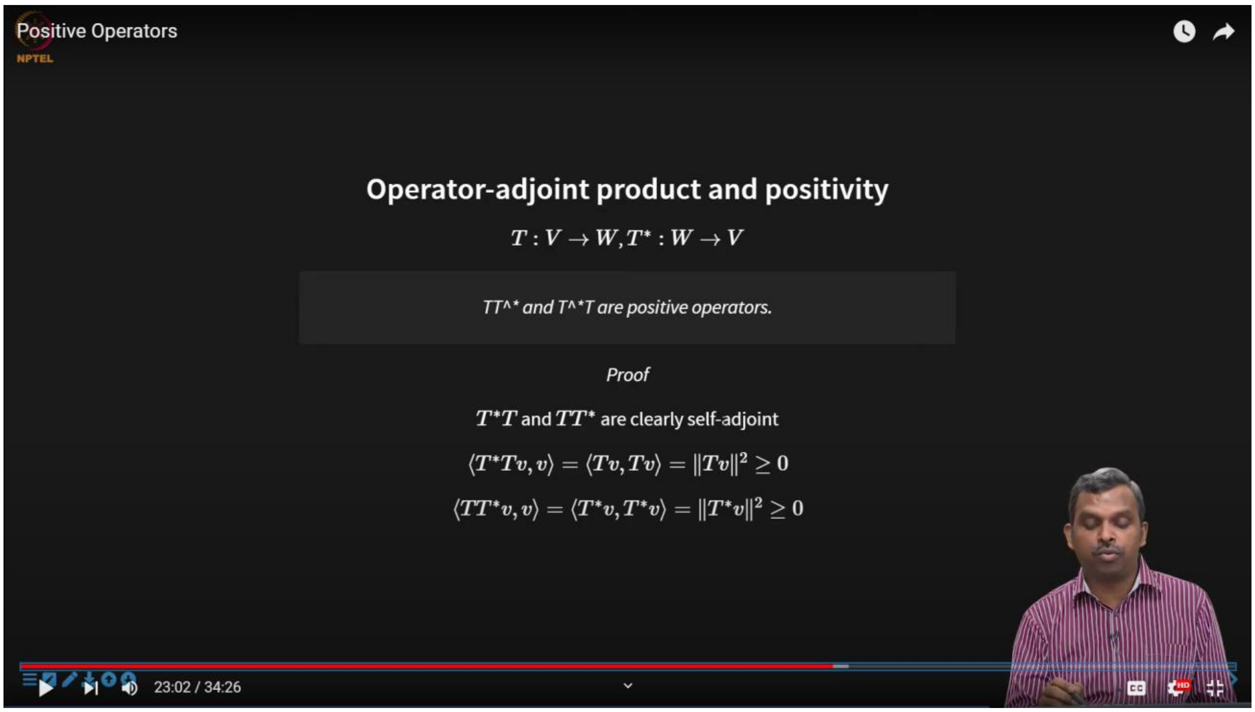

It turns out there is. And here is the characterization, okay? So this uses all the ideas that we collected so far, various types of definitions, and uses it to completely characterize positive operators. What operators are positive? Again we will present this result in a very familiar form, this following are equivalent form, okay? So this is a constant form in which a lot of results will come. When I say equivalent and list a bunch of conditions, it means any one condition implies all the other conditions, right? So that's what it means. So for instance if T has a positive square root, then T itself is positive. Then T is self-adjoint with non-negative eigenvalues. Then there is an operator R such that $T = RR^*$. So all of this is true. Any one thing is true implies everything else is true, okay?

So how do you prove this following are equivalent? You just go through all the results in some order and then ensure that everything implies everything else, you are done, okay? So what we will prove is: we will prove first 1 implies 2, okay? This is not very hard. So if, first of all, if T is positive, T is self-adjoint. That is just by definition. We are taking the definition like that, there is no problem there. How do you show the eigenvalues are non-negative? Here is this simple little argument. If you have an eigenvalue λ with eigenvector v, then you look at this product $< Tv, v >$, okay? This quadratic form inner product, we know that this is greater than or equal to zero. But now what is Tv? $Tv = \lambda v$ and λ will come out. So you get this nice result that $\lambda < v, v >, < v, v >$ is positive, equals something that is positive, okay? So λ itself has to be positive, okay? So it's a very simple argument to show that eigenvalue of a positive operator has to be positive, it cannot be negative, okay? Well, greater than or equal to zero, but, you know, that's what it is. Non-negative, okay? So we have shown 1 implies 2 and we'll also show 2 implies 3, 4, 5, okay? Right?

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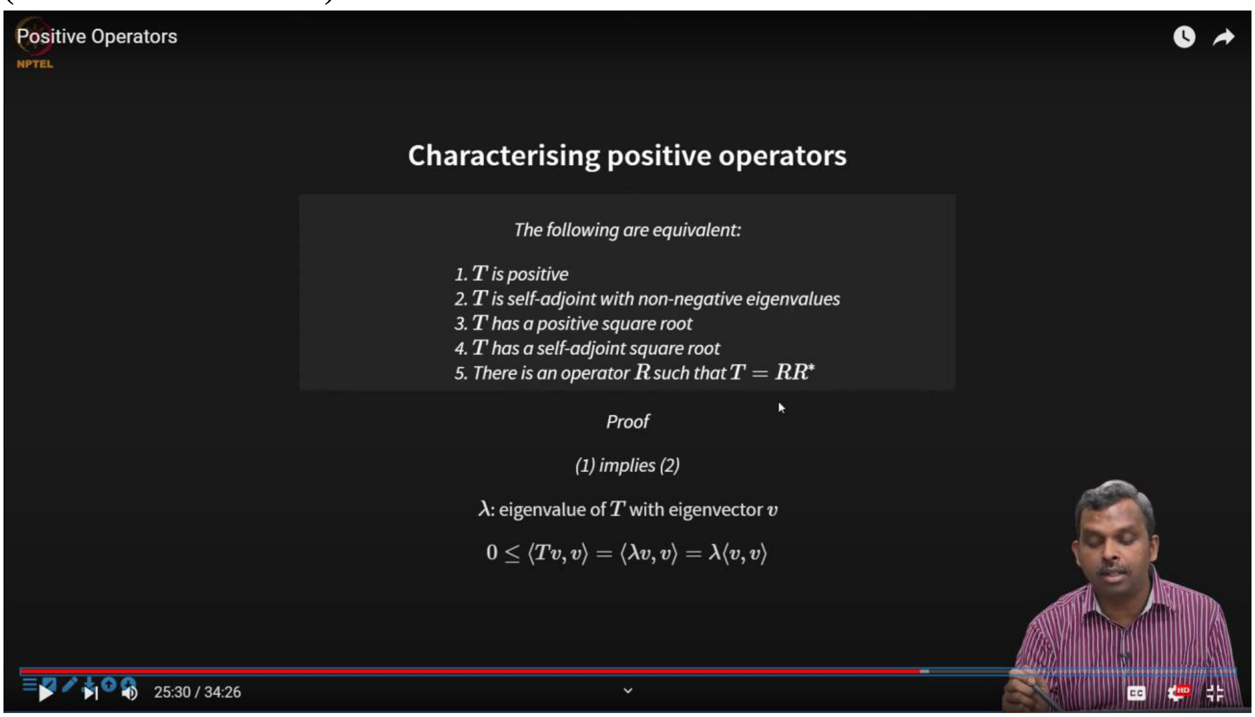

What is 2? So once you have a self-adjoint operator with non-negative eigenvalues, it implies that T has a positive square root, T has a self-adjoint square root and T has a, there is an operator R such that $T = RR^*$. All these three are implied by T being self-adjoint with non-negative eigenvalues, okay? And it comes from the spectral theorem, okay? So you see the power of the spectral theorem here. Once you have self-adjoint with certain types of eigenvalues, I can fully characterize what T is. Once you characterize T, you see square roots etc. etc., all of these are true, okay? So how do you do it? Here's the spectral theorem. There is an orthonormal eigenvector basis for T, I will assume some coordinates in some standard basis $\{e_1, \ldots, e_n\}$. Then I know I can write Te. When I say equal to, I mean, you know, T is represented by something like this. So maybe I shouldn't say equal to, I should put, okay… So you know what I mean when I write that. $\lambda_1 e_1 \overline{e_1^*}$. Or even hermitian, I have used the notation, I have not used it here. $\overline{e_n^*}$ and all these $\lambda_i \geq 0$. So that is what is very important here, okay? So once I have all the λs being greater than or equal to zero, I can meaningfully define an operator R with square root, okay? And positive square root of all the lambdas like this, right? So I can define this R, okay? Is that okay? So clearly you see that R is also self-adjoint, okay? And clearly you can see R is the square root of T, okay? You just do R^2, every other cross product will vanish because of the orthonormality. And the same thing when multiplied will give you root λ_1^2 which should be λ_1, $\left(\sqrt{\lambda_n}\right)^2$ which should be λ_n. So these are easy things to verify once you write it in this form, okay?

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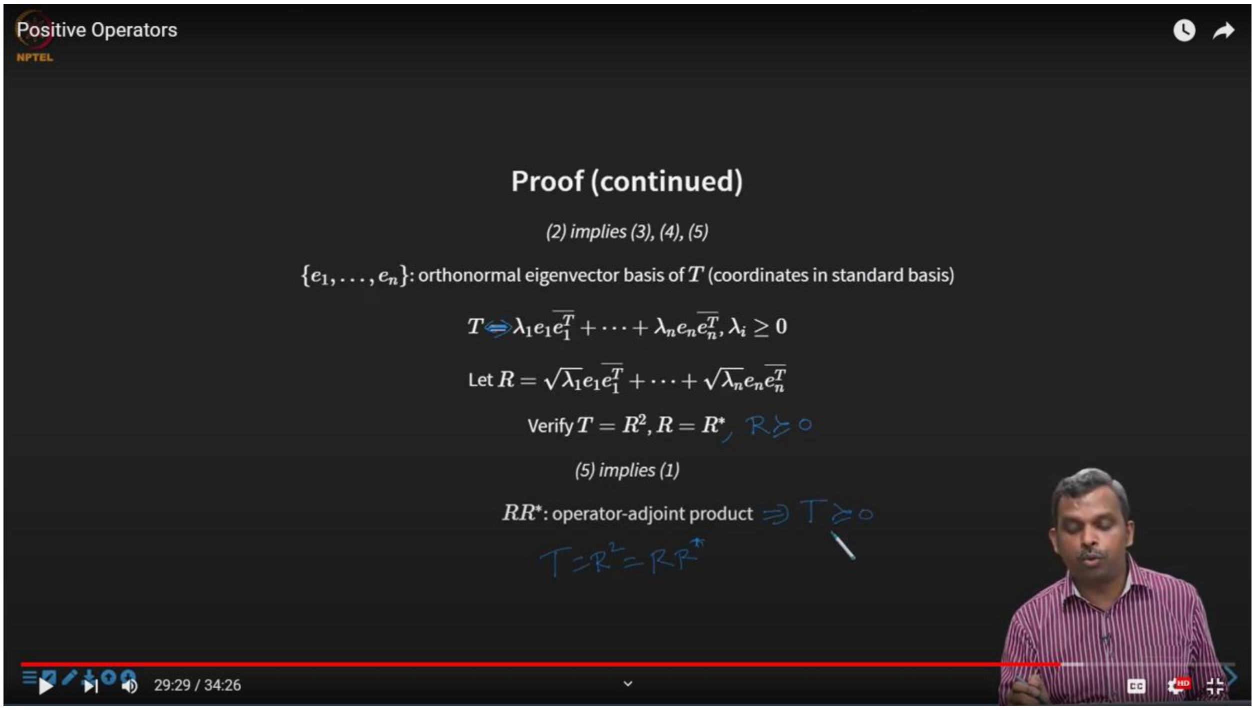

So notice the power of this form, isn't it? So the spectral theorem and the form in which you can write T really gives you a huge benefit, okay? So I have written it in terms of coordinates and all that but, you know, there is a way to write it without coordinates. You can use linear functionals or something and then write like this. But I am not going into all that trouble, I am writing it like this, okay? So you can see that R is root of this like this, okay? And you can check that this R satisfies everything that you want. So T first of all is self adjoint. T has a square root, square root is also positive, right? Square root is positive, you can check that, okay? So you need to check that R is positive, okay? I will leave that as an exercise, okay? So you need to check this out.

How will you check it? You can take any inner product, right, so you put x^H, x on this side, you will see, you know, it will just work out perfectly positive, right? Every term this dot product will be like a norm square type of thing and you will get, everything will be positive, okay? So you can check R is positive, I have not written that down here. R is positive, you can verify that T is R^2 and verify that R is R^*, okay? So all these things you can verify. So 2 implies 3, 4, 5 through this clever little device of going to the orthonormal basis and simply taking square root, positive square root, okay?

And the fact that 5 implies 1 is sort of obvious. R is R^*. So T is equal to R^2, T is equal to RR^*, okay? So maybe I should write down this also very clearly. So along with this, you are going to check that R is positive, okay? And then since R is R^*, $T = R^2 = RR^*$, right? So all of this is satisfied by this. So once you have $T \ = \ RR^*$, so that that gives you 5 also. So how does 5 imply 1? RR^* is an operator-adjoint product, right? Product. Once you have an operator-adjoint product, it is clearly positive also, okay? So that implies T is positive, okay? So that implies 1 and you are done, okay? So this is a proof, a complete proof for the characterization. So let me take you back to the characterization, okay? T is positive means, you know, T is self-adjoint with non-negative eigenvalues. So once you have self-adjoint and non-negative eigenvalues, you can simply take square roots of those eigenvalues over the same orthonormal eigenvector basis and that gives you the positive square root, the self-adjoint square root, and the operator R such that T is RR^*, okay? It's very nice. And if you have T being R^*, clearly it's an operator-adjoint product, so T is also positive. So you go back to 1. So all of that is done, okay? So this T being RR^* is a very nice and complete characterization of positive operators, okay? So that is a nice result, okay?

Finally, to conclude, I want to talk about this ordering of operators, what are called partial ordering, and this is a terminology that is used quite extensively, particularly in the optimization area. So I think it is good for you to know that. So what we have called as positive is usually called positive semi-definite in the typical optimization terminology, okay? So this quadratic form being non-negative is A being positive semi-definite, that's the name for that, okay? If you say it's strictly greater than 0, people use the terminology positive definite, okay? So for us, positive means positive semi-definite, that's how we are doing it. There is a similar definition for negative definite and negative semi-definite. So you can see this is just the negation of this, but these are important characterizations. So these are just definitions, extensions of what we did before, okay? And then you can also do an ordering of A and B, okay? So you say A is sort of greater than B in some sense, you know? This sort of curly greater if $A - B$ is positive, okay? And you can also do other relations of $A < B$ if $A \ - \ B$ is negative and all that, right? This positivity of $A \ - \ B$ gives you a lot of, positive definiteness of $A \ - \ B$ gives you this ordering for A, B. So this kind of ordering is useful in optimization. Maybe we will see later on some applications of this. But for now, just, this is just a definition I am making for you to be familiar with some of the optimization notation. So later on, if you do a course in optimization, all these things will come and help you then, okay?

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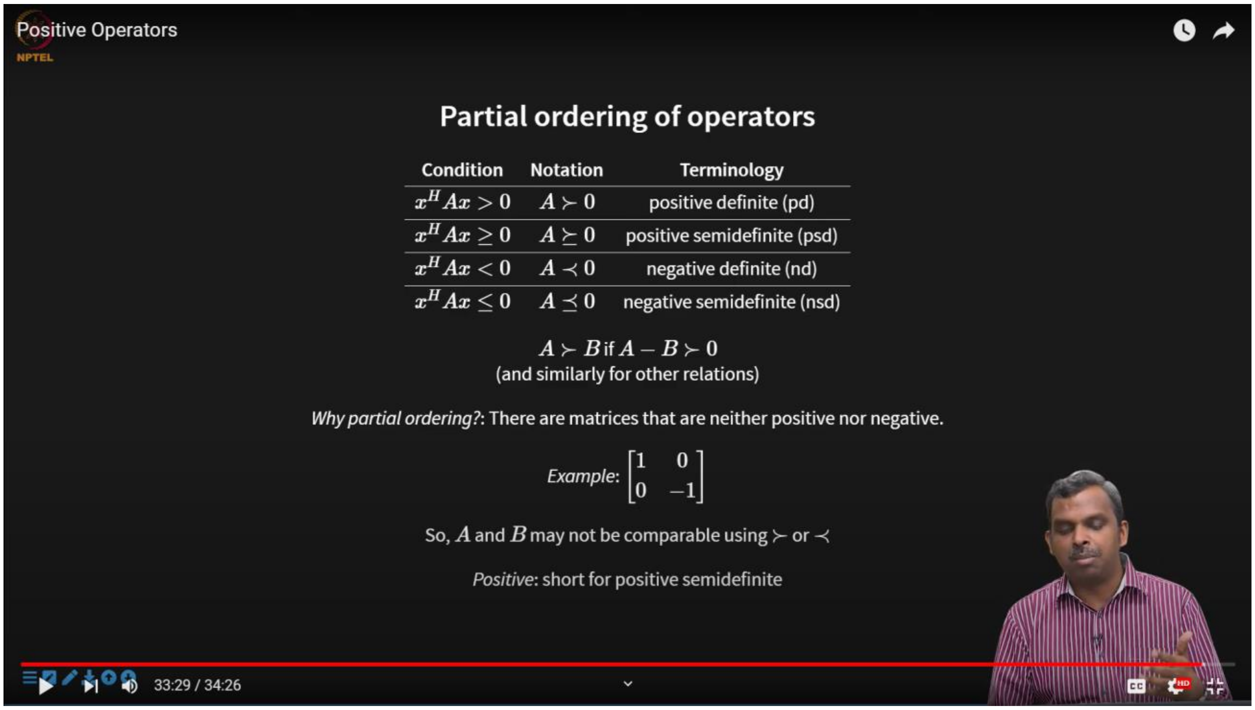

So why do we say partial ordering? Why is it not a complete, what's the difference between complete ordering and partial ordering? If you say a complete order, then any two, any two things should be comparable with respect to that order. For instance, if you take real numbers, right, this > sign is a complete ordering because any two real numbers are either greater than each other or less than each other. So there's no question of not being comparable, okay? But on the other hand, if you go to complex numbers, this greater than sign, less than sign applies only to the real number subset, it's not like a complete ordering of complex numbers in some sense. So there's no ordering as such, okay? So that is there. So likewise, this positivity is only a partial ordering of operators, it's not that any two operators have to be either greater than or less than, that's not needed, it's only partial for instance because there are matrices that are neither positive nor negative. So, for instance, $[1\ 0;\ 0\ -1]$, it's neither greater than zero nor less than zero, okay? So you, so if your $A - B$ becomes something like this, $[1\ 0;\ 0\ -1]$ then you cannot say $A > B$ or $B > A$, okay? So there can be two matrices which are not even comparable using this positive definiteness or negative definiteness ordering, okay? So it may happen. So such kinds of things are called partial orderings, okay?

So that hopefully gives you a picture. For our definitions, positive is short for positive definite, okay? So that brings us to the end of this lecture. Hopefully this last slide, I mean it is not useful in a linear algebra course, but it's an application to optimization and later on you'll see this definition come up quite often. Wherever you see, people will use these kind of different terminologies. And I'm just making it here so that you're not surprised when you see it, okay? So hopefully this gave you an idea of positive operators, their characterization. The most important characterization is: a self-adjoint operator with non-negative eigenvalues is positive, okay? And you can just take the square root, go to the orthonormal basis and take the square root of the eigenvalues, you get the square root of that operator. Non-negative self-adjoint, positive square root of the operator. And then that gives you a full nice characterization of what positive operators are. And we saw that these very special operators like, you know, operator-adjoint product is positive always. Projections are positive always. So positive operators seem to be showing up in very many interesting places. And quadratic forms play a very important role in the understanding of positive operators as well, okay? So we will see applications of all this in a later lecture. Thank you very much.