# Image Signal Processing
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**Lecture 40**
## Extending 1D Unitary Transform to 2D- Example

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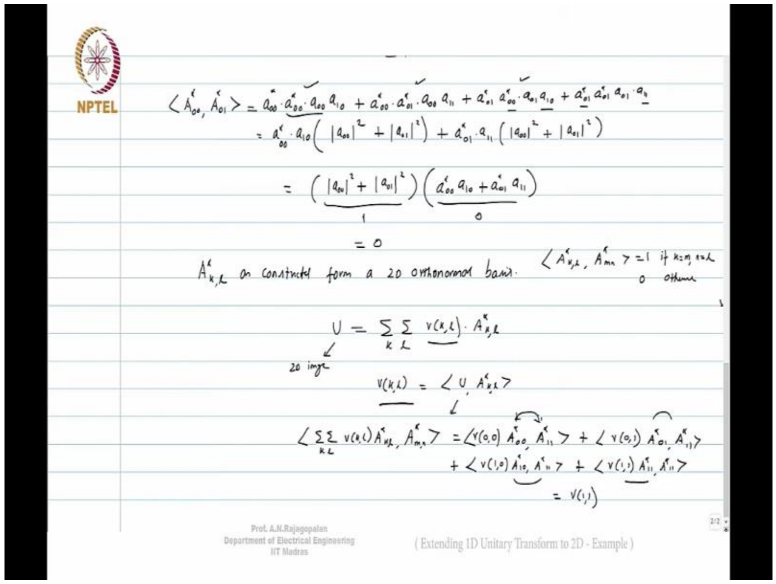


Let me just take one example okay just to just to make kind of matters a little more clear and then and then we will kind of go to the next level. We just look at one example.

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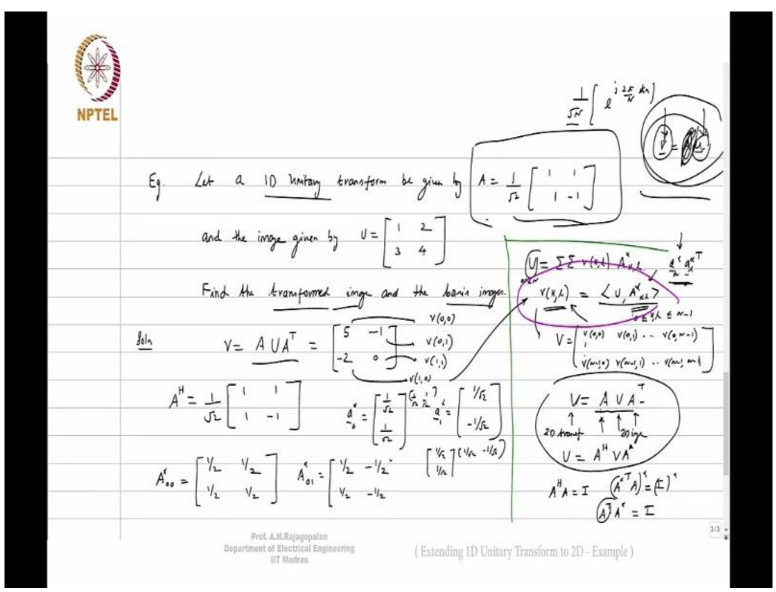

So, let us take this, this is a very simple example but you know helps to make matters clear. What it says is let us the 1D let a 1D unitary transform be given by not the a 1D unitary transform A is equal to 1 by root 2, 1 1 1 minus 1. Have you seen this somewhere? Well I mean in terms of Signal Processing.

We will not want to go that far. What is this? Why do you want to go that far? Is not this simply a DFT matrix of size 2 except for the scale factor. When you have e j pi you know, what is that e j 2 pi by N kn. So, k is equal to 0, n is equal to 0 and 1 by root n, k is equal to 0, n is equal to 0 so 1 by root 2 is anyway out. So, k is equal to 0, n is equal to 0 is 1. And k is equal to 0, n is equal to 1 1, k equal to 1, n equal to 0 1; k equal to 1, n equal to 1 is e j 2 pi by 2. So, e j pi is minus 1.

So (())(1:56) we have not even talked about it. Well they are the only thing that when they actually put this 1 by root n here it is called as a unitary DFT. Alright the normal the MATLAB I think it does not compute unitary skips that 1 by root n. But unitary we need because you all about unitary you do not want some alpha times identity.

So, A is equal to so coming back to this problem right and let the image be given by see what I want you to notice. This is a 1D unitary transform okay that A, this is not a basis image. The A is still coming from so all this A is you know as far as possible I will stick to the same sort of a notation. When I write this A means it is the 1D unitary matrix okay of the 1D unitary transform.

And what does it say let the image be given by you do not want songs here close that thing. Yeah 1 2 3 4 so U is 1 2 3 4, this is a very simple example. So, what is required is find the if you transformed image and the basis images. And of course, as far as we know basis images have to be independent of the image itself right, they have to be independent of their input okay. So, the basis images have to constructed or even otherwise independent of U.

But then they say transformed and we say what do you mean by the transformed image? One more thing I forgot to tell. Okay yeah right so let me just put this on a side. Now, this equation right that you have which is this what is that U or whatever V. And I am going to leave it as an exercise to you. So, U is equal to when I wrote double sum v k l A star k l right, this is what wrote. And V is equal to okay v k l in fact is equal to in a product U with A star k l.

So, then I wrote this okay you can also, see for example here you get you get every transform coefficient A comma l. So, so if you were to enter all those entries just as your U has your entries

U 0 0, U 0 1, U 1 0 U 1 1 and so on right depending upon its size. Similarly, we will have the same size as U because we are transforming the image just as you transform a 1D sequence of length and you go to a transform domain wherein you have n coefficients.

Similarly, here you have N cross N image, in case if it is a rectangular, you will get N cross N whatever. So, let us assume a square for simplicity, so whatever is that size right then we will also acquire the same size. So, what this means is that your k and l right will have to go from 0 to right n minus 1 okay if U is N cross N. If U is N cross N image then then we will have to compute your v k l for all k l going all the way from 0 to n minus 1.

So, if you put that up in a matrix form, we will get your v as some v 0 0 where this these are all these are all right these all they come from v k l expression. So, V 0 1 all the way up to you know v 0 n minus 1 and then we will come down then v n minus 1 0 v n minus 1 1 all the way up to v n minus 1 comma n minus 1.

One way one way is to kind of find it out like this but I am going to leave it to as an exercise to show that v is in fact right so v, one way is to compute like this okay which is of course the matrix. I wanted to show that this matrix when you when each of these entries is defined in this manner in a product of U and A star k l. Write this back as in the form A U A transpose.

So, one is to look at our look upon it as a summation in a product and all of that, the other way to look upon it as simply a matrix operation. Now, the I mean so so this small exercise you can take some example and show this okay just expand and then you will be able to get this. But the nice thing about this is V is really a 2D transform because if you see your image it is actually a 2D image. Your 2D image is sitting here correct and you are actually asking for its transform okay. Now, that transform can be obtained by actually pre multiplying it with your 1D Gy.

So, your A is not really a 2D transform okay because A is the same that you had for 1D because A is kind of pre multiplying U and then A transpose is kind of a post multiplying. There is no complex conjugate there. There is only A transpose, so A U A transpose gives you V. So, the point is another way, but then like this may not be still so kind of insightful and I'm going to just leave it as an exercise to you to show that it is indeed true.

You can just mathematically check this out that the same v k l, so if you check any entry here suppose, you write is A 0 0 A 0 1, U 0 0 all that A transpose U, right you multiply each of those

V 0 0 V 0 1 and all you will get right for every entry. You can check that will have the expression that you will get if you were to evaluate this particular you know this one in your product.

And you can see A star k l right oh A star k l is not something that is dropped from the sky. A star k l is in fact A k star A l star, right A k l star A l star transpose an A k l star A l star that are themselves coming from A correct. So, this so this relationship to a is also is kind of this is one way to interpret it. Another way to look at it is it is getting indirectly embedded here. Because when you look at A star k you may wonder where is A sitting there. Then A is coming through these A k s.

Because A star k l is itself a matrix that you get by taking A k star a l star transpose. And this A k star itself is coming as the column of kth column of A Hermitian. So, this A and right so you can see that so you can see that we should have entries of the A of course when you try to write down v k l through the matrix expansion, you will get the full sum of you should be able to show that there is nothing vey you know simple.

Now, you can also write down what can I write U as then? If you can write is the reason is okay some and some books they like the like this form in some books there is another form and that form is the one that I like the most because even now if you look at it now, it looks like you have a 2D transform of the left, and then you have the U image which is getting sandwiched somewhere between two 1D transforms and it is not clear. Why cannot I write something like something like you know in the same form in which I write 1D. In 1D I write A times U okay.

But then when it comes to 2D it looks like suddenly some kind of crazy thing has happened and then it does not look like I can have this form But the truth is that you can have this form. Only thing is the interpretation of this A will have to change the interpretation for your V will have to change, the interpretation of U will have to change even though these still represent the coefficients of the image. These still represent the image in some form and then this should represent something that should have a have a relationship with the underlying 1D A.

But it cannot be the 1 D A itself, but it should have, it should bear a relationship with 1D A in some form. That form is the one that is the most neat and then of it sort of it is like saying that oh does not matter whether I work in 1D whether I work in 2D whether I work in 3D. Wherever

I work I can always write it up in this form which is what is what we are all familiar with right in 1D signal processing. So, this is the form that you are most familiar with.

So why in 2D right, we should have we should have an expression like that? Nothing wrong with this is also true but one still hopes that you know, one would like to see something like the ones that we have seen for 1D. And once you have that then that matrix becomes something special this Gy that sits there and that no and then what properties it has and so on and all those things we can tell much.

So, at this point of time, I just want to say that, do not say that. So yeah. So, what is U then? This is V and so U is A Hermitian V. And then A what cannot be A no, A star is it not? A Hermitian A is identity no? So, A star transpose A is i, you take star you take star. So, what do you get A transpose A star is I star is I.

So, A transpose inverse is what do you need? You need an A transpose inverse of A star. So, okay. So, now the point is this. So, we want to find this now I am going to arrive at this transformed image in various ways. Okay, so one the most simplistic thing would be to simply do this V is equal to because transformed image is what your V right, you are trying to transform it.
So, the simplest way would be A is equal to B is equal to A U A transpose where U where A is this 1D Gy. It is not really a 2D basis image is, but that is the 1D unitary transform. I will simply pre multiply post multiply and let us not do it here, just leave it to you to show that this is 5. Okay. I have the answer here minus 1 minus 2 is 0 okay, if you solve this what you get is this now so now.

So, the point is it some people like to see it as a complete matrix operation the whole transformation and therefore for them right doing something like this looks elegant. So, 5 minus 1, So this is V 0 0 this is V 0 1 this is V 1 0. This is V 1 1 okay where each of these v k l is that Gy okay correct.

So, we can either say that okay. That is our V or the or the but then that you would also like to show that all these things also hold and whatever we wrote here if we want to show that all those things also hold. So, to show that now suppose we go back now if I want to solve this inner product, then I need A star k l no. I need I need all the four of them A star 0 0 everything I need.

So, suppose I ask what is your A this one Hermitian , but then that will mean that I should first take this A, compute A Hermitian. But in this case it does not matter. It is real and then it is also symmetric so this is simply 1 1 1 minus 1, but this need not be true in general. Okay, need not be equal to A  in general. This is a very very simple case.

So, then what is your what is your A star or A zero-star? That is 1 by root 2, 1 by root 2. What is your A 1 star 1 by root 2 minus 1 by root 2. And then that is if we start looking at what is A star 0 0. Then because I need each one of those here okay. But I need k l going from both going from 0 to 1 right. So, A star 0 0 is then 1 by root 2.

So, what will you get 1 by root 2 into 1 by root 2 is 1 by 2, 1 by root 2 into 1 by root 2, which is 1 by 2 by 2 1 by 2. It is okay no, this way multiplying with itself 1 by root 2 into 1 by root 2. Then what about A star 0 1, okay so 0 1 will be this with this with this with this is like what you get A star 0 1, 1 by 2. So, this is like this Gy. So, A 0 star 1 by root 2, 1 by root 2 then 1 by root 2 minus 1 by root 2. So, what will you get 1 by 2 minus 1 by 2; 1 by 2 minus 1 by 2.

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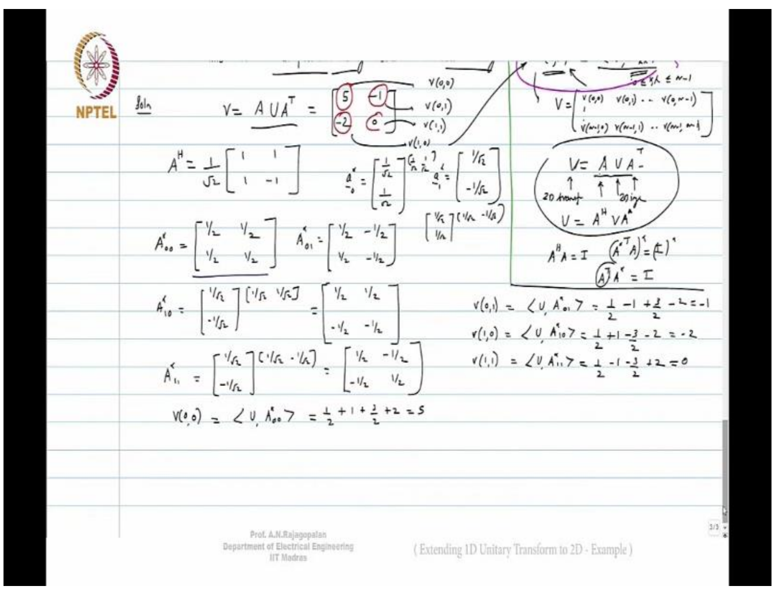


Then what about your A star 1 0? So, 1 0 is 1 by root 2 minus 1 by root 2 by 1 by 2, 1 by root 2. You get 1 by 2, 1 by 2 minus 1 by 2 minus 1 by 2. Then finally A star 1 1, so that is 1 by root 2 minus 1 by root 2 into 1 by root 2 minus 1 by root 2. To do this 1 by 2 minus 1 by 2 minus 1 by 2, 1 by 2. Of course, you can all check that if I were to take the enough product in that we have already shown if you want to go through all that. Now, the point is suppose, suppose I am interested in v 0 0, v 0 0 is what enough product U with A star 00. Now, in this case, everything is real so there is no need to take any.

So, U is what 1 2 3 4 no, so if you look at A star 0 0 is a 0 sitting here, and our U is there with this Gy. So, this will be 1 into that is your A star 0. So, 1 into 1 by 2 plus 2 into 1 by 2, which is 1 plus 3 by 2 plus 4 by 2 which is 2. So, what is this one and a half 3 plus 2 is 5, it is what you got here no. It is what you got here as V 00 okay, you should of course match right, there should not be any surprises there. So, v 01 is in a product U with A star 0 1. So, look at A star 0 1 is here so this will 1 into 1 by 2 plus 2 into minus 1 by 2 is minus 1 plus 3 into 1 by 2. So, this 3 by 2 first 4 into minus 2.

So, what do you get minus half, one and a half 1 minus 2 is minus 1. So that is the same as what you had here for v 0 1. Okay, then come to v 1 0 is in a product U with A star 1 0. So, 1 into 1 by 2 that is 1 by 2 plus 1 minus 3 by 2 minus 4 by 2 is minus 2. So, minus 2 right so these two cancel up. Then V 1 1 so as you can see right, this is the same as whatever you got through the direct set of matrix multiplication.

U with A star 1 1 is here. So, 1 by 2 minus 1 minus 3 by 2 plus 2, half minus half is minus half minus half minus half minus one and a half is minus 2 plus 2 is 0. So, that is again the same as what you had here okay. So, this is one way to check that all your v k ls right whether you come through a U a transpose or whether you come through the an expression like that wherein you take the inner product construct the basis images okay which is actually a 2D basis now.

You can even come you can come either via a 2D basis image and then compute your coefficients or you can still use the 1D unitary transform itself to the matrix multiplication and get there. And then of course you can clearly show that you can reconstruct U right? So, if you try to get this expression now, so V 0 0 which is like phi times? So what will it be like?

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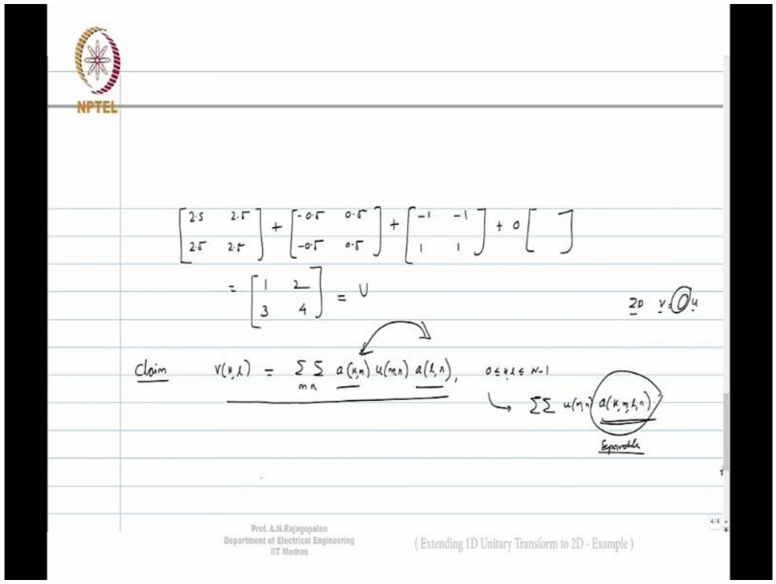


So, if I try to reconstruct right, so phi into A star kl can somebody tell that so phi into A star is phi by what is it 2.5 minus plus half yeah. If you can tell me 2.5 minus 2.5 what was it 2.5, 2.5. What was it? 5 times A star 0 0. What was the last one? All are 2.5, no no minus 2.5 is there no? Oh okay all are 2.5, I am looking at the wrong one. Yes, all are 2.5 correct, I am looking at plus. Okay v star 0 1 was how much, minus 1. So, that we need to multiply with A star 0 1.

So, that will be minus half, what is it? Minus half plus half and, minus half plus half. Then third Gy is what minus 2 or something we saw no? Minus 2 multiplying A star 1 0, what is that? Minus 1 minus 1 plus 1 1 and then the then the final one is anyway 0 right because that coefficient itself is 0.

So, if we add all this, 2.5 minus 0.5 is 2 minus 1 is 1, 2.5 plus 0.5 is 3 minus 1 is 2; 2.5 minus 0.5 2 plus 1 is 3; 3 plus 1 is 4 so which is equal to U okay. So, this is the only a small example to show that one can get a one can get a V as around okay whichever way we want. You can either come via in a product or we come via matrix or we can come via whichever we actually want.

But now the point is right we would like to we would like to kind of take this forward right and look for a formulation right where even in 2D we would like to be able to write something like v is equal to something like U multiplying U which can be interpreted as a 2D transform okay. That form right we would like to kind of right we like to see.

In order to see that okay before we can see that okay, we will have to look at what is really called what is called a Kronecker product. Okay now before that I still have to have to write a few more, before we get there no I think prior to that there is some of the work that needs to be done okay. Now, I have got to make a claim okay, I want you to check this out. This is again yet another form okay, if time permits, we can do it today.

So, claim is this the same v k l right which we had which we said is in a product with U and A star k l that is one way we wrote it. Or we said just this you know the whole matrix coming out of a V U A transpose. Yet another form for v k l is as double sum a m n okay and note that the small a are again the same entries of the matrix A of the 1D Gy, okay these are the same a.

So, I am not changing anything along the way so that we can always map back to what we started off with. So, no this is not a m n, this has to be a k m k comma m, u of m comma n and then a of l comma n. And this sum runs over the m comma n and then k and l.  So, well 0 to m minus 1.

Now, the point is okay, something like ideally read what you would have thought you would have thought it may look like this Um n is my image then multiplying a 2D transform that is of the form may be a comma m comma l comma n okay. Now, most transformed okay that we encounter that the ones that you are familiar with are all transforms wherein this Gy is actually separable. This Gy separates as A so it splits the A k m and into a l f.

For example, if you take a Fourier transform, okay in 1D it is e to e ratio minus a 2 power by nkm. Look at the 2D formed by b to the power j 2 pi by n k plus you see n l, so crater splits. So not only this (())(24:00) every one of these? There are some which are not separable like it is not true that everything is separable but most of the transforms that you use in your daily life at and also right there is also a reason for it okay why we like such separable transforms is because whatever sort of equipment you have already developed whatever tools you have developed for 1D you can all those you can actually you know directly use for use for (())(24:25) 2D.

For example, if you look at the 2D Fourier transform, if you want to cancel it you might say it is an image in there for computing a 2D DFT looks like a lot of work right. Because we will have n square portions to compute but then you know that there is a fast Fourier transform in 1D. So, you might want to ask them can I sort of employ that 1D FFT because it is already so quick right I kind of use that to do a computation of let us say 2D Fourier transforms.

So, all that right you know is possible provided this kind of a separability notion is there. And in fact, when most of the transforms have that you would come across are of this separable okay. And see these separable transforms in fact when I did A star came to A star l transpose. I assumed that separability is already there okay, that is you know because see when oh, in books right when we write when they right this. This separability notion is made is made is kind of see explicit, when they write the summation because ideally you would have thought it might look something like that.

Now, instead of but then the point is if we write like this and move on right then there is so much of matrix structure that lies underneath right that you would miss you would not see any of that okay. If I had simply introduced DFT as U m n e power minus j 2 pi by n k m plus n l except write this as a separable transform and therefore right that is how that is how a 2D basis image will look like.

Then you will actually miss the point that you know, irrespective of which transform you looking at provided all these transforms that we are looking at are basically unitary and you know separable then there is said there is a nice structure at which binds all of them which binds 1D to actually to 2D and the same vehicle right that you used to go from 1D to 2D becomes actually very obvious that if you want that tomorrow ready.

If you wanted to do a 3D processing. Let us say around instead of just image you have a complete you know this one a video which is kind of varying with time. Now, if you say you wanted to do you know 3D DFT. Now, you would not have to wonder now as to what should I do next. What are the basis images will I get and so on because this this vehicle is so structured that it is automatic, why 3D right wanted to go to 4D, 5D right does not really matter.

So, that structure you tend to miss okay which is which is that all these forms are correct. It is not to say that you know one form, nothing like any form is inferior. But I am just saying that you know going through a matrix sort of operation right throws up so many things which you do not see which we don't see otherwise, we will simply miss all of that and think that think that you know, and there we even see diagonal relation capability of certain matrix and all, none of that we can see.

It will all look like each one is a standalone a DFT will have the summation the Dc will have the summation whereas if you just take some A right and do not really do not really worry about what is the DFT a TCP or (())(27:14) somebody said (())(27:16) we do not really care about any of that.

We know that all of them are unitary all of them are separable and if you can say develop a common framework which allows you to seamlessly go right. You should not make an effort to get there right, it should be a seamless thing. If you can go seamlessly from 1D to 2D. Then it automatically means that from 2D to 3D should be another seamless task right.

And you know that is what we are hoping to do okay but in the but in the process, we should also be aware that if you see books, they might have expressions like this and you start wondering what is this right I never spoke about this, so I am saying that all these all these mean the same thing; they are all these equivalent each one.

Now, if you look at it as a matrix vector operation it may look you know it may take a certain form. If you look at it as a summation, it may take a certain form but I always like to go back to the kind of right matrix form because that is where the entire structure is embedded okay. And these are these are kind of right other ways of interpreting the same equation.