Applied Linear Algebra
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Week 04
Determinants

Hello and welcome to this lecture. In this lecture, we are going to study determinants. So in fact we'll do a very quick job of looking at determinants, they are not very crucial today to linear algebra as it was before. In fact many applications of determinants in numerical linear algebra, purely in the linear algebra part of it are not so popular anymore. So you don't need to introduce determinants early and do the theory with respect to that. However, determinants are very useful in other areas of mathematics. They are used quite often in other applications to capture a lot of properties of systems, so it's an important thing to know. So we'll quickly go through the properties of determinants and their connections to linear maps. Later on, we'll come back and revisit it and define them in another way and look at other properties, okay? So let us proceed.

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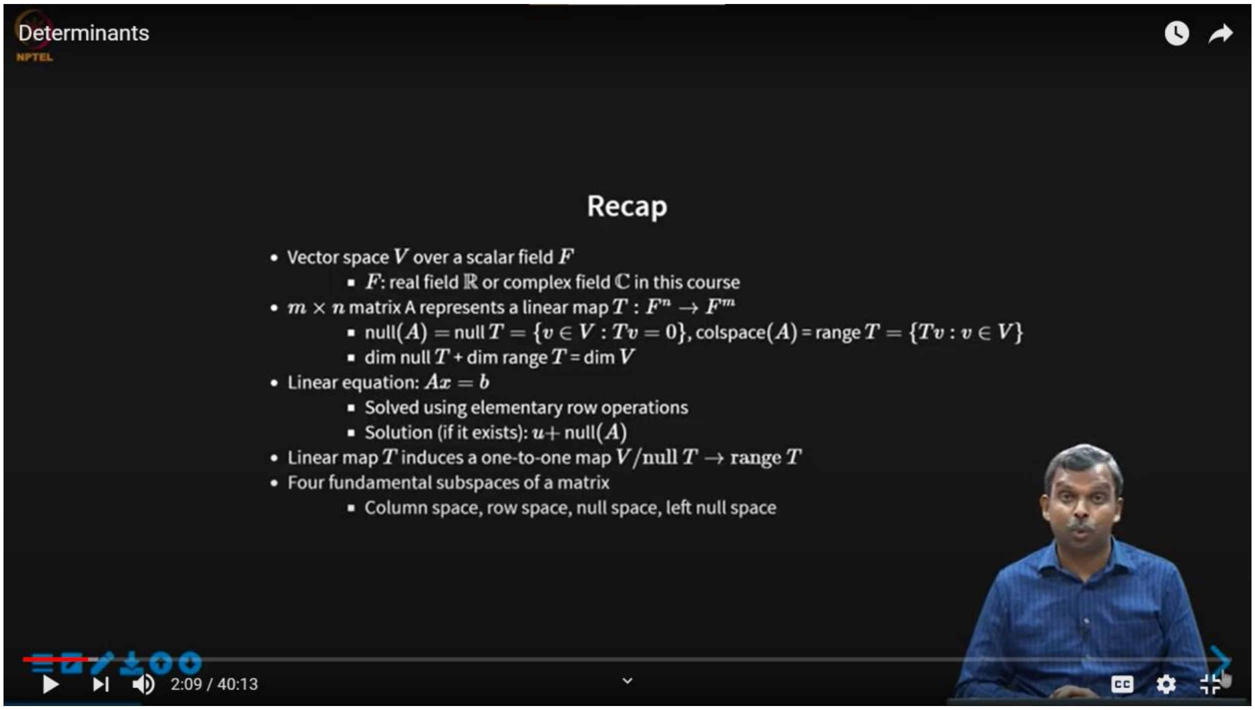

So a quick recap. As usual, we are looking at vector spaces. We looked at, in the last few lectures we looked at linear equations, how to solve them and how there is this quotient with the null space which is very interesting to understand a linear map. And then we saw these four fundamental subspaces of a matrix, right? So column space, row space, null space, left null space. We looked at connections between all these spaces, you know? And we saw that column space and row space have the same dimension for a matrix. That was something, some sort of a special property. And then we saw these elementary row operations which give you a certain simple form, the row echelon form is very, very succinct. In fact, if you have an invertible matrix, invertible square matrix, if you do row reduction, you can get it down to an identity matrix, right? So that's also something that's possible. The form is very simple once you do a row reduction, okay? So I'll pick up from here and we'll introduce determinants. We'll see it in a slightly informal sort of manner, we won't be very rigorous, complete in our proofs and description. And that's more or less enough, the crucial properties which we will use later on will be more or less... This introduction is enough. So it's a quick lecture introducing determinants. Let us jump right into it.

Okay. So for the purpose of introducing and defining determinants, I will use the following notation for a matrix. If I have an $m \times n$ matrix A, I will call each row as you know v_1, v_2 etc. and then I will put the semicolon to separate the rows. So if I say $[v_1;\ v_2;\ v_3]$ so on, the first row is v_1, the second row is v_2, like that, okay? So just to know, put a succinct notation for the matrix, I am using semicolon to separate the rows in the notation, okay? So just a quick notation. We will use this over and over again to describe determinants, okay? So most of you are familiar with the formula for determinant. I think, like I said, this is the second course you are doing in linear algebra so you must know determinants, this notion of minors, cofactors expansion along rows and columns, that is known to you. But I will sort of introduce determinants in a slightly different way, slightly more abstract way, if you will. Without actually defining it, I will define some of the defining properties of determinants. What are the properties the determinant has, okay? Determinant is a function, it is a function which takes as input a square matrix, $n \times n$ square matrix over some field and its output is a scalar value, okay? So it puts out a scalar value from the field. The most common field we will be looking at is the real numbers, of course. So if you take an $n \times n$ real matrix, determinant will do something to the numbers in that matrix and produce a real number at the output, okay? So that is what determinant is. It has the following properties which we will assume that this function has, okay?

So at this point, as far as this lecture is concerned, we may not even know if the determinant has a formula or not. But I do not care. Let us say it is some function which satisfies these properties. That is how I care about determinant. From your practice, you know that there is a function for determinant. But let us say for now you forget all that you learnt about determinants and somebody were to introduce determinants for the first time. I am going to introduce it like that. I am going to say determinant is some function from square matrices to the scalar field which satisfies the following properties. There are four properties that I'll list down. First property is for the identity matrix, I should have a determinant of 1, okay? So that's the first property. The next one is row scaling. If I take a matrix and scale its row by c, I should get the determinant without the scaling. So hopefully you see this relationship. So I have the original matrix $[v_1;\ v_2;\ \ldots;\ v_n]$. I scale the j^{th} row by c and then I evaluate this function, determinant. What should I get? I should get c times the determinant of the original matrix. So that is one of the properties that this function should have. Next it should have this linearity property with respect to the row. What is linearity? Supposing I write the j[th] row as $v_j + v_j'$. Two vectors that add to give the j[th] row. Then it should be equal to the determinant with v_j alone in the j[th] row plus the determinant with v_j' alone in the second matrix. So I can make two matrices. All the other rows remain the same, okay? Nothing changes in the other rows, only in the j[th] row I have $v_j + v_j'$ as the original determinant that I want. I can equate it to the determinant without the v_j', just the v_j, plus the determinant with only the v_j' without the vj, right? So that's the formula, that's the requirement of row linearity. There are other terms that people use for these things, I'm just saying... You know? Some people call it multilinearity, etc etc, okay? And then the last one is equal rows, which you know. If two rows are equal then the determinant should go to zero, okay? So these are the properties that one needs for a function. Like I said, assuming you do not know anything about determinants, you don't know the formula, you've forgotten the formula, so how do you define a determinant? Determinant is some function which satisfies these four properties, okay? So like I said at this point in the lecture, I haven't even told you such a function exists, okay? So we don't know if such a function exists or not. Why do you give such an intricate definition, what's the point of this and is there such a function? Like I said, once again I'll remind you from your practice and from your previous experience, you know that such a function exists. But let's say, even if you don't know, let's hypothesize that such a function exists. And then you'll see it's very useful, that you can do a lot of nice things with a function which has these properties, okay?

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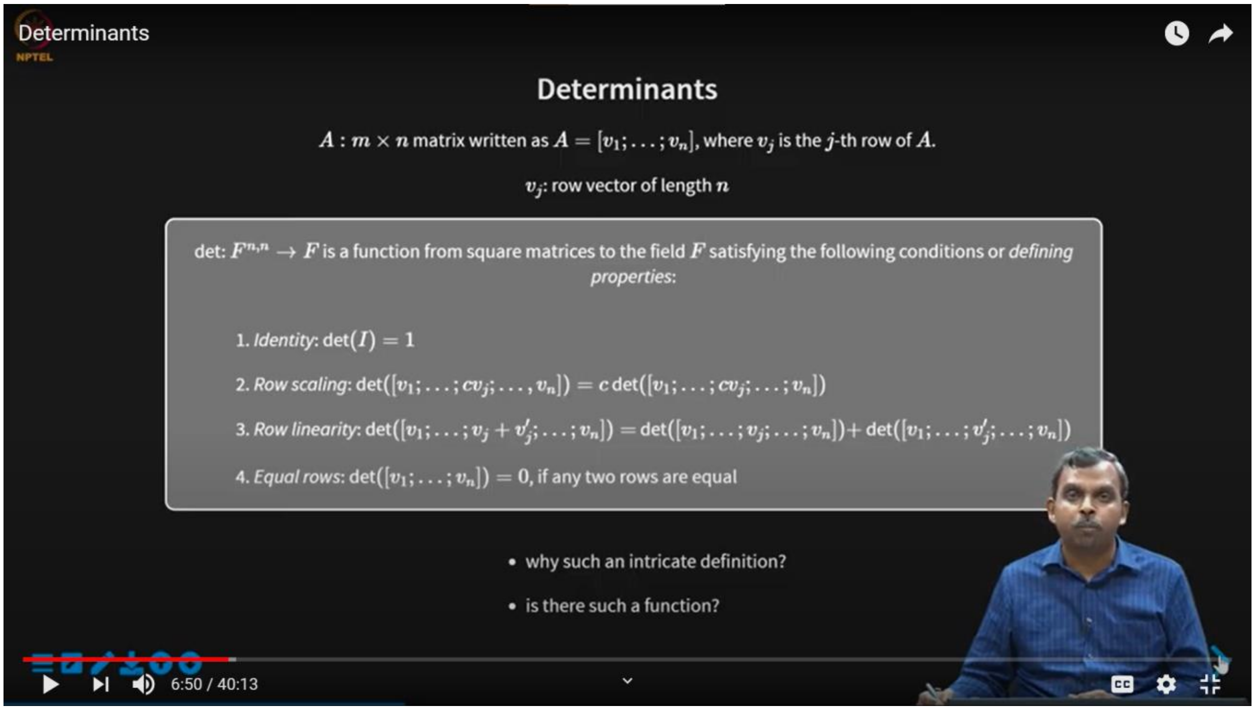

So what are those nice things we can do is what we're going to focus on and why this definition you'll see... I mean there's lots of ways to motivate it. One interesting way to motivate it is through this geometric connection, okay? So if you look at… We'll take small examples. So I'll start with a 2×2 example, probably the simplest example. Of course there is a 1×1 example which is too trivial. So we will leave that out. I will go to the 2×2 example. If you look at $[a\ b;\ c\ d]$, okay, it turns out you can define the determinant which is $ad - bc$. You may be aware of the formula and that's equal to the area of the parallelogram formed by (a, b) and (c, d) okay? So if you want, you can draw this little picture here which might be interesting to you. So if you have (a, b) here, let's say somewhere here. (a, b). And then you have (c, d) here. The parallelogram sort of defined by (a, b) and (c, d) would look something like this, isn't it? This point would be $(a + c, b + d)$, okay? So that is the sum of the two vectors and that forms the parallelogram, okay? Hopefully that's okay. That's okay. (c, d), (a, b). Then you get that, okay? So $a + c$ and $b + d$. So you, if you want to compute the area of this, okay, it is actually the absolute value of $ad - bc$, okay? So you might have seen this. If you have not seen this, this is one of the motivations for why the determinant is sort of defined in this fashion, it's the area. You look at the rows, look at the parallelogram created by these rows and the area of that is the absolute value of the determinant. You can prove this $ad - bc$. So you can make this, you know, you make a rectangle, you know, big rectangle and then subtract out all the other areas, you will get this, okay? So it's easy to show this. This is one motivation for the 2×2. There is something similar for 3×3, you know? 3×3 also holds, it turns out it's true.

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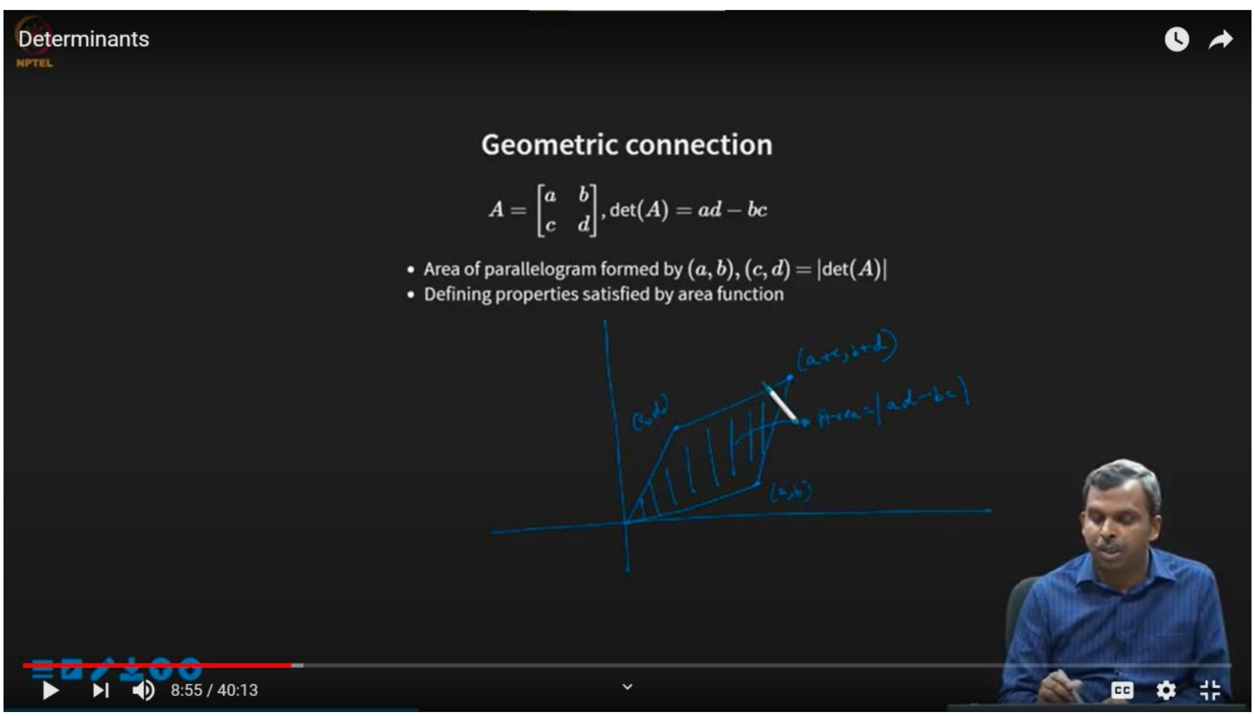

So if you go to 3×3 now, okay, you have this ABCDEFGHI. So if you compute the determinant from your formula, you will get this. So this is a valid formula for 3×3 and you can check all the four properties in case, you know, just to show you that this is a valid determinant. You can go check all your four properties, okay? Identity matrix will be 1, you know, row scaling will happen. Any two rows are equal, it will become zero and then the row linearity will also happen. This formula will, you know, all those formula properties will be true. You can check that. So it's a valid determinant, okay? And it's also the same determinant that you know from before. And it turns out it's also... If you like, take the absolute value of this guy, it's the volume of the cuboid formed by $(a, b, c), (d, e, f)$ and (g, h, i) in $\mathbb{R}^3$. So let's say I make v_1 as ABC, v_2 as DEF, and v_3 as GHI. So the cuboid is the set of all v... Or let me write like this. Set of all $a_1 v_1 + a_2 v_2 + a_3 v_3$, $a_i \in (0, 1)$, okay? So that's the cuboid. This is also the parallelogram, you can go back and see the parallelogram. Parallelogram was set of all points, see it's all linear combinations. But this a_i has to be between 0 and 1, okay? You are not allowed to let this a_i go wherever it wants. It should be between 0 and 1. So if you make all possible combinations, you will get the cuboid, okay? So that's the definition for the cuboid.

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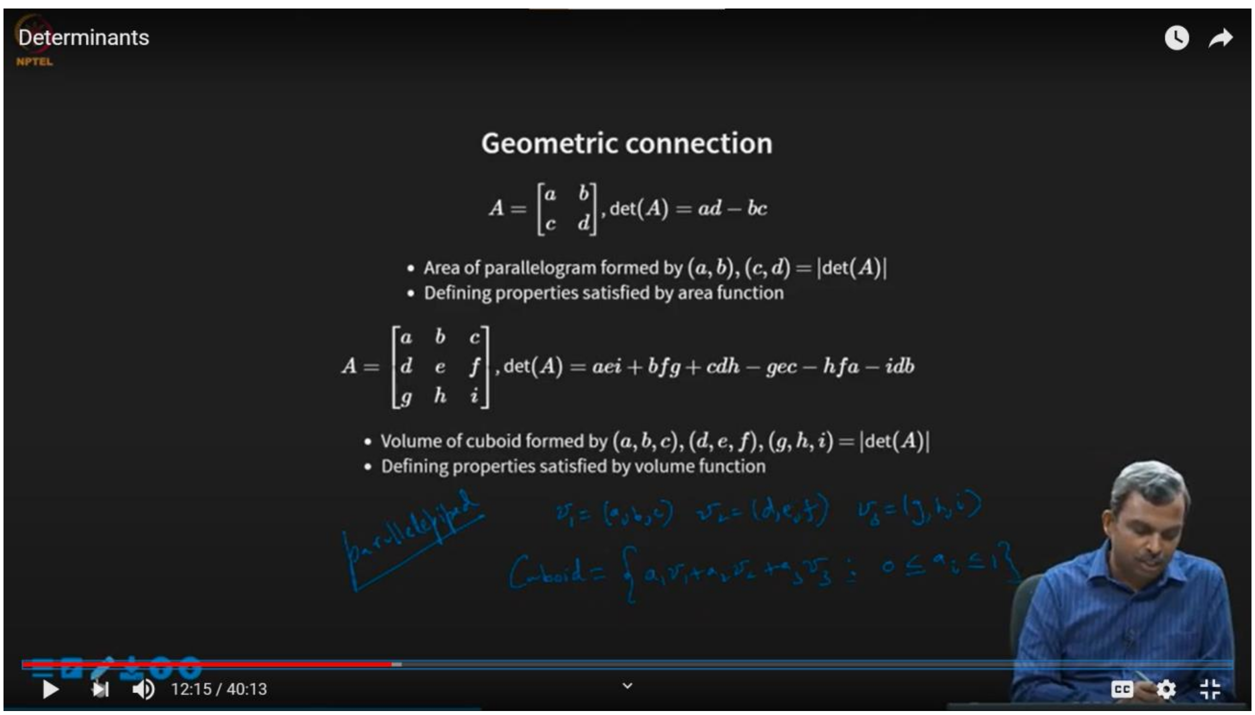

You can also geometrically picture it, you know? You have a, b and c, you know? Three different… I mean not a, b and c, v_1, v_2 and v_3. And you take all possible combinations, you get the whole cuboid, right? So it will be the proper cuboid that you can get. So more than a cuboid, maybe a better term is this parallelepiped. Parallelepiped, I think, it's probably a better term for this. But, you know, three dimensions cuboid is reasonable, you know what, you know what I mean when I say that. So the volume of this cuboid will be the determinant of A. So it's not really a proper cube, you know, I mean it's not, it's like a parallelogram, it doesn't have 90° and all that, so it's like a, some three-dimensional parallelogram extension, okay? So that's what it is. So you can see that. So that's why cuboid is maybe not a very good term. But you know what I mean. So you can sort of picture that this is a three-dimensional solid object, sort of parallelogram-ish extended to three dimensions. Its volume, if you compute the volume, is given by the determinant's absolute value, okay? So this is a nice geometry connection and one can extend this to n dimensions also. If you do $n \times n$ determinant of A... I mean I haven't really talked much about it, but the volume of the n dimensional parallelepiped formed by the rows satisfies all these defining properties. So if you imagine the parallelogram, right? So if you look at row scaling, what am I doing? I am keeping one vector the same, the other vector gets scaled by something. So clearly the area will also get scaled. Same thing with the cuboid, right? If you take the base, if you keep the base the same and you keep increasing the height alone, the volume will, you know, increase linearly. If you take two vectors and add them, you know, and you will get the volume of the first parallelogram and the area of the first parallelogram adding to the area of the second parallelogram, you know, all of these properties can be verified with the volume intuition. So those properties that we had in terms of linearity...

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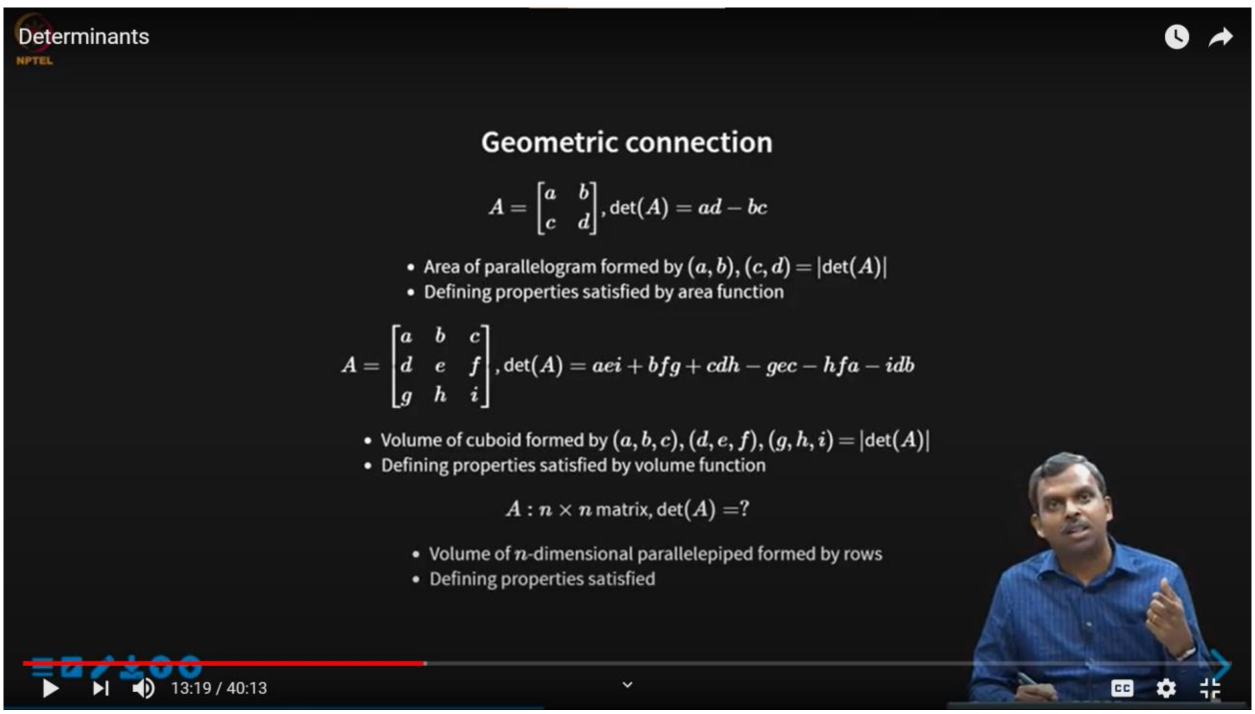

In fact, what about the third property? Two rows being the same? Supposing you want to define a cuboid where two rows are the same, then you don't really get a cuboid, right? You don't really get a volume, you'll only get a two dimensional figure. So its volume is zero, okay? So that also makes sense. In a parallelogram if you make the two lines meet, then your area is zero, okay? You don't have an area anymore. So same thing happens with this volume of the parallelepiped that you define with the rows of the matrix given to you. All those properties make sense from a geometric point of view. So that's the connection to why a determinant should exist. The volume of this parallelepiped is the determinant's absolute value. Of course there is a sign to worry about, let's not worry too much about it. There is some orientation involved in the sign, but this is the geometric intuition and connection in case you are wondering why these, all these properties are interesting. But for us, determinant is more interesting from the linear map perspective, right? So what does it say about the linear map, we are most interested in that, okay? So supposing you have a matrix A, square matrix A and supposing I compute its determinant or the determinant function I want to use or figure out something, what will it say anything about the linear map, what kind of properties can I deduce from it, that's most interesting to us in this course. And let's focus on that in the remainder of the lecture, okay? So for that we will, there are further properties of the determinant. We will only use the four defining properties and manipulate them and then we'll get further more interesting properties.

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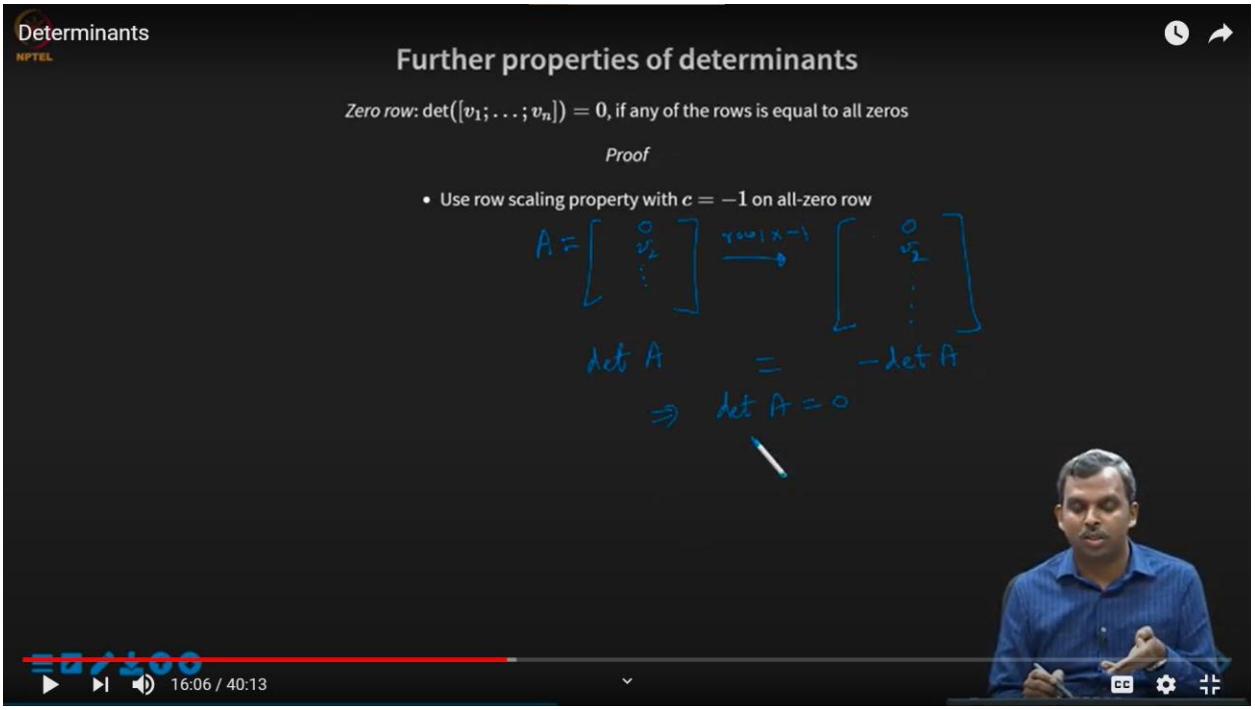

So look at these properties. The next property is zero row, okay? Supposing one of the rows of the determinant is equal to all zeros, entirely zero, okay? Then it turns out determinant is 0. Why is that? So you have A... I'm sorry. You have a matrix A, let's say just for argument's sake the first row is 0, the next row and all is some v_2 etc, okay? So like that, okay? So I am going to look at the matrix which is, you know, -1, you take -1 and multiply the first row with -1, okay? So row 1, so I do this here. row 1 into -1, okay? So if I do that, what do I get? I get the same matrix, right? Why is that? Because the first row is 0. If you multiply -1, you get the same matrix. But what does the property of determinant tell me? Determinant of A here, property tells me minus determinant A, right? Because if you multiply any row by a scalar c, the determinant should get multiplied by that value. But the two matrices are identical. When you have the zero row. So these two have to be equal. Which implies determinant of A equals zero, okay? So the property that, you know, scaling of rows results in the scaling of the determinant enforces this result. If you have an all zero row in your matrix, that matrix determinant has to be 0, okay? So that is a property that one can quickly derive. That is the zero row property as I call it, okay?

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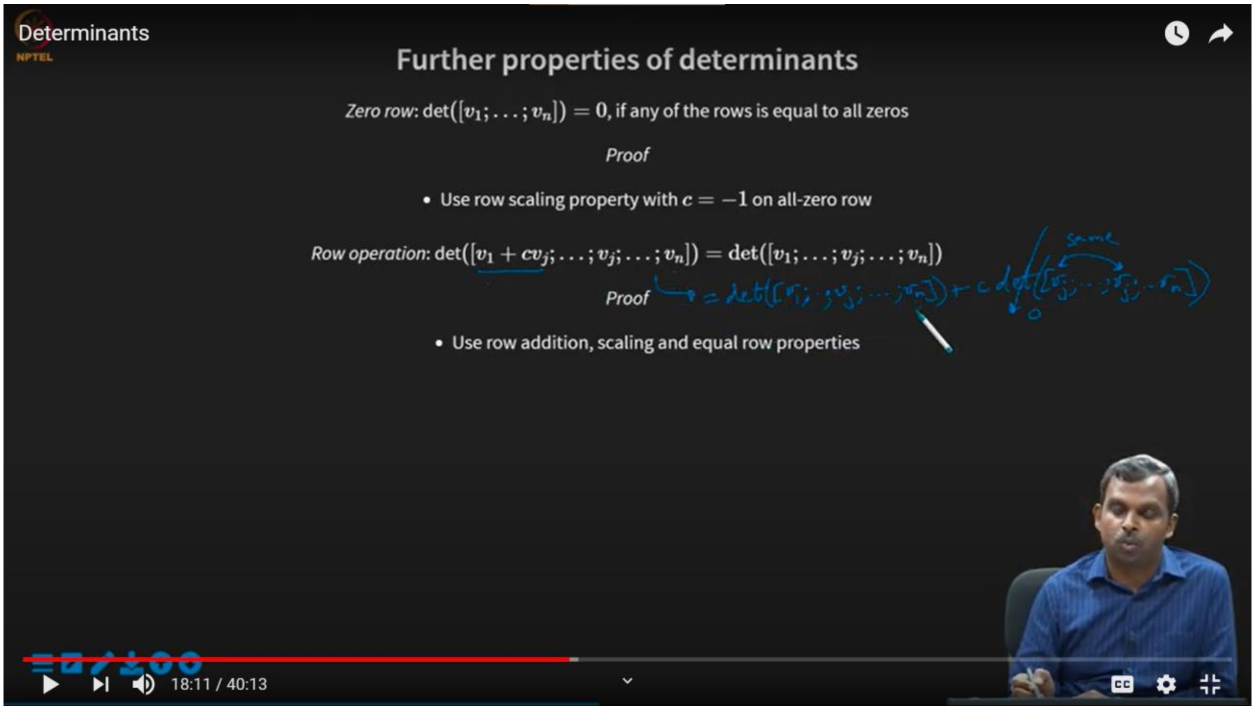

So let us go to the next property. The next is this row operation, okay? So we saw this elementary row operation where you did something like this, right? You took row 1 and multiplied c times $row\ j$ and added it to row 1, right? So this is the operation $v_1 + cv_j$, okay? Now what happens to the determinant when you do this, okay? That's the question that one can ask. So it turns out determinant does not change when you do an elementary row operation of this form $v_1 + cv_j$, okay? Of course there are other elementary row operations like, for instance, multiplying a row by c. If you do that, determinant gets multiplied by c, okay? So this particular operation where you do one row plus c times another row... I have put v_1 here, nothing is sacred about v_1, it could be v_i, okay? You take the ith row and add c times the jth row, okay? $v_i + cv_j$ also, the determinant does not change. It remains the same. When you do this row combination. Again, it's just a sequence of properties, right? So how do you prove this? This guy is the same as determinant of, you know, $v_1\ldots$ I used first the addition property $[v_1; \ldots v_j; \ldots v_n]$ plus the linearity property, right? The row linearity property, and the scaling, okay? So then you will get c times v_j. Here the c will come out, c determinant of $[v_j; \ldots v_j; \ldots; v_n]$, right? So this is a property that you are able to use. Use the row linearity plus the scaling. Now here these two rows are the same, okay? They are the same. So if you have two rows same, you have that last property of the determinant. So this whole thing goes off to zero. So you see the determinant of this guy is the same as the determinant before the operation, okay? So this row operation does not change the determinant, okay? That's good to know.

Okay. So from here one can extend to dependent rows, okay? If you have a determinant of a matrix and the rows or columns are linearly dependent, then the determinant is zero, okay? So forget about this “or columns” for a minute, we'll come back to that soon enough. We'll see why that is true. But let's first look at the rows, okay? If the rows are linearly dependent, the determinant ends up being zero, okay? This function which has the simple properties of, you know, row scaling, row linearity, two rows being equal is zero, determinant of identity is one, just those four properties make sure that if the rows of the matrix are linearly dependent, then the determinant will end up being zero. So how do you do it? It's not very hard, you use your linear dependence lemma. You know that v_j is in the span of the previous vs and you put out a linear relationship. You replace v_j by this combination, okay? And then you evaluate this determinant. You will see that, you know, first of all this is a zero row, right? When you replace v_j by this guy, you will get a zero row, okay? So that determinant will become zero and then when you expand it out like this, all the other guys will go to zero, only the v_j will remain. And you will get this is equal to zero, okay? So I am not going into details of this proof. I think hopefully you can see that. So you take this matrix, replace v_j with this combination. But remember that combination is actually 0, okay? So on the left hand side you will get 0. 0 equals, you will start expanding with your addition formulation. In the addition formulation, only the first value will survive, only the v_j the determinant will survive, everything else will become 0 because of the row being the same, two rows being the same. And then you will only get the determinant being equal to zero on the left hand side, okay? So that is a simple little method here to see that.

So what about columns? Why do columns enter the picture? Because we know row rank and column rank are the same, if the columns are linearly dependent in a matrix, the rows are also linearly dependent, is that right? Is that right? In a square matrix that is true, right? If it is not a square matrix then you can have other possibilities. But in determinants, remember we are dealing only with square matrices, okay? So keep that in mind. Square matrices, whenever we talk of determinant, we are talking about square matrices. So in a square matrix, if the columns are linearly dependent, the rows have to be linearly dependent. Why? Because row rank and column rank are the same, okay? So if the columns are linearly dependent, then the column rank is less than n which means the row rank is also less than n. So if the row rank is less than n, then the rows are also linearly dependent, okay? So whether the columns are linearly dependent or the rows are linearly dependent, determinant will end up becoming zero, okay? So that's a nice, nice property to know.

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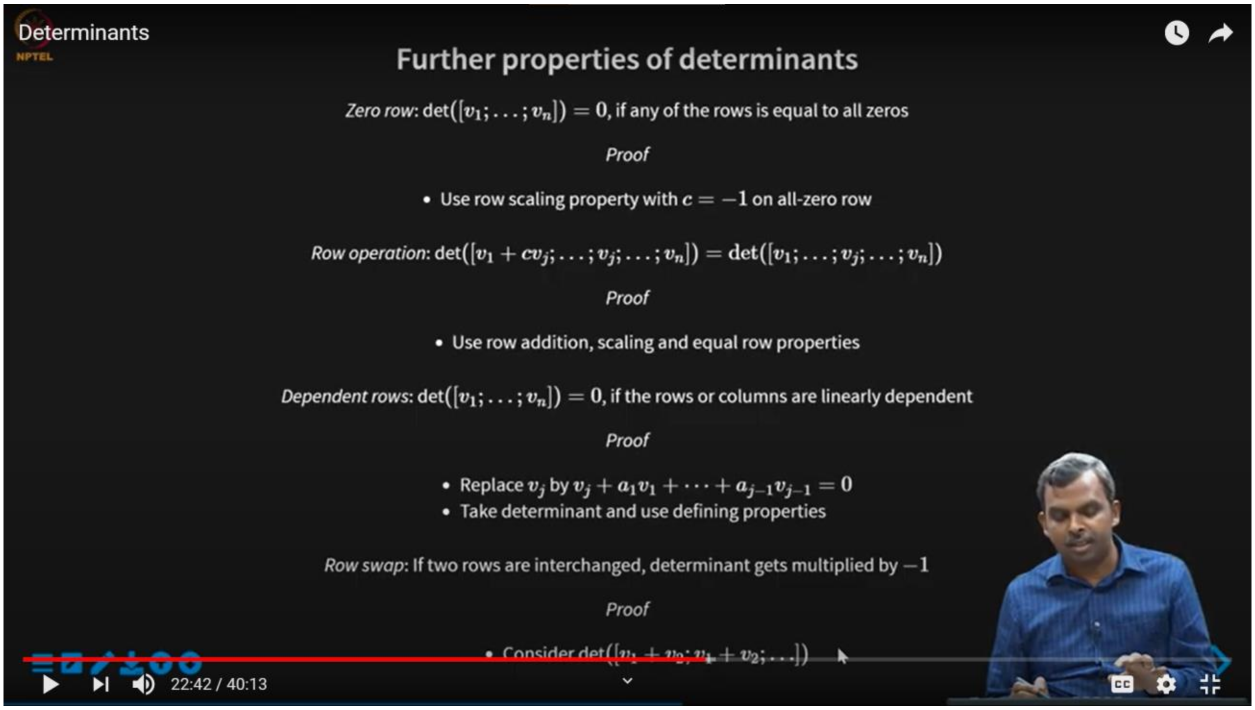

And the final property is a row swap, okay? So we were doing these row swaps as part of the row operations, right? If two rows are interchanged, then it turns out the determinant gets multiplied by -1, okay? So how do you prove this? This might seem like a little bit difficult. But what you do is - you take the original matrix $[v_1;\ v_2;\ \ldots;\ v_n]$, you look at this other determinant. What is this determinant? I look at $[v_1\ +\ v_2;\ v_1\ +\ v_2;\ \ldots]$. First two rows instead of being v_1 and v_2, I will replace them with $v_1\ +\ v_2$ and $v_1\ +\ v_2$ and look at the determinant of that, okay? Now use your row properties, okay? Row linearity properties, you will get four terms on the right hand side. So this determinant is equal to four terms on the right hand side. On the left hand side, this determinant itself is zero. Why? Because the two rows are identical, okay? On the right hand side, two of the rows, two of the determinants will be zero. Again because rows are identical. The two others will be $[v_1;\ v_2;\ \ldots]$ and $[v_2;\ v_1;\ \ldots]$. The sum of those two determinants is zero which means determinant of $[v_2;\ v_1;\ \ldots]$ equals minus determinant of $[v_1;\ v_2;\ \ldots]$, okay? So that's the simple proof here. Once again I am skipping the detailed writing down of every step. But hopefully you can see where this comes from. So once again, I want to emphasize, we just defined the function determinant where I did not really fully specify what it was. I am just relying on your experience to know that such things exist. We motivated it with the geometry connection, but there were these four simple properties and those properties are helping us derive more interesting ones. Particularly the dependent rows is really interesting, right? So when the rank goes less than n, determinant goes to zero, okay? So that's a nice property to know.

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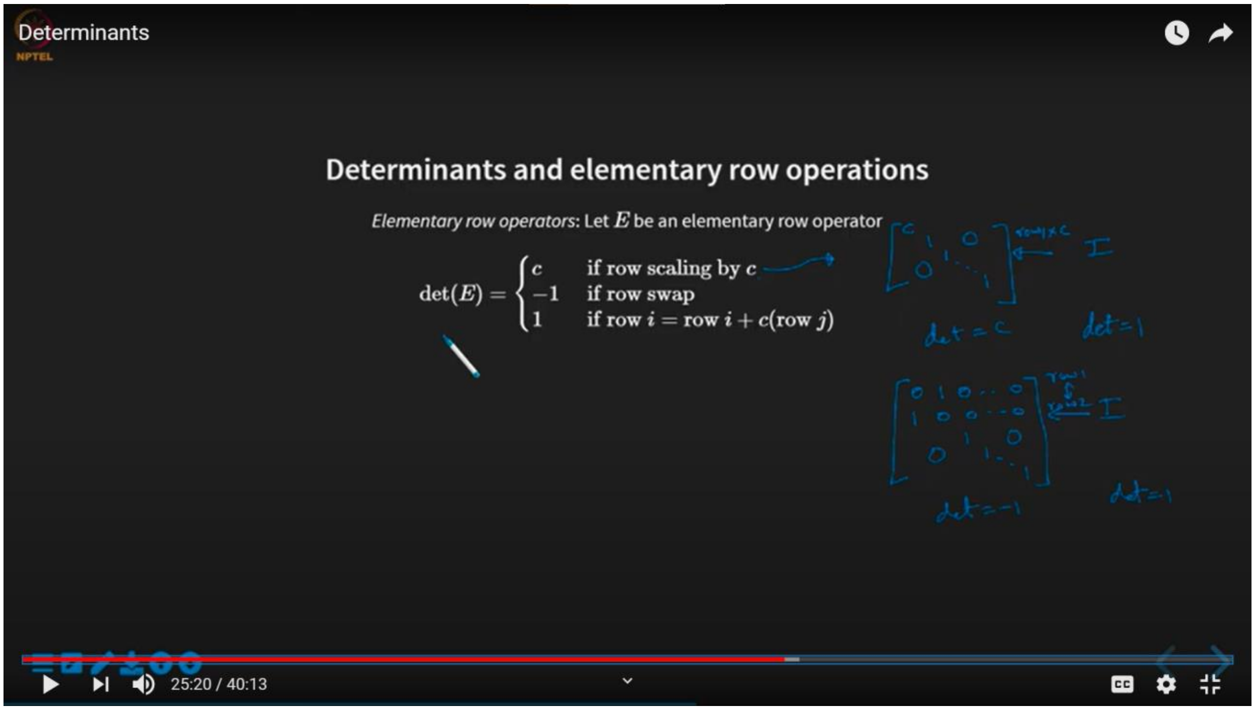

Okay. So it turns out one can say a lot more about determinants, these elementary row operations are sort of very easily tied to determinants, okay? The first thing is - one can compute the determinant for the elementary row operators very easily, okay? So let me just show you the first one, okay? So if you look at the determinant of the first one, the row scaling by c, so what is the matrix corresponding to row scaling by c? Let's say, let me say the first row is scaled by c, everything else remains the same. So you get a diagonal matrix with 1 on all the other terms except for the first row where there is c, okay? Now the c could be any other place also but let us say it is in the first place. Now this you see, you know. How do you get to this? You take the identity, okay, multiply row 1 by c, okay? So then what should happen? Here you have determinant equal to 1, here you have determinant equal to c isn't it? It is c times this determinant. When you multiply the row by c, so you see determinant of E for any row scaling by c is c itself, okay? So that is very nice. What about row swap? Why does it become -1? So if you look at I again... Everything starts with I when you look at elementary row operations. What happens when you swap rows? You're going to swap the first and second row, let's say $(0\ 1\ 0 \ldots 0)$ and $(1\ 0\ 0 \ldots 0)$, everything else remains the same, right? Everything else is 1 on the diagonal, okay? How did I come to this? You exchange row 1 and row 2, okay? Any row swap is like that, right? When you want to swap two rows, the matrix corresponding to that, operator corresponding to that is simply the swap of identity, swap of the corresponding rows, okay? So here determinant is 1. From your row swap property, you know that the determinant here has to be -1, okay? Same thing with the last one, okay? When you do $row\ i$ equals $row\ i$ plus any c times $row\ j$, we know the determinant does not change, okay? And the matrix corresponding to this operation is simply the same operation applied on I, okay? So you take I for which you know determinant. You apply the same operation, you will get that the determinant is 1, okay? So this is a simple way to compute the determinant for elementary row operators. For the elementary row operators, the three of them, okay, row scaling, row swap and row combination, right? $row\ i\ =\ row\ i\ +\ c(row\ j)$, we see that the determinants are easily computed because they operate on the identity matrix, okay? That's easy to write down.

Now we can do more. It turns out when you do product of elementary row operators also, it is easy to characterize the determinant, okay? So look at this formula for instance, right? So why is this true, okay? So this you can verify, okay? What happens when you do E times A for each of these operators, okay? When you do EA, if you do row scaling by c, right, if E represents the row scaling by c, when you look at $EA\ldots$ What is EA? EA is basically that operation done on A, right? If you do row scaling by c, row 1 is multiplied by c, EA is simply that row scaling by c, okay? So you know the determinant of EA when you do row scaling by c is going to be c times determinant of A, okay? Now but what is c? Now if E represents row scaling by c, right, this c is actually determinant of E, okay? So you will get $\det(EA) = \det(E)\det(A)$. Now what if E is row swap? Then again you see determinant of EA becomes $= -\det(A)$. So, -1 is again determinant of E. Now what if E is just row combination? This $row\ i\ =\ row\ i\ +\ c(row\ j)$? Again you see determinant of E is 1 and $\det(EA) = \det(A)$. So just by looking at it case by case, you can conclude in general that when E is an elementary row operator, $\det(EA) = \det(E)\det(A)$, okay? So that's a crucial result. It seems simple enough. But once you do this, you can start iterating on it, okay? You can do an induction on this result and quickly see if you have a product of several elementary row operators, okay? If you do a product of several elementary row operators again the determinant will multiply like this, okay? So determinant of a product of several elementary row operators times A equals the product of the determinants of this E_i and the determinant of A. You can do an induction argument here. It's not very hard, right? If it's true up to i, how do you show it for $i+1$? In $i+1$ you have only one row operator multiplying. When you have one row operator multiplying, already the result is true, right? So it is very easy to do an induction, it is a basic induction here. So you get this result, okay?

So this it turns out is a very powerful result, okay? So why is this very powerful? Because we know elementary row operators can do wonders for your matrix, right? Elementary row operators we know can do wonders for your matrix. That we know, okay? So using the previous properties, we can now start really computing determinants for a whole bunch of matrices. The first matrix for which very easily you can compute determinant is a diagonal matrix. If you have a diagonal matrix $[d_1\ 0\ \ldots\ 0;\ 0\ d_2\ 0\ \ldots\ 0;\ 0\ \ldots\ 0\ d_n]$ then the determinant is simply the product of the diagonal elements, it's very easy to compute, you know? You just start with the identity matrix, do simple row multiplying multiplications you will get this, right? So you can do the, you know, row scaling by d_1 to d_n and this multiplies this... You start with the identity matrix. You know that this is true, it is very easy to do this.

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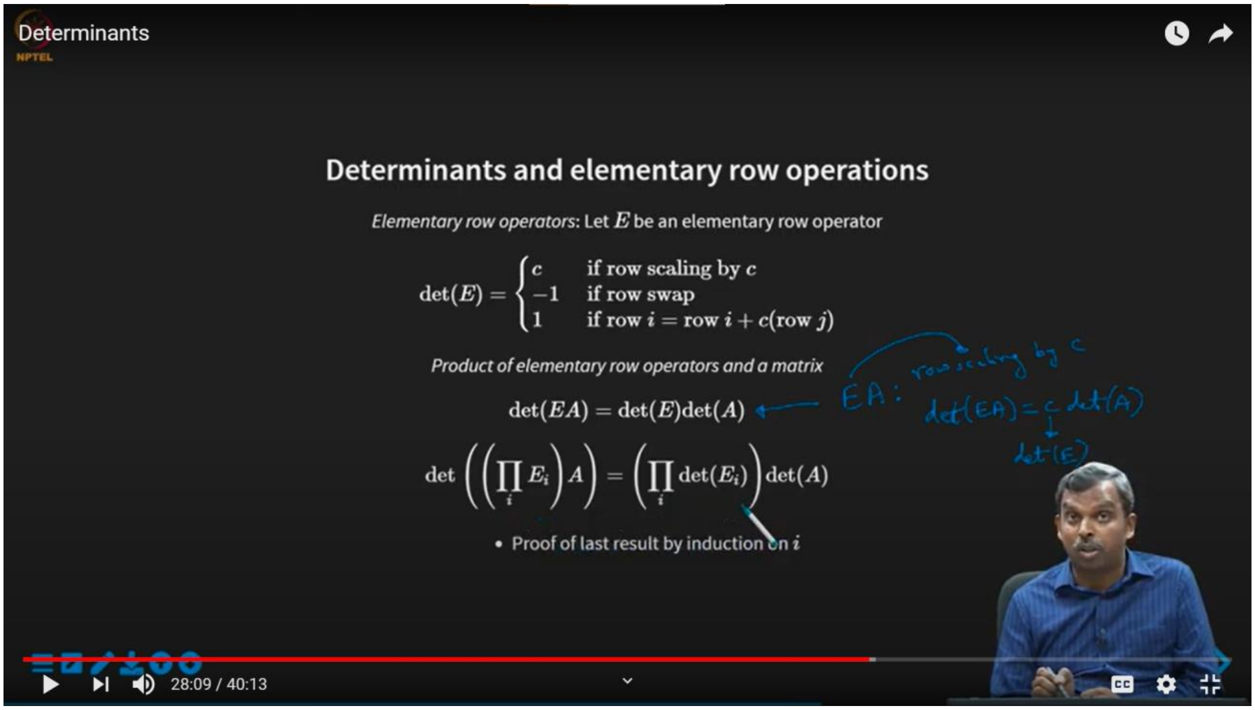

For any non-invertible matrix, determinant will be zero, okay? This is easy again, right? So if the matrix is not invertible, then the columns are dependent, rows are dependent. If the rows are dependent, determinant becomes 0. It's easy to compute. Now if you have an invertible matrix, it turns out you can do row operations to reduce it to identity, okay? So this is an important result. Go back and think about how we did row operations. We did row operations earlier to reduce it to an upper triangular matrix, right? You know you can always reduce it to upper triangular. And remember this is an invertible matrix which means the rank is n. And you have $n \times n$, which means every step of the way you will find a non-zero pivot, okay? The rank does not reduce. So you will get the full thing and then you will end with an upper triangular matrix. How do you go from upper triangular to identity, right? So you can do the row operations again, right? So you can do the corresponding row operations from row one minus etc., you can clear out all the zeros, all the non-zeros on the top by row operations and you will get it to identity, okay? So this is an important result. So go back and check and convince yourself that this result is true, okay? So basically, you know, you go from, because it's invertible there are n non-zero pivots, okay? And rank is full and you can reduce it to upper triangular and then further do row operations to reduce it to identity, okay? So this is very important, okay?

So what are we saying here? So there exist elementary row operators E_i such that the product of all these $\sum E_i A = I$. So this is crucial. Once you get this, you can take determinants on both sides.

When you take determinants on both sides, notice what happens. On the left hand side, you have this familiar form, it's just product of row operators times A. I know my determinants go inside, right? When my determinants go inside, what can happen, okay? Out of these E_is some of them will be row swaps. So let's say n_s is the number of row swaps that you did. What will happen to the determinant of a row swap? You will just get -1, right? So you get $(-1)^{n_s}$. Determinant of A will be there, you will get $(-1)^{n_s}$ because of that, okay? This comes from row swaps. I mean of course it’s on the left hand side but when you bring it to this side also it retains the same $(-1)^{n_s}$. What are these guys? These are the row scalings, okay? How many ever row scalings you had, whichever row you scaled, all those scalings will come into the denominator here, right? So, because all of them will have c_j as the determinant. What about row combinations? Those things don't show up because you know that they don't change the determinant. And determinant of I is just 1 so you just get this formula, okay? So your matrix is either invertible or not invertible. If it is not invertible then the determinant is zero. Straightforward, done. If it is invertible, you keep doing the elimination, find your pivots, you keep track of your row operators, you know how to compute determinant. It’s very easy, as simple as that, okay? So you just need to do elimination. You can compute determinant as well. And if it is invertible, it is, I mean you can see non-invertible case also when you find a zero pivot. Anyway it goes off to zero, okay? So that is the logic queue. So once again, using the simple properties of determinant we are able to easily compute it for any matrix that we want, okay? Not only this, you can do a lot more interesting things with these row operations and other things.

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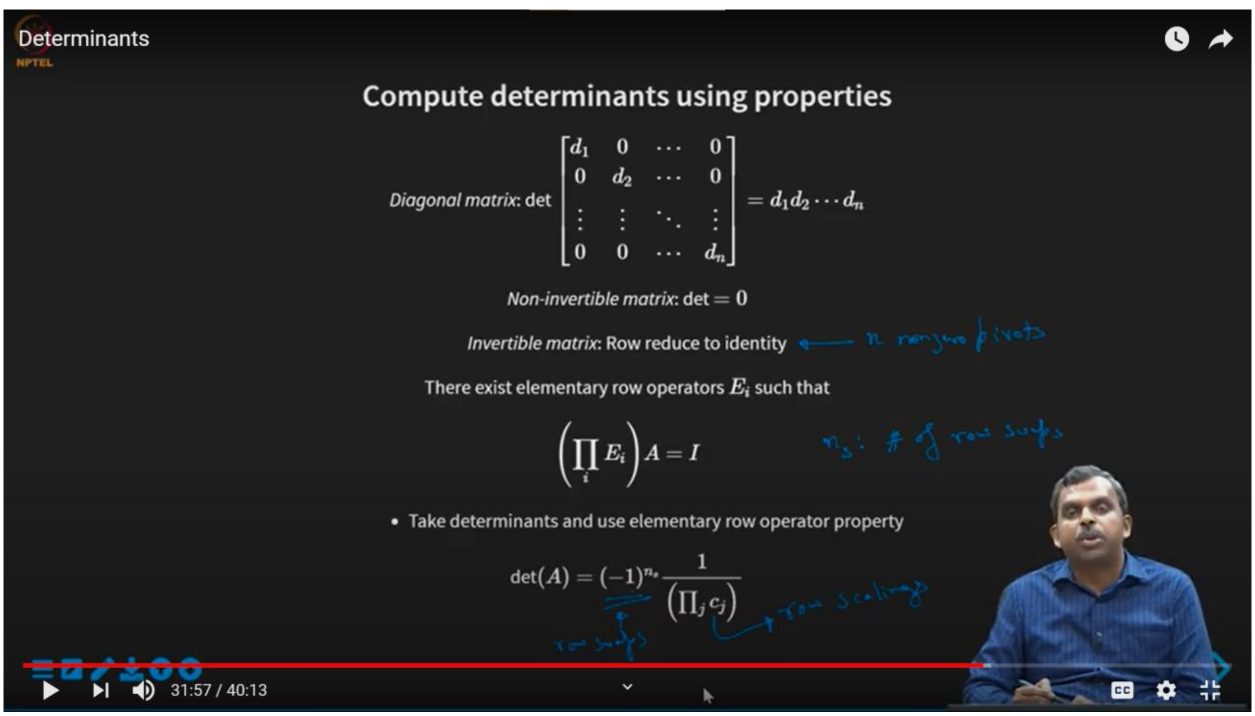

Here is another fantastic result that is very non-trivial and interesting for determinants. If you have two square matrices, determinant of the product of the two matrices is the product of the determinants. $\det(AB) = \det(A)\det(B)$. How do you prove this? First case you take is when A and B are not invertible, okay? If A and B are not invertible, it turns out AB is also not invertible, okay? So I will leave this as an exercise for you, it's quite easy to prove, okay? You prove this, okay? If A or B is not invertible, AB will be non-invertible, okay? So it is quite easy to prove. You can use many methods. you can just look at the, find the range space or null space and show in either case… Null space will be non-trivial, so it will be non-invertible. So AB is also non-invertible. So once AB is non-invertible, A or B is non-invertible, you see $\det(AB) = 0$ and $\det(A)\det(B)$ is also zero. So it's, both of them are equal, okay? The zero case is taken care of.

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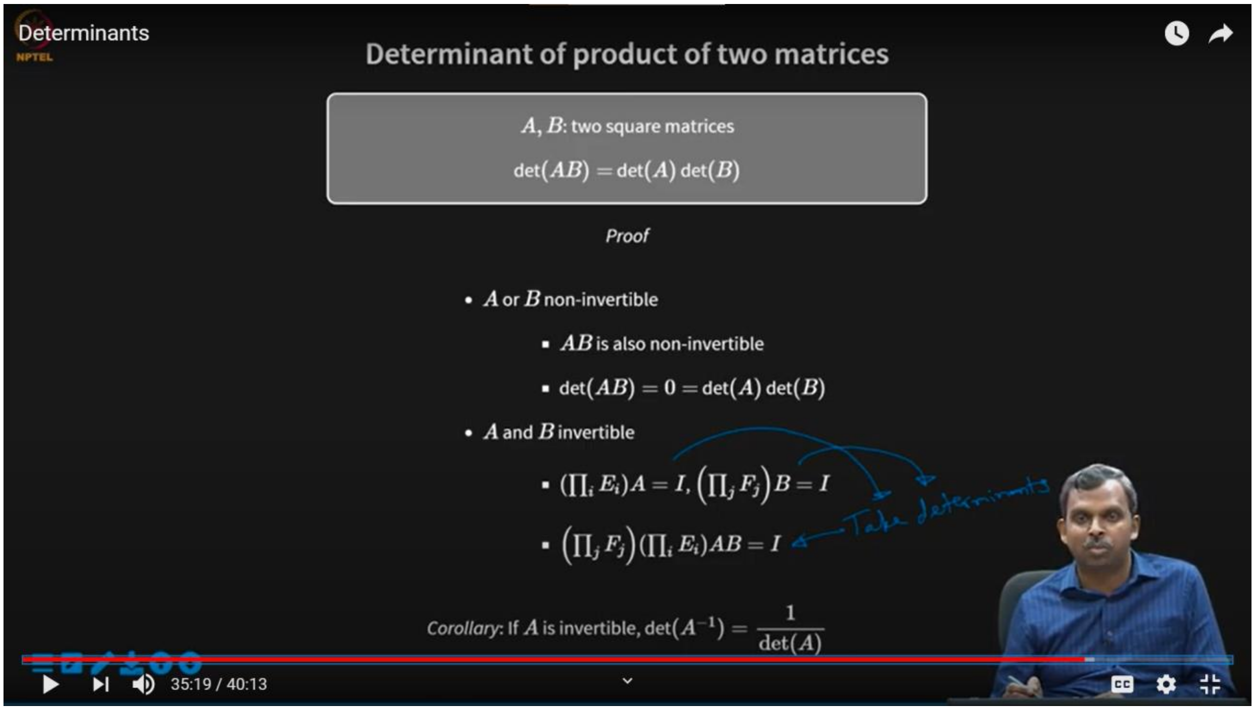

Now when A, B are invertible, when A, B are invertible I know that there is a sequence of row operations which take A to identity, sequence of row operations which take B to identity, okay? Both of these are true, isn't it? So you can do a sequence of row operations take A to identity, sequence of row operations take B to identity. And notice what I can do here. If I take the operation sequence for A and then put the operation sequence for B to the left of it and look at this product with AB, I get I, okay? So you can see this, you know, E will first operate on A, you will get I here. And then F will operate on B, you will get I here. So this is like the inverse that you can do. So this is true, okay? When A and B are invertible. Then what do you do? Once you have this equation, okay? Once you have this equation, you are through, right? So you take determinants, okay? So let me write that down. So you can take determinants here and then take determinants here and take determinants here. You will see that $\det(A)\det(B)$, when you multiply those two you get $\det(AB)$, okay? So it is quite an easy result that you can quickly do. So this product is what is most important. You simply take determinants on both sides, you will get… Because this is all elementary row operators, right? So we know elementary row operators, you can take the determinant inside and then you will get $\det(AB) = \det(A)\det(B)$, okay? So it's easy to do. I am not writing down the full steps here. So this is the crucial idea. Once you do this, you are done.

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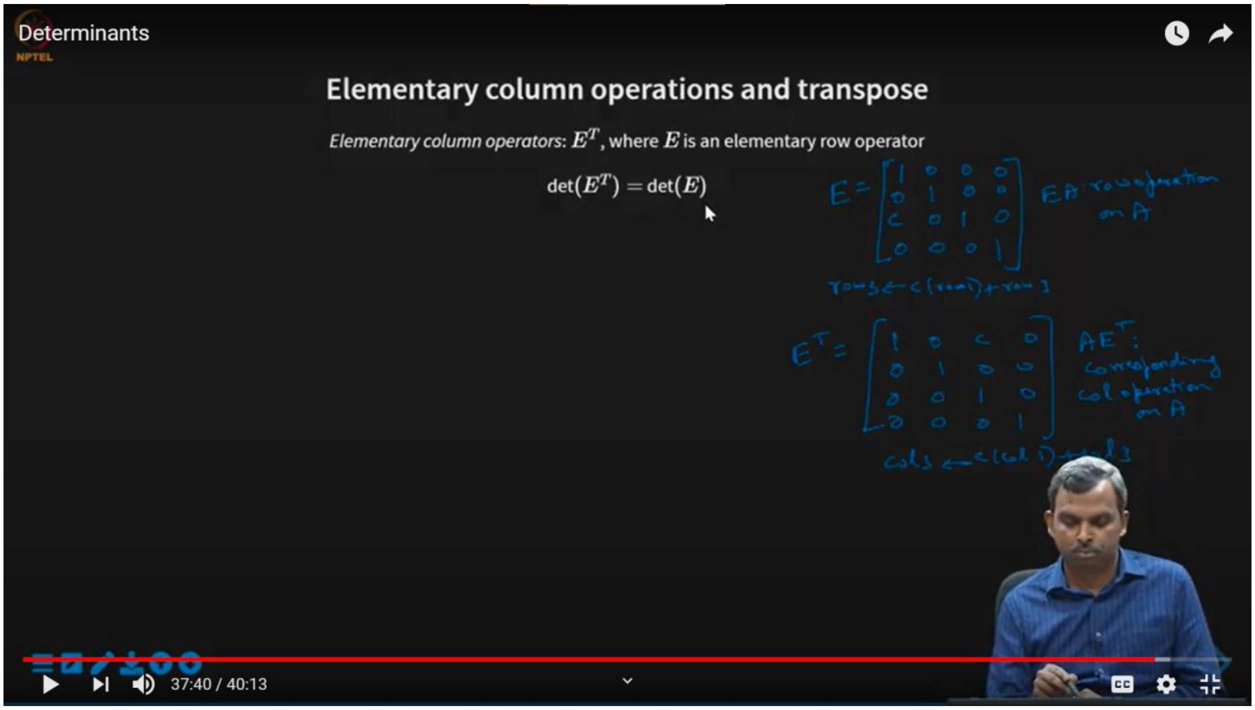

One of the nice corollaries of this result is the following. If A is invertible, you know $AA^{-1} = I$. So if you use the $\det(A)\det(B)$, you get $\det(A^{-1}) = 1/\det(A)$, okay? So it's a nice little formula. When you need determinants for the inverse, you don't have to compute the inverse and then use some crazy formula, you simply find the determinant and then take its inverse, you get the determinant of A^{-1}, okay? So that's a nice thing to do. What about column operations and the transposing, okay? So we've been always talking about row operations, but column operations and row operations are very closely connected. If you see, if you look at the elementary row operator and look at what happens when you do column operations... So let's look at E, which is, you know, row scaling is very trivial, right? It's diagonal so whether you do columns or rows, it's the same. The matrix doesn't change at all, okay? What's interesting maybe is the, maybe row swap is interesting, I don't know. But you know, what's maybe more interesting is the, you know, the row combination. So let's take a 4×4 matrix and then maybe, you know, row three became some c times row one. So let's take this matrix for instance, okay? So this basically represents $row\ 3 = c(row\ 1) + row\ 3$, right? So this is this matrix. If you look at the transpose of it, E^T, it basically represents, you know, the c would go there, right? So you get [1 0 c 0; 0 1 0 0; 0 0 1 0; 0 0 0 1] everything else remains the same. So this represents $col\ 3 = c(col\ 1) + col\ 3$. Remember when you do EA, it is row operation on A. How do you do column operation on A? You have to do AE^T, okay? AE^T is the corresponding column operation on A, okay? So this picture is good to keep in mind when you do, you know, when you do $row\ 3 = row\ 3 + c(row\ 1)$. You want to do the same thing for column, you take that E, do a transpose, multiply on the right, then you are doing the column operation with that, okay? So that is one thing. And also I think you can very quickly convince yourself. There are only three cases, each of them you calculate and you can convince yourself that determinant of E^T is the same as determinant of E, okay? This is not very difficult to prove, okay? So the column operations also determinant can be computed quite easily.

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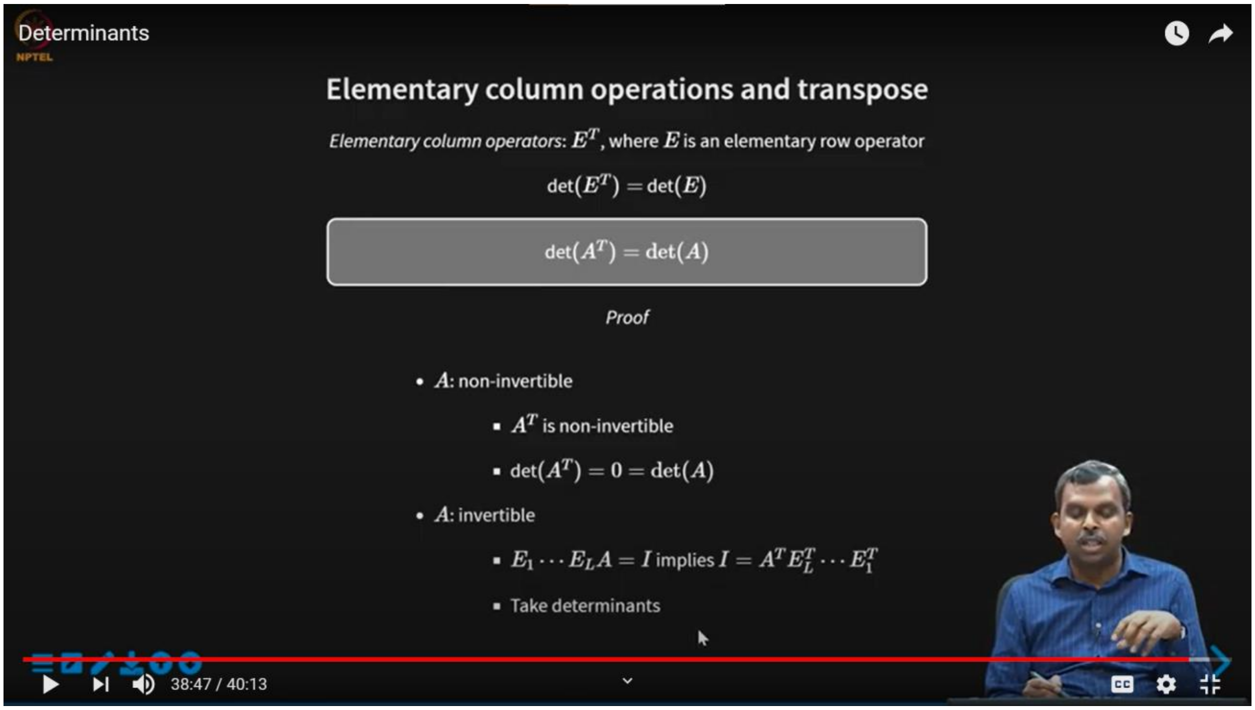

And once you know that, you can quite quickly realize that this result is true. $\det(A^T) = \det(A)$, okay? So proof is quite easy. If A is non-invertible, then A^T is also non-invertible. We have seen this proof, right? Column space, row space have the same dimension and determinant of A^T and determinant of A are both zero. If A is invertible, then there is a sequence of row operations which take A to I and you can do a transpose now and you see there is a sequence of, you know column operations which took A^T to I, okay? Now you take determinants, you will see that one equals determinant of A^T times all these determinants. All these determinants are same as determinant of A, so determinant of A^T and determinant of A are exactly the same, okay? You use this result. The determinant of E_i^T is the same as determinant of E_i, and then you use these two relationships, you will get the answer, okay? So these powerful properties the determinant has, you know, with respect to product, with respect to transpose, makes its calculation very easy and one can quickly do these things, okay?

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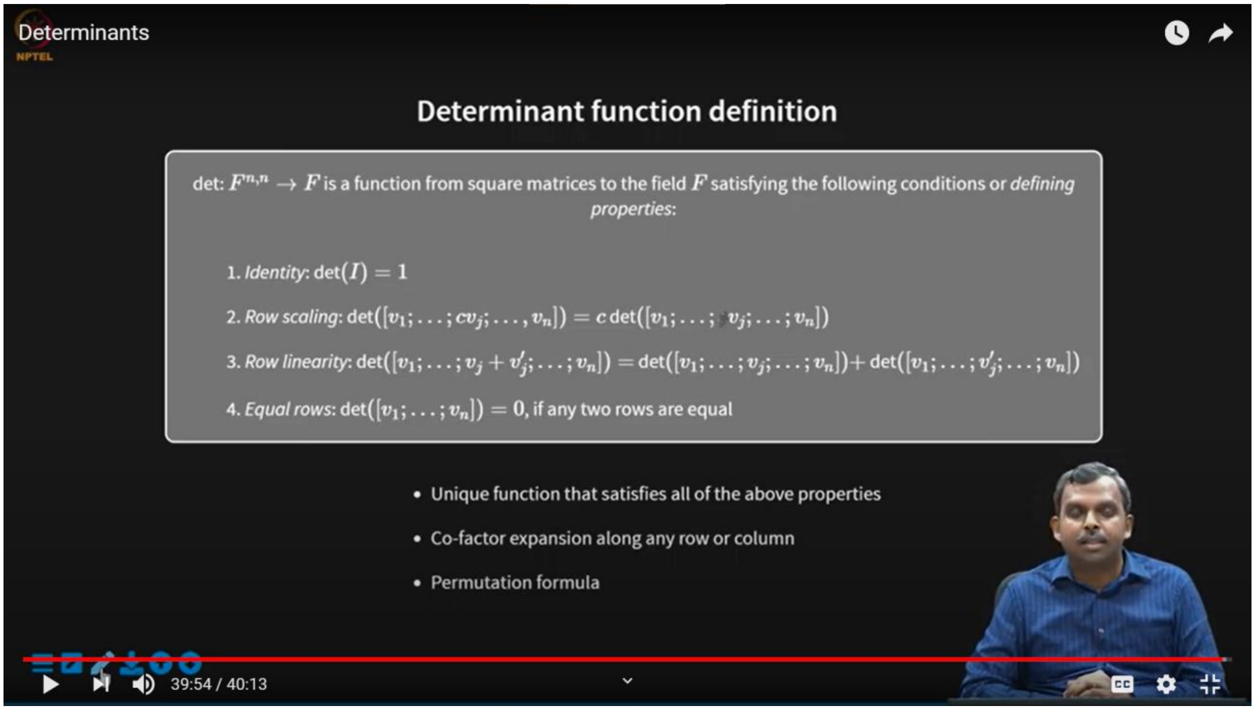

Okay. So here's the last slide on determinants. Once again we did a very quick job of looking at determinants from a high level. Once again, I'll assume that you know quite a few things about determinants from before. These are the four crucial properties that we saw the determinant function satisfies, okay? So it turns out one can show the following three results, which I won't go into any detail in this class at least. There is exactly one function which satisfies all these properties. You cannot come up with more than one function which can satisfy all these four properties and that is the function that you've been studying all along as cofactor expansion along any row or column, right? So familiar cofactor expansion formula for determinant satisfies all these four properties and you can also show it as a unique function. There's no other function which can satisfy these four properties. Once you have these four properties, you know all the other properties come quite easily, right? So this is important. There's also a very interesting permutation formula for determinants. That also we will not cover in this class, you can go look it up and read more about it and it is also equivalent to the cofactor expansion. Has to be equivalent, right? There is only one function which satisfies these things and that is your familiar determinant function, okay? That's the end of this lecture. Thank you very much.