Applied Linear Algebra
Prof. Andrew Thangaraj
Department of Electrical Engineering
Indian Institute of Technology, Madras

Week 9
Adjoint of a linear map

Hello and welcome to this lecture. We are going to start looking at adjoint of a linear map and this starts a new week in this course as well. Week 9. So let's get started by looking at it. This word adjoint, if you have had, if you looked at determinants and all that before this, the same word adjoint is used for something else, another matrix. We will not use that as far as this course is concerned. So for us, adjoint will be what I am going to define in this lecture, okay? So keep that in mind.

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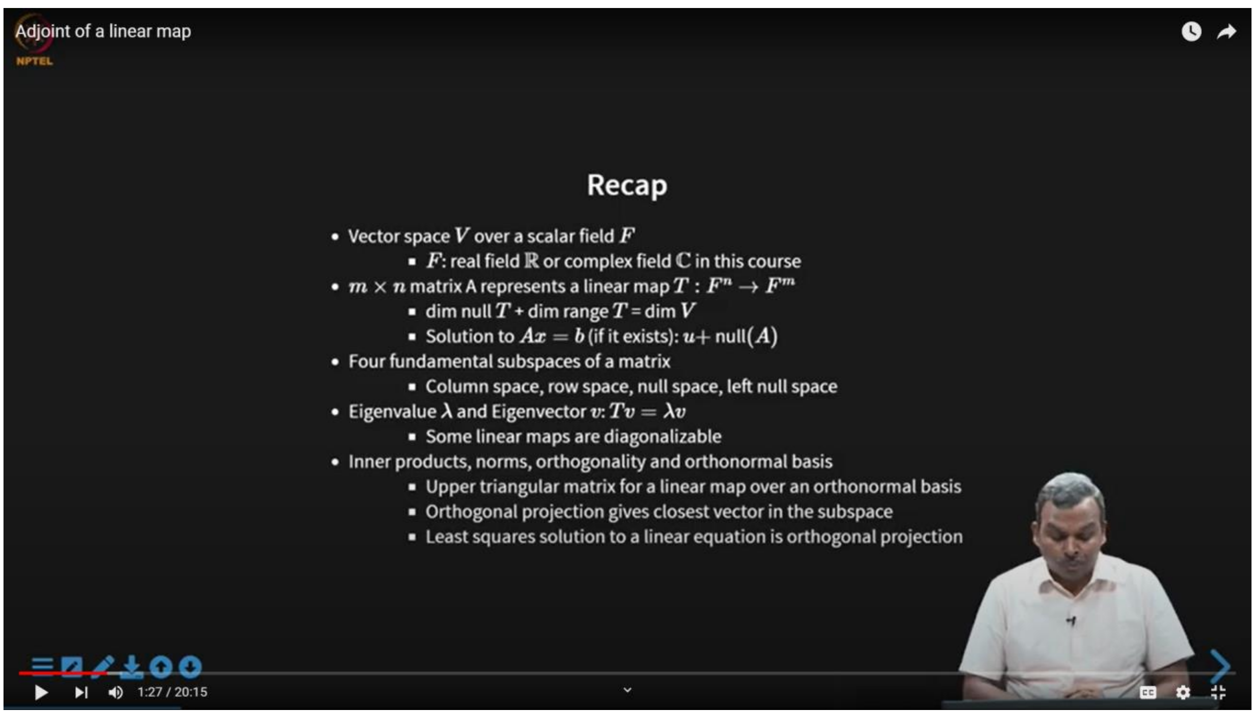

Okay. A quick recap. We've been looking at vector spaces over the real or complex field. We saw how linear maps are very important to understand what goes on in vector spaces. And linear maps have this nice fundamental theorem. There is a matrix representation and four fundamental subspaces for a matrix. In particular these invariant one dimensional subspaces, eigenvectors and eigenspaces and eigenvalues are very important to understand linear maps particularly well. Diagonalizable linear maps are easy to deal with and we saw how, you know, inner products and norms and this notion of orthogonality helps us tremendously to simplify things in understanding linear maps. So in particular, we saw in the last few lectures orthogonal projection which gives you the closest vector in a subspace and it's associated with these least square solutions for linear equations. And both of these are very good applications even in learning and very recent topics of interest, okay? So this is a quick recap of where we are in this course and from now on we are going to study operators in inner product spaces, okay?

So we have these inner product spaces and what are the special types of properties that we can give operators in an inner product space? Look at the effect of how operators affect inner product, okay? So that's sort of the way in which we will look at it. So you can think of how it will work, you know. For instance you have two vectors u and v and you have an operator. There is an inner product between this $< u, v >$. And then after you have operated with the T, they go to, you know, some other domain. And you can also do an inner product there. So are there connections between these inner products, are there some interesting things we can do around these inner products is the central question here. And there is a nice study of that we can do. And these adjoints are very important in that study, okay? So the properties of the adjoint of an operator will tell you a lot about the operator itself, okay? So that's where we are going, okay, from a high level. So let's get started.

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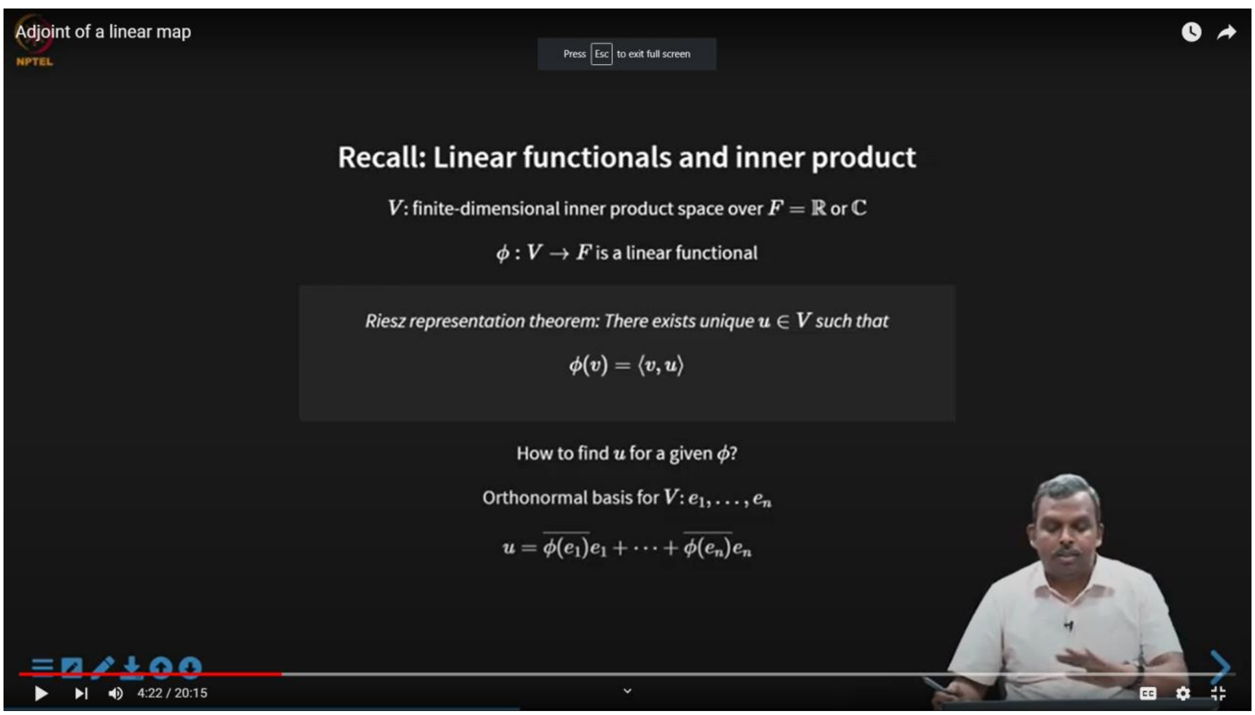

Okay. So let us recall one connection between linear functionals and the inner product that we saw. So now we are going to study how linear maps affect inner products or how inner products shed light on linear maps, both ways. We will look at it now, beginning with this lecture. But before that, we've already seen one nice little property, particularly with respect to linear functionals, right? So if you have a vector space, I will take a finite dimensional vector space because this result holds in that. And if you have a linear functional from the vector space to the scalar field $\mathbb{F}$, then we have this Riesz representation theorem which says there is a unique u such that this $\phi(v)$ is given by the inner product of $< v, u >$. So all linear functionals are inner products with some chosen vector, one vector. There is nothing else that is a linear functional. So when you have one dimension, there's this nice property. And how do you find this u, if you have to find this u? We use this orthonormal basis. Take an orthonormal basis and simply apply ϕ on each of these things and then just reorder and use the properties of the inner product. Now because in an orthonormal basis, the coordinates, you know, you can write any vector u, any vector v as, you know, inner product $< v, e_1 > e_1 \ +$ so on. And then you just use the properties of the inner product. Now take the phi inside etc. And then you will get this answer. So there is a very direct and simple way to compute this unique vector u given the linear function, okay? So this is the background. And we will use this in a very nice way to study, you know, inner products and linear maps.

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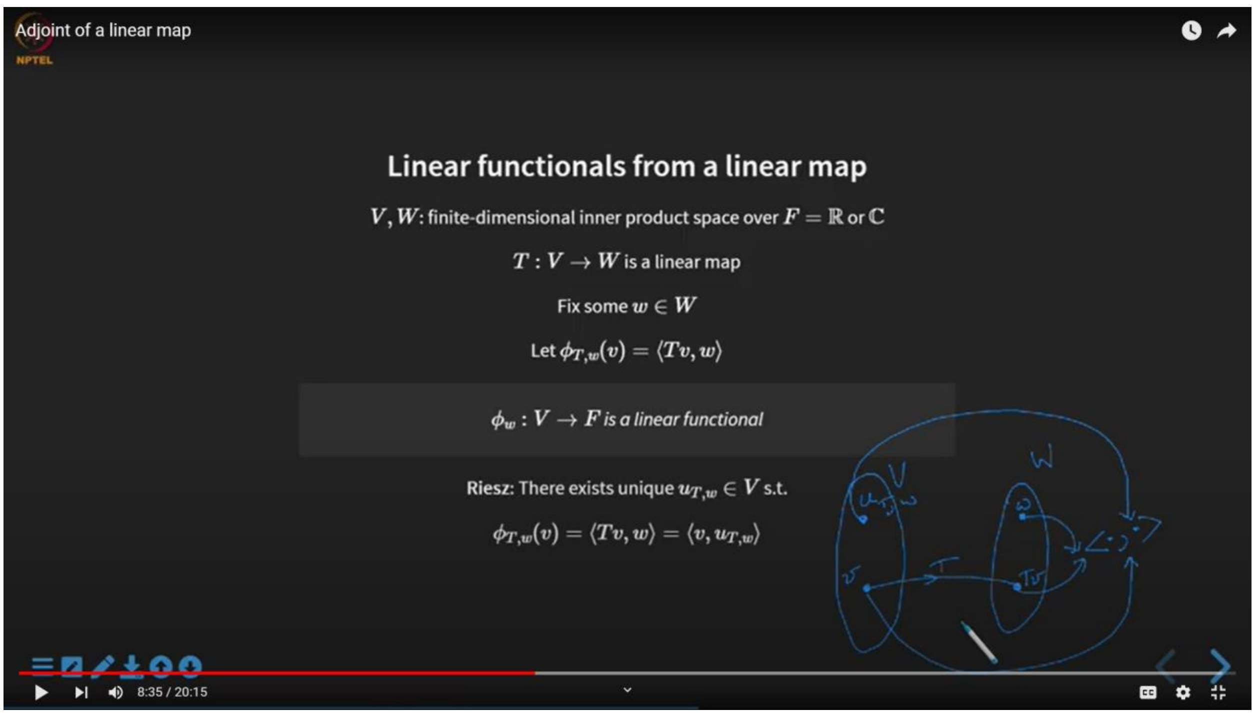

Okay. So for that we will first look at extracting some interesting linear functionals from a linear map, okay? So a linear map is given to you. I am going to think of a linear map now from a vector space V to another vector space W, okay? Both are finite dimensional inner product spaces. So this is the new thing. So so far we have not brought in the inner product. I'm going to now bring in the inner product, okay? So there is, both of them are inner product spaces over the same field $\mathbb{F}$, okay? And then $T: V \to W$ is a linear map, okay? So this is the setting. We will fix some w in the range space in W, okay? We'll pick one vector w there and then I will define this linear functional from V to $\mathbb{F}$, which is defined as mentioned here. So you take the inner product of Tv, okay? So Tv, if you take a v, which is a vector in V, you apply T to it, you go to W. And I already have this other w sitting there, I take inner product of those two, I get a number, okay? So it's almost as if, you know, this $\phi_{T,w}$ captures, you know, what part of W is in T in some sense, right? So T takes any vector v to Tv. And then what is the connection between that and W? You take the inner product to figure that out, so that's the functional, okay? So the functional itself has a very simple definition. It is not wrong to define this. Now what does Riesz say in this picture? In this picture, using the Riesz representation theorem, you know that there is a unique u in V such that this linear functional inner product $< Tv, w >$ is always equal to $< v, u_{T,w} >$, okay?

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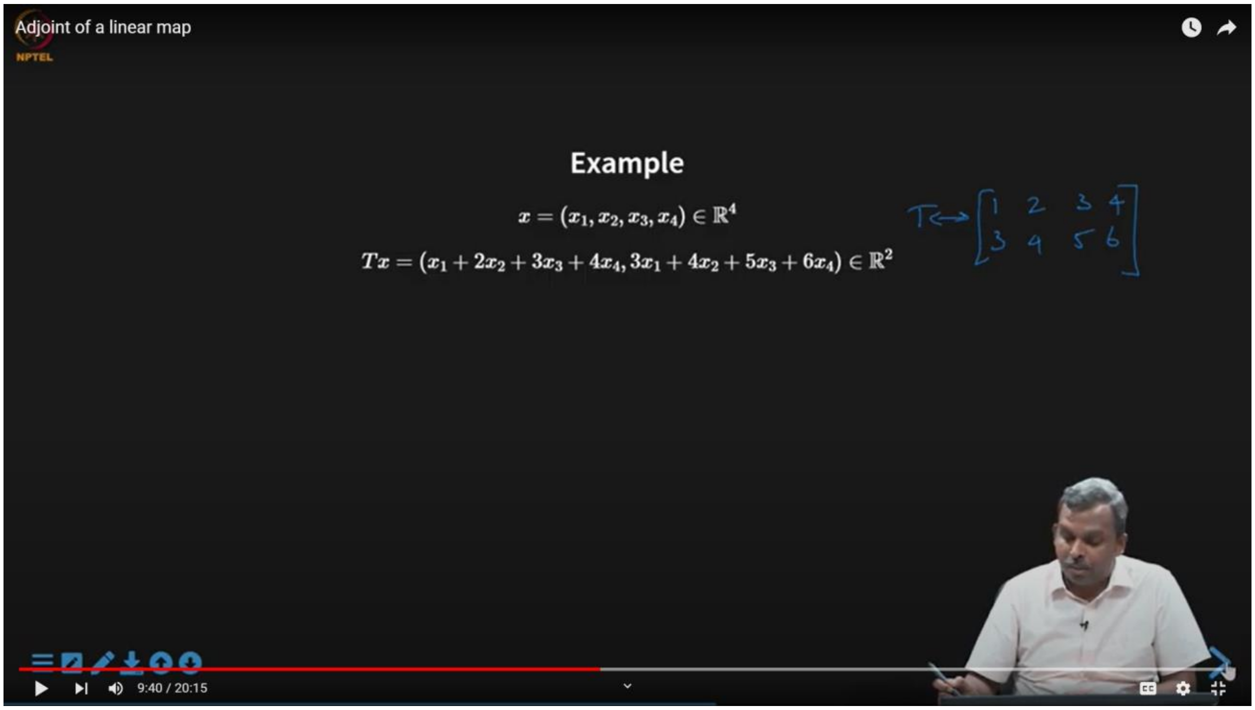

So at this point, let me draw a picture to illustrate what is going on. This will help you maybe picture this. So you have V and you have W and there is this w that I have identified, okay? And you have a v which takes you by T to Tv, okay? There is an inner product that you can do between these two, okay? You will get a linear functional. Now once I fix a w and this linear functional is well defined, I know that there is some $u_{T,w}$ in V, right? And this inner product and this inner product have to match, isn't it? So that's what Riesz representation theorem is telling you. So you pick some w, right? You pick some w and there is this T given to you. And you define this linear functional using just w and Tv. You know by Riesz representation theorem that there is a $u_{T,w}$. And any time you do this or you do the direct inner product of v and w, you should get the exact same answer. That's what Riesz representation theorem is telling you, isn't it? So this picture I think is a good picture to remember. And you can also see we are taking our first steps towards, you know, mixing up inner products and linear maps, right? So how do you mix inner products and linear maps? As in, mixing in the sense how do you study the effect of a linear map or inner products or how do you study how inner products shed light on linear maps, right? So these are the same sort of ideas you can do. So you have inner product before the linear map inner product after the linear map and there is this connection between the two, right? So you take a linear map and then evaluate inner product with a fixed vector w. It's the same as doing an inner product before, okay? So the inner products through linear maps are preserved in some interesting fashion in this way, okay? So this is the Riesz representation theorem. So let us see an example, a concrete example. You will see how easy this is. I mean this sounds like a very abstract idea, but when you break it down to a concrete example particularly in finite dimensional Euclidean spaces, this idea is very very simple, okay?

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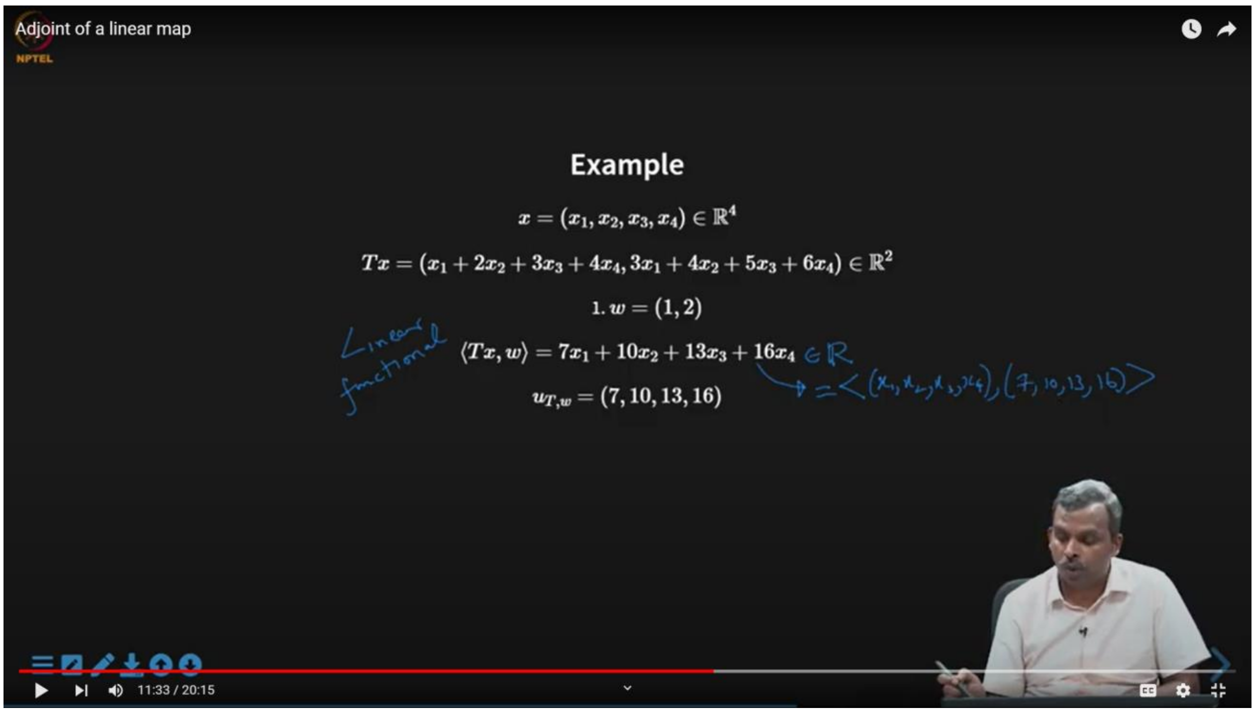

Here is the illustration, okay? So let us take $\mathbb{R}^4$ and this x is a vector in $\mathbb{R}^4$. And let us look at a very simple linear operator Tx which is what I have defined here. It's one of my favorite operators. If you want you can write down the matrix here. So this matrix representation of T is you, know, $[1\ 2\ 3\ 4; 3\ 4\ 5\ 6]$. I've used examples of this sort throughout this course. So maybe there is some familiarity with it. This is the basic matrix representation. So Tx just simply goes to this. If you pick the standard basis for instance, you will get this, okay? So that is T for you. A very simple operator. So now suppose w is (1, 2), right? So you remember T goes from $\mathbb{R}^4$ to $\mathbb{R}^2$. And I've picked one particular vector $(1, 2)$ in $\mathbb{R}^2$. And then I am going to look at $< Tx, w >$, right? The inner product of Tx and w. So $(1, 2)$ inner product with this is going to be simply this, right? It is a specific expression and it belongs to the real numbers, right? So it belongs to $\mathbb{R}$. So this is just, you know, one times this plus two times this. This is a very simple expression. I hope I have not made any mistakes here. Yeah, so it looks okay to me. Okay. So it's just two times the second coordinate plus 1 times the first coordinate, that is $< Tx, w >$ and clearly I didn't mention it here. It does belong to $\mathbb{R}$, right? So it is a linear functional, $< Tx, w >$ right? Isn't it? Now from this you can quickly read out what $u_{T,w}$ will be, right? See if I have to write $< Tx, w >$ as a $< u_{T,w}, v >$, this is what it is, right? So this guy is equal to inner product $< (x_1, x_2, x_3, x_4), u >$ right? This u is $(7, 10, 13, 16)$ isn't it? For any x it is true that $< Tx, w >$ is simply an inner product of that x with $(7, 10, 13, 16)$ and this is all that Riesz representation theorem is saying, okay? So when you take a simple example, it's a very easy and rudimentary thing to see, okay? So $< Tx, w >$ there's always a u. Once you fix the w, there's always a u, a fixed unique u such that $< Tx, w >$ becomes an inner product with this, okay? It's as simple as that, okay?

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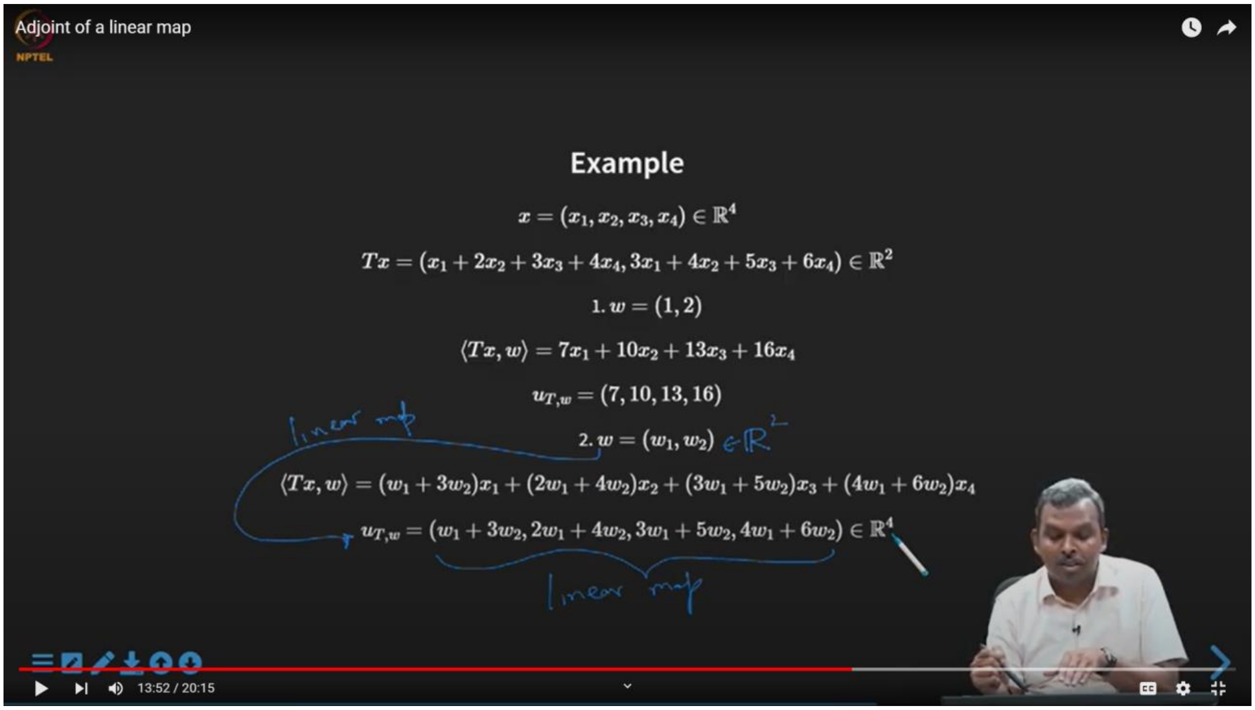

What if you make this w as an arbitrary (w_1, w_2), okay? I took a particular example here. $(1, 2)$. Maybe you do not like this example, $(1, 2)$. Maybe it's not good enough for you, it's not clear enough for you. You take an arbitrary (w_1, w_2). Now what happens when you do $< Tx, w >$? You can go back and plug in w_1 times the first coordinate plus w_2 times the second coordinate and simplify and you will get this. Do you get this? So now this is also an inner product of $u_{T,w}$ and x, right? And you can pull that out here. See these coefficients come out. $(w_1 + 3w_2, 2w_1 + 4w_2, 3w_1 + 5w_2, 4w_1 + 6w_2)$. So for an arbitrary w in $\mathbb{R}^2$, okay, for an arbitrary w in $\mathbb{R}^2$, $u_{T,w}$ is a linear map into $\mathbb{R}^4$, is that okay? Okay? Do you see how this worked out? For an arbitrary w, this thing of finding $u_{T,w}$... So now you want what is $u_{T,w}$. Write it out and identify $u_{T,w}$. And if you look at the map from w to $u_{T,w}$, this map is a linear map, isn't it? This is just a map, you know, I mean a map from w to u_(T, w) and that ends up being a linear map from $\mathbb{R}^2$ to $\mathbb{R}^4$, okay? Once again backtrack. You have a map, a linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$ and you found a linear map from $\mathbb{R}^2$ to $\mathbb{R}^4$ which sort of captures your inner product $< Tx, w >$. Is that okay? So this gives you a very clean and nice example. And this example is good to keep in mind as you go through the rest of the theory which defines adjoint and all that. It's a very simple definition at some level but when you do it in theory sometimes it looks like I am pulling out all sorts of notation and terms. And this picture, this little example here should help you clarify what is it that we are actually talking about, right? So there is this nice linear map which we seem to have discovered from $\mathbb{R}^2$ to $\mathbb{R}^4$ in a very interesting and simple way. And the adjoint simply is a generalization of this example, okay?

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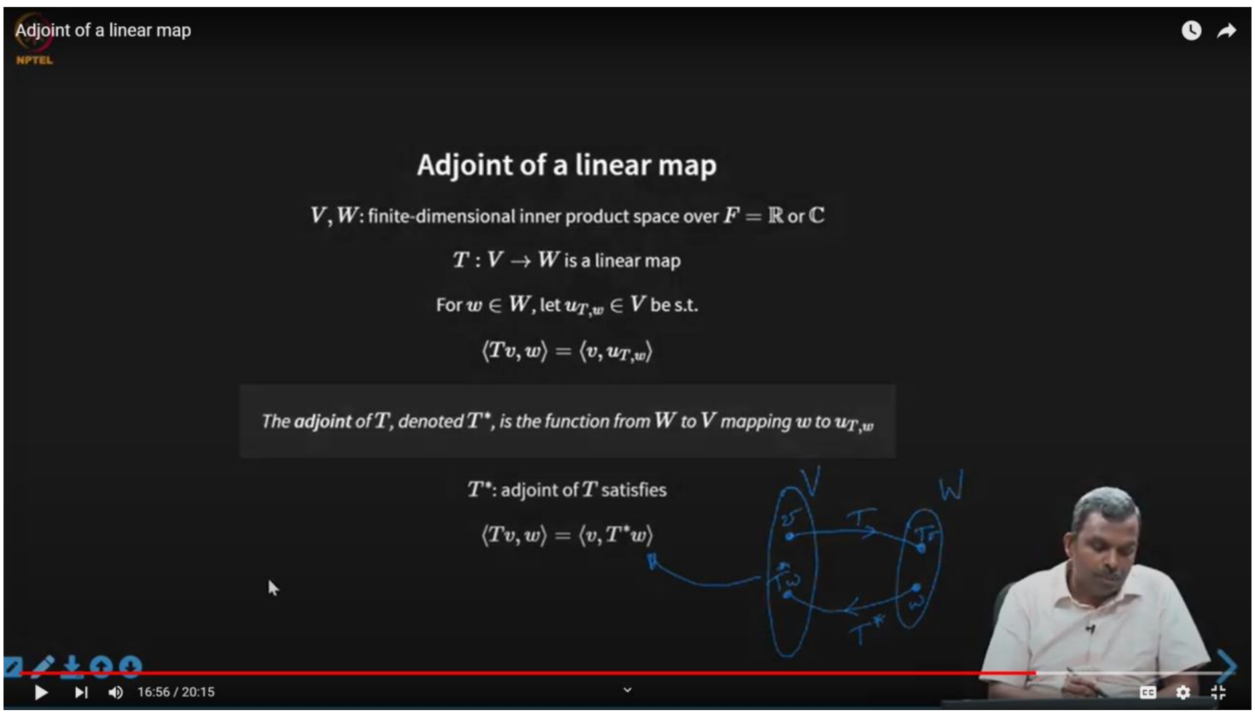

So let us see that. Here is the definition for the adjoint of a linear map. We have two vector spaces V and W, both finite dimensional inner product spaces over a field $\mathbb{F}$. And we have a linear map from V to W, okay? The adjoint, okay, so if you pick a w, small w and capital W… Using Riesz we know that there exists a unique $u_{T,w}$ in V such that this inner product $< Tv, w > = < v, u_{T,w} >$. So this u we know is unique and this comes from the Riesz representation theorem. So this is, left hand side as a linear functional, right hand side has to be an inner product and this u will have to be unique, okay? So this is what we know already in the adjoint of T which we will denote as T^*, okay? In this course we will use the notation T^* for the adjoint. It is that function from W to V which maps w to $u_{T,w}$, okay?

So we saw on the previous example how this map ended up being a linear map as well. We will show it's linear soon enough. But this is the definition. At this point I am defining the adjoint as simply a function from W to V which maps w to $u_{T,w}$, okay? Now notice this is a very very very important property and we will use it again and again and again to make our understanding of linear maps very clear, okay? $< Tv, w >$, the inner product between Tv and w is the same as the inner product between v and T^*w, okay? So notice this T^*w is my new notation for $u_{T,w}$, whatever I had as $u_{T,w}$ is T^*w, my new definition of adjoint. And it satisfies this wonderful relationship. Once again a picture. I have drawn this picture before too, but it is good to draw this once again now that we have adjoint and all that. So you have V and you have W and you have a v that takes you by T to Tv. And you have any w here, you would have the adjoint which takes you to T^*w. And what is so great about T and T^*? They preserve this product, the inner product between Tv and w in W is equal to the inner product between v and T^*w in V, okay? So this is sort of the abstract generalization of the previous example that we saw in defining T and T^* and this adjoint exists in this fashion, okay? So later on, when you do advanced courses, people would generalize this to other types of spaces and all that. And the adjoint mapping in philosophy will work in the same way, okay? So this is the definition of an adjoint, okay?

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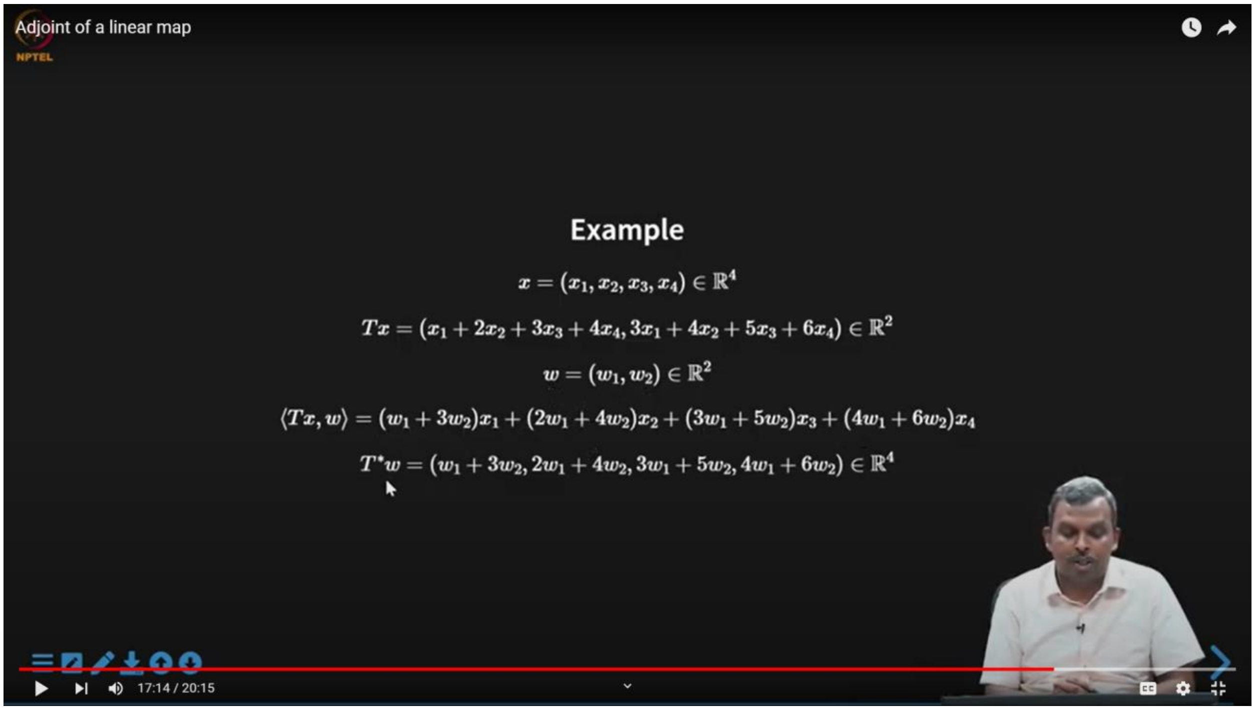

And we saw in the previous example that the adjoint ended up being a linear map in that case, and that is in general true. So that is what we will prove next. And let's see, let's… Before we go there, let's revisit that example and make sure we understand exactly what it is and we see that, you know, w is (w_1, w_2). And you look at this inner product and you read off T^*w as this map, okay? So you can see T^*w is a linear map, both of these are linear maps. One takes you from V to W, another one takes you from W to V and they are both given in a very simple way, they are given in this fashion, okay? So in fact there is matrix representation for these things, and later on we will look at matrix representations. At this point I will simply stop with the definition of adjoint.

Okay. So the general property that one can quite easily prove is that adjoint is a linear map, okay? So if you take any finite dimensional vector spaces V and W and T is a mapping from V to W, T^* is the adjoint of T and that is a linear map from W to V, okay? The proof is not very hard, it just involves that same property, you know? If you were to look at $(w_1 + w_2)^*$, you can show it is $w_1^* + w_2^*$, okay? So how do you prove that? So if you look at $< v, T^*(w_1 + w_2) >$, it's it… So this is the property and you know this is equal to $< Tv, w_1 + w_2 >$, you use the same property again and again. But once you come into this world, you know this will split, this is distributive, right? $w_1 + w_2$. So you write like that. Then each of these things is… Again you go back to the adjoint and then you again use the distributive property, you get v comma this, okay?

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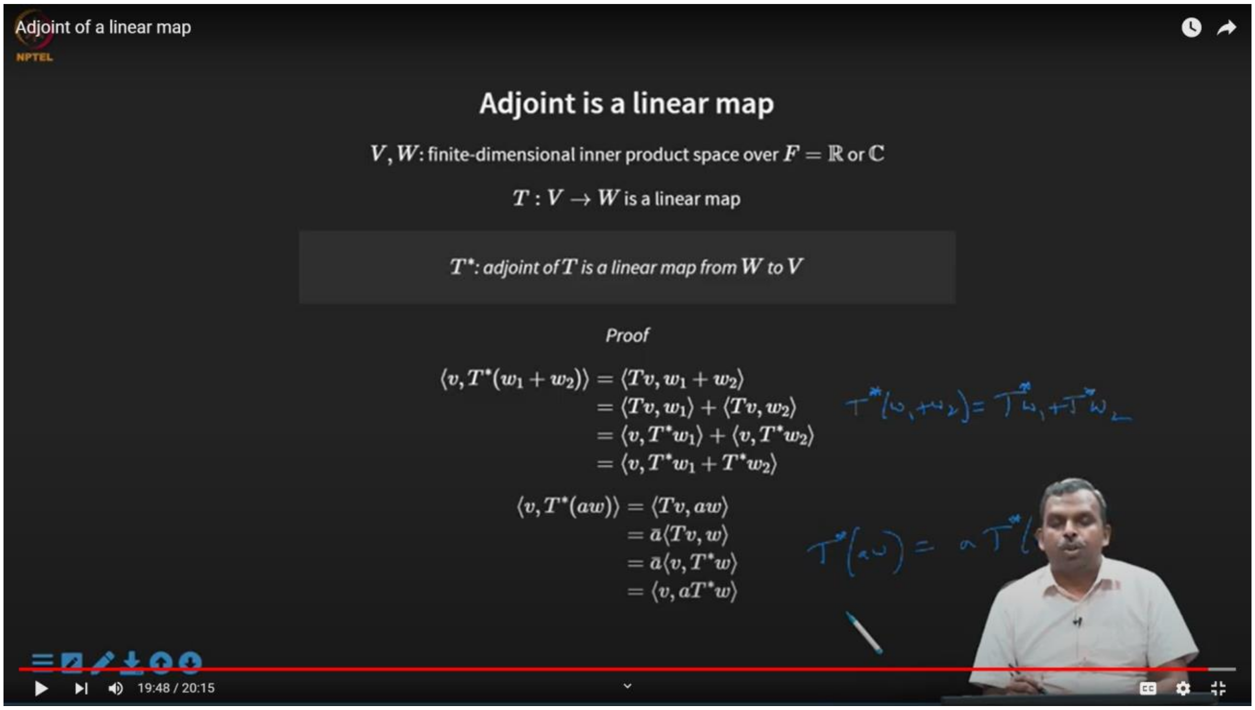

Now you know that this T^*w is unique in some sense, right? So this and this have to be the same. So $T^*(w_1 + w_2)$ equals this, okay? So that's what this means. Same proof you can do for $T^*(aw)$, okay? You do the same thing, there'll be some conjugation going back forth etc. and you will get this. So what these two things prove is that $T^*(w_1 + w_2)$ equals $T^*w_1 + T^*w_2$. And $T^*(aw)$ equals $aT^*(w)$. So these two things together show you that T^* is a linear map. So this is a way to prove this result. So what we have done in the short lecture is defined adjoint for a linear map as something that, you know, maps the inner product from one part as the inner product in the other way using this linear functional argument. And Riesz representation theorem we know that this adjoint is a map from W to V. Like if T is a map from V to W, T^* is a map from W to V. And T and T^* seem to be intermingled and they are connected wonderfully by this inner product in both of these spaces, okay? So this is a general linear map. When we specialize to adjoint of an operator, you will see many more interesting things will come. But even in this case, there is a nice little way to define various things around T and T^*. So you will see a lot of interesting ideas can be developed around that, okay? So that’s the end of this lecture. We will take it up, take up more interesting properties of adjoint and study it further in the next one. Thank you very much.