A Brief Introduction to Micro Sensors

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Lecture - 05
Basic Mechanics - Part 02

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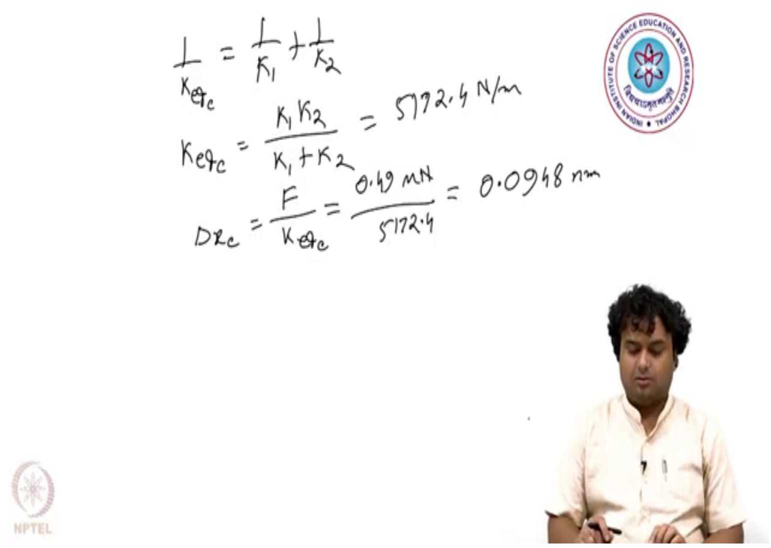

So, in this class, we are going to now talk about little bit advanced concepts like where you cannot use a for a like simple beam theory for calculating the stiffness constant etcetera. Now for that before, we go to that we will just brush up a little bit about the whatever the basics we have learned and some more things which will be required for this study ok.

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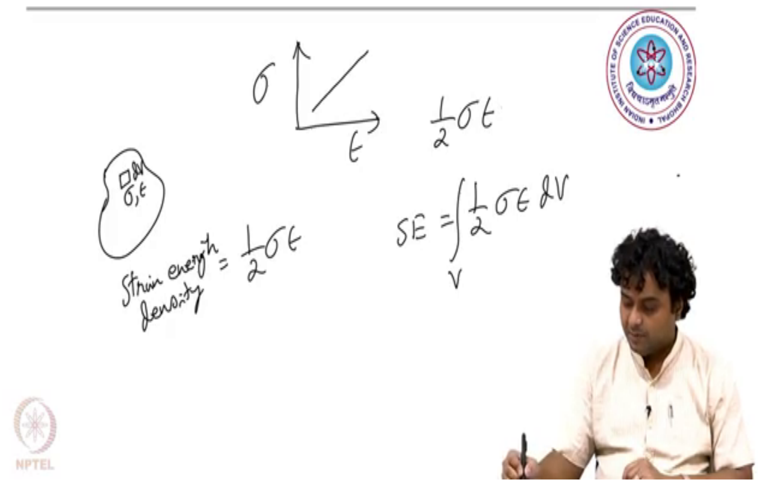

So, now, you know that according to laws of elasticity stress and strain are linear right. So, if the sigma is the stress and let us say epsilon in the strain, then you get a linear curve. Now what is the area under this curve is half into sigma into epsilon and this is also known as the strain energy. This is also known as strain energy.

So, now, let us say I have somebody and in that there is some element volume element dV which will which is having a it is stress as sigma and strain is epsilon. Then half sigma epsilon is the a stain energy density of that particular body and then strain energy density.

Now, to calculate the total strain energy, let us say we call it SE total strain energy is equal to strain energy density that is half of sigma epsilon into and then we have to integrate it over the volume. So, this is a volume integral of half sigma epsilon dV now this half sigma epsilon you can also relate it with the spring mass system.

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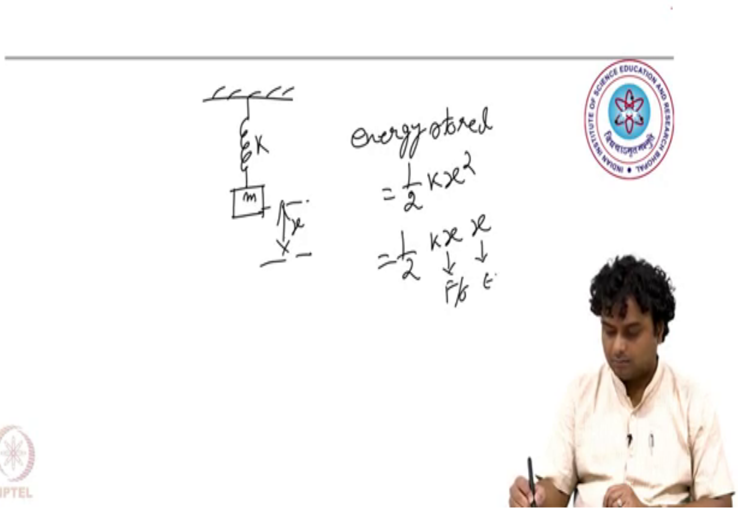

Because for a spring mass system you know that let us say you have a spring of spring constant K and any mass m is there and let say the deflection is x. Then while it is at a deflection x are stretched still x, then the energy stored in it is equals to half K x square and this you have already learned probably twelve standard or some basic mechanics course.

And now we can see that here half, you can write it as half into K x into x where this K x is the force and this is the deflection. So, this is a like the sigma and this is the epsilon right ok.

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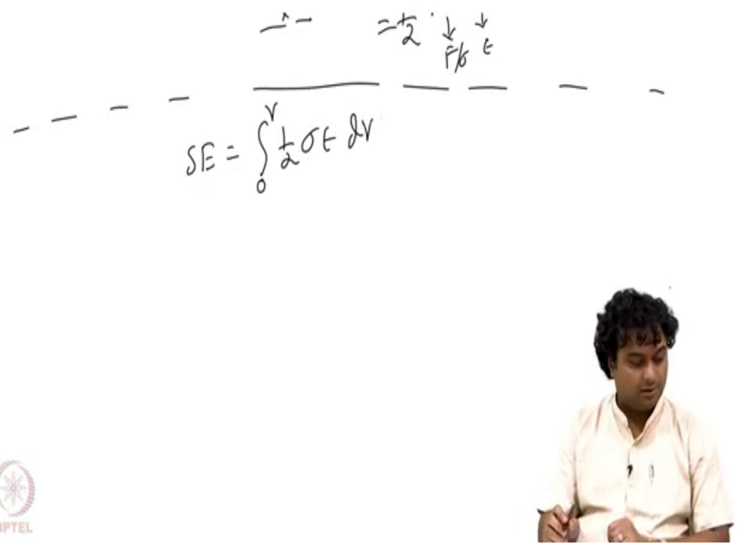

So, total strain energy SE is equal to integral 0 to volume half into sigma epsilon dV ok. Now we will discuss a theorem called Castigliano's theorem.

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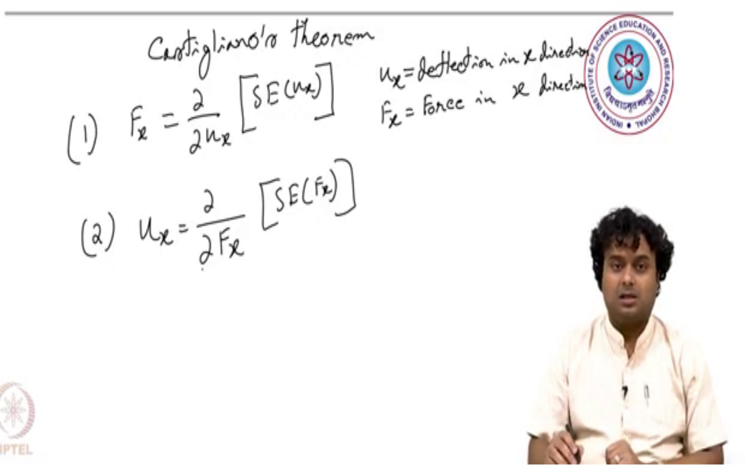

So, this is very specific to mechanical engineering and in this part of the course or in the scope of this course, we are not going to discuss what is the there in Castigliano's theorem and how exactly that is derived and etcetera. But we are going to use the theorem like the result of this theorem and because that will be useful for us for solving different complicated beams.

So, what does this theorem say? Theorem says is force at the direction x is equals to partial derivative with respect to deflection at that in that direction of total strain energy expressed as a function of deflection. So, here u x is deflection in x direction and Fx is force in x ok.

So, this is one of the expression and the second expression is just the vice versa that is u x or the deflection is partial derivative with respect to the force while total energy total strain energy is expressed in terms of the force. Now these two expression, we will use for solving our for solving different complicated structures. But we are not going to discuss that how these two expressions come ok.

Before we go to that exactly how we apply this theorem, we will consider the one of the constant which we discussed in last class that is guided beam.

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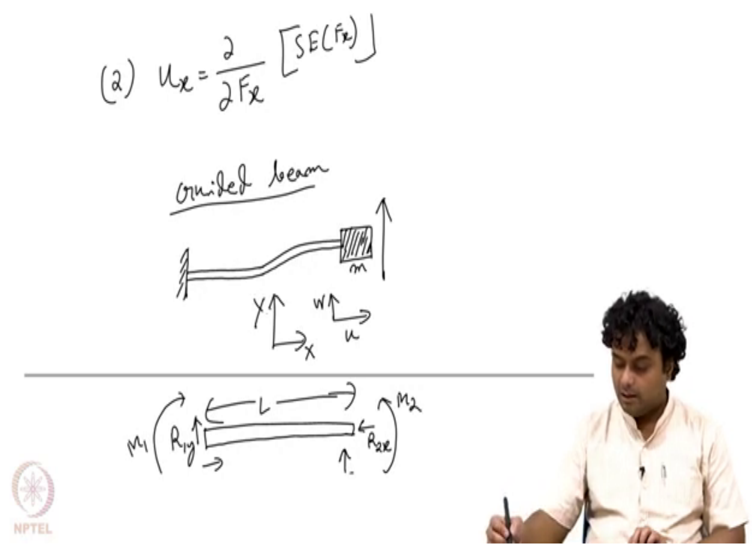

So, if you remember guided beam was guided beam was the case where one side is fixed and another side was moving without any rotation at the set or any movement in this direction ok. Only this end is left side or this end is fixed and this end is moving, but only in this direction up and down. So, it is something like this. So, let us assume that this side is a of the cantilever is fixed and then this is the mass which is connected to the other end and that mass is moving only in this direction.

So, there is no x axis moment or no rotation in the frame. Now for that we have seen the. If we draw the free body diagram then the constants are like this right F side is fixed.

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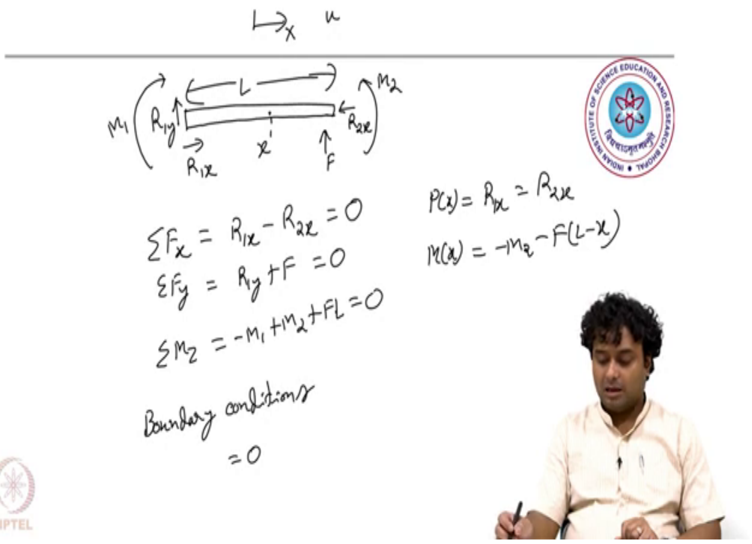

So, anyway there will be all the different reaction forces R 1 x R 1 y and then we have the moment M 1 and this direction actually we are applying the force. So, we have the force F which is an external force and then the reaction force R 2 x and then M 2 in the reaction moment.

Now you see that in the y direction we have not mentioned any, but this end or in the y direction, we have not mentioned any reaction force because in that direction, the mass can move. So, there will not be any reaction force. So, by balancing all the force, we can write that summation over F x or equals to R 1 x minus R 2 x equals to 0.

Then summation over F y equals to R 1 y plus F equals to 0 and then summation over moment M z equals to minus M 1 plus M 2 plus FL equals to 0. Now, see here I have put I am negative sign before M 1 because here I am considering that the clockwise moments are negative.

So, this is pure convention whatever convention we fix and accordingly we should move with that particular convention, we should not change it. So, for this purpose, I am considering only that anticlockwise moments are positive and clockwise moments are negative.

Now, and another thing is let us assume that the beam length is L ok. So, this distance is L. So, minus M 1 plus M 2 plus F L is equal to 0 right because of this force will have a movement of F L and M 2 and F L are both in that same direction that is anticlockwise and both are positive ok. So, now we are going to discuss what are the boundary conditions ok.

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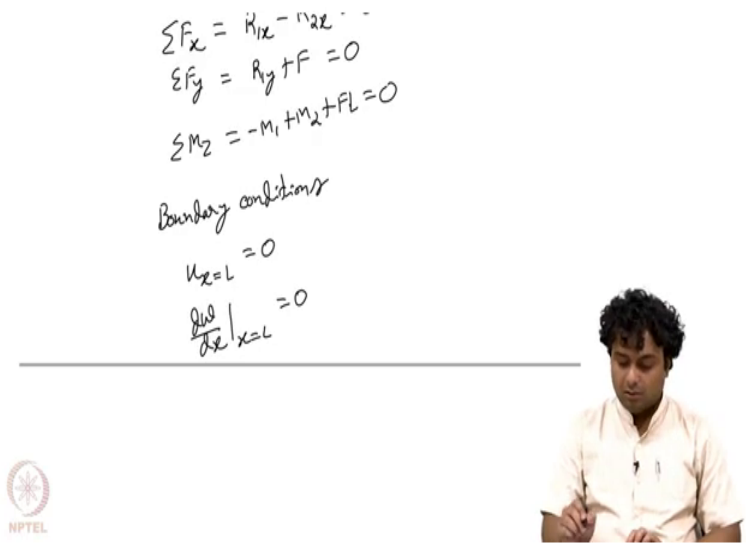

So, let us see the boundary conditions. For this particular boundary condition, we can write that u x at x equal to L is equal to 0 right and then dw dx at x equal to L equals to 0. So, what I am considering here that x direction the deflection is u and y direction the deflection is w. So, this is u and this is w and this is my x and y.

And how are we getting that? You are getting that because u at x equal to L x equal to L means x equal to L means this point here right and in at this point, we do not have any movement in the x direction right; we do not have any movement in the x direction. So, at that point or at the tip of the cantilever the x directional deflection is 0 and then another thing is as this is guided beam as I am saying that there is no rotation.

So, guided beam no rotation means you can see that there is also no slope. So, this is in this direction or the y direction the deflection is w. So, what is the slope in that direction? It is dw dx. So, dw dx and dw d x at x equal to L will be 0; these are two boundary equations.

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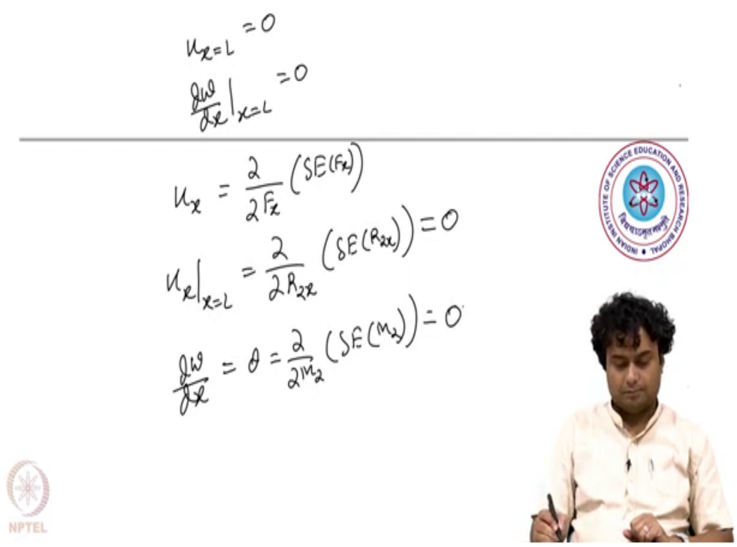

So, now, we see that what we get this boundary equation from this boundary equations in terms of Castigliano's theorem. Now according to Castigliano's theorem, we see that u x is equal to del del Fx of the strain energy right in terms of Fx.

Now what is this F x? Fx is the force in x direction F force in x direction right. Now what is the force in x direction? At x equal to L that is the reaction force R 2 x. So, u x for x equals to L is equal to del del R 2 x of strain energy same here R 2 x and this equals to 0 according to our boundary condition one.

Then what is the another condition? Another condition is that dw dx. Now, dw dx dw dx is what? Dw dx is the slope right and this slope means the angle theta and what does it make angle theta, the moment. So, if the force makes deflection and moments make makes the body rotate right. So, dw dx this dw dx is the slope of the beam right and dw dx means some angle, let us say theta.

If theta is the deflection, then what is the force? Force is the actually the moment right and at x equal to L, what is the moment that is M 2. So, according to Castigliano's theorem del del M 2 of strain energy in terms of M 2 is equal to 0 ok.

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$$\frac{dw}{dx} = \theta = \frac{\partial}{\partial M_2}(SE \ldots$$

$$SE = SE_{axial} + SE_{bending}$$

$$= \frac{1}{2}\int_0^V \sigma_{ax}\epsilon_{ax}\,dV + \frac{1}{2}\int_0^V \sigma_{bending}\epsilon_{bending}\,dV$$

$$\frac{\sigma}{\epsilon} = Y, \quad \epsilon = \frac{\sigma}{Y}$$

$$= \frac{1}{2Y}\int_0^V \sigma_{ax}^2\,dV + \frac{1}{2Y}\int_0^V \sigma_{bending}^2\,dV$$

Now we need to calculate what is the strain energy. So, strain energy SE equals to SE axial plus SE bending. Now here I would like to point out one earlier case while the cantilever transfers force deflection we have calculated there you see.

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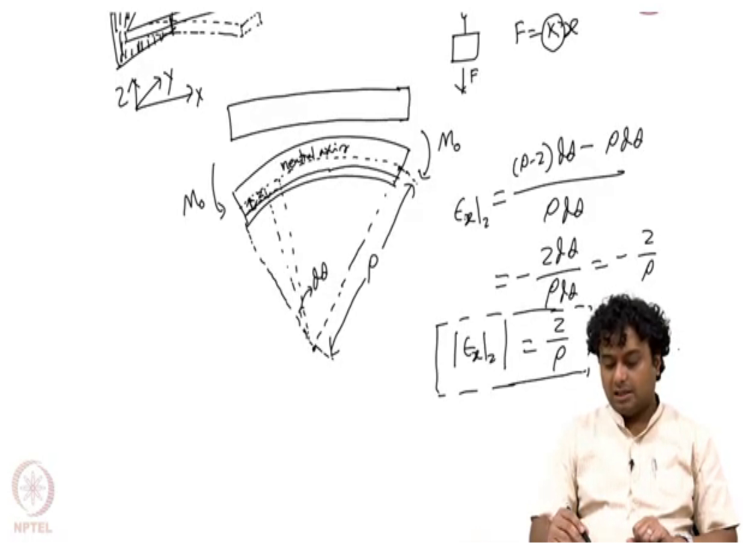

Here you can see that this is the case where it is pure bending. So, there is no lateral deflection associated with it right; neutral axis and according to the neutral axis is pure bending case for this transfers force cantilever right. But here what we are considering now? We have axial stress also and transfer stress also because this is guided case and then we have to calculate the strain energy for both the cases.

So, first of all we will calculate one by one that is strain energy axial and strain energy bending. So, what is strain energy? Strain energy is half in to integral of 0 to V sigma a x epsilon a x dV plus half of we can polynomial integral 0 to V sigma bending epsilon bending dV ok.

Again we know that we know that sigma by epsilon is equal to y. So, what I can write is epsilon equal to sigma by y. So, here we can write that half of 2 y into 0 to V sigma x square dV plus half of y also comes out 0 to V sigma bending square dV.

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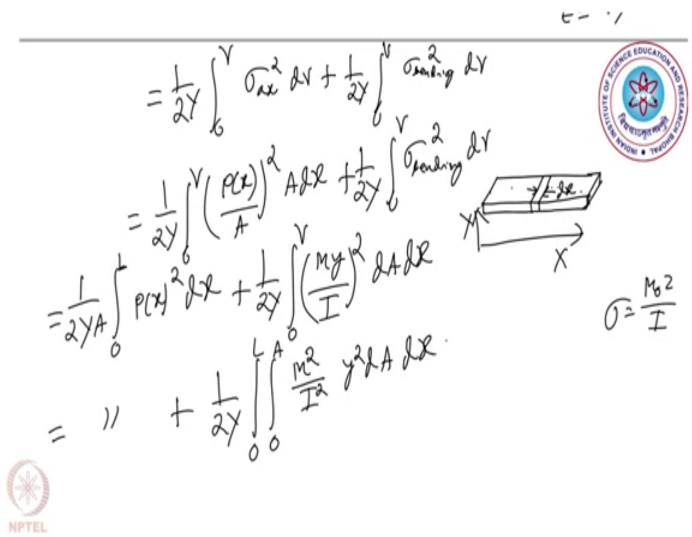

Now, 1 by 2 y to 0 to V, this is sigma x. Sigma x means that is stress in the axial direction right, now axial direction. What is the stress? It is a force in the x direction divided by its area. So, let us assume the force at any point x in x direction is px ok. Then it will be px by A whole square dV or we can write dVs adx, why? Because this is our beam and this dV is like this cross section and here this is x direction let us say and this is y direction.

Now see that the at different different position of the beam, there will be different different A force. But area throughout the beam is same according along its length the area is same. So, A is a constant right. So, this dV we can write as A into dx where A is the area of this volume, this small volume and dx is the this dx.

And then we have 1 by 2 y, this is to V sigma bending and square fine. It is again 1 by 2 y and one of the A is goes off. So, it is 1 by 2 y, A as it is constraint I take it out. Now this volume integral becomes a line integral and then it is limit will be 0 to L because L is the length of the beam.

And here for sigma bending, we will refer to one of the old earlier result what we got or this bending case pure bending case, we got that from a distance z from the neutral axis the stress is E z by rho where E is the Young's modules, z is the distance from the neutral axis and rho is the radius of curvature.

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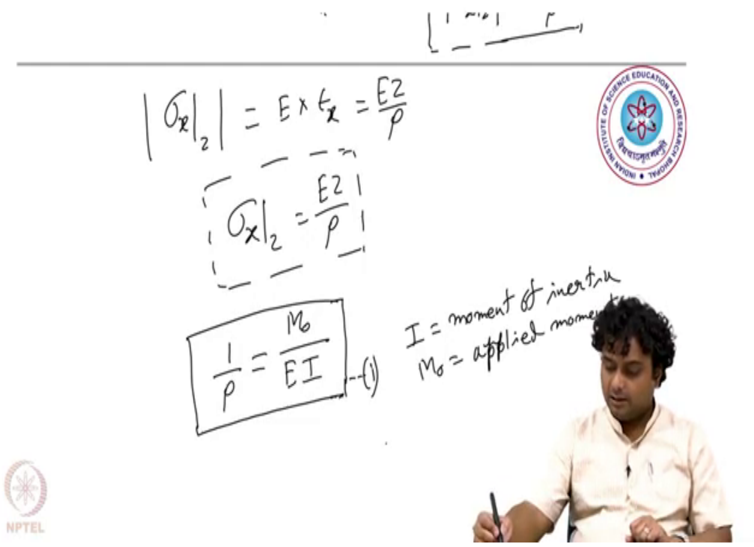

And then also you got that 1 by rho is equals to M by E I which is like M is the moment and E is the Young's modulus and I is the inertia and if we combine these two, then we get that sigma x equals to M 0 z by I.

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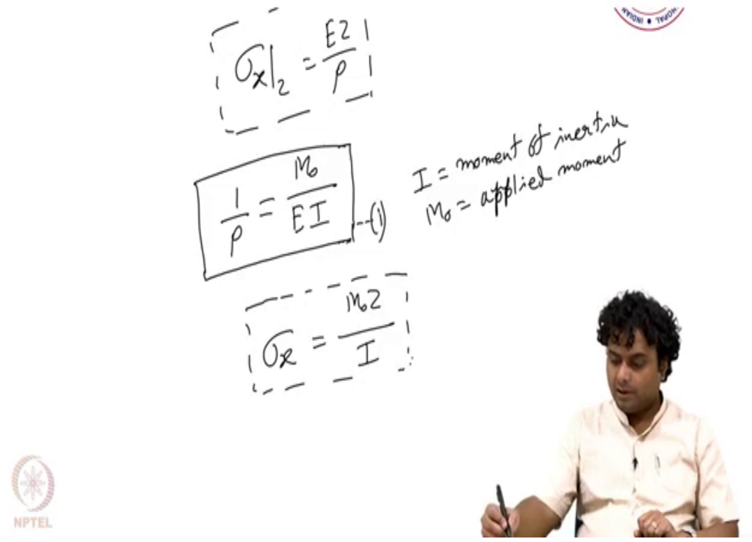

Now we are going to use this relation for our current solution. So, M 0 z by Y where M 0 is was the moment and z was the deflection from the neutral axis. And here in this case, we are considering this as w right because in Y direction or in the vertical direction, we are considering the deflection as w.

So, now let us use this relation for our present solution and what do we got? We got that sigma is equal to M 0 Z by I. Here in y direction, let us say if the deflection is or let us say if we take a point where the layer is at a distance small y from the neutral axis, then we can write that by to Y; this Y is the Young's modulus capital Y and then 0 to V this sigma is like M 0 by z by I.

So M 0 is here, our general M or the moment at any point x. So, M and then the Z is actually here, let us say y because we are taking the we are taking the layer in Y direction. So, let us say at a distance of y from the neutral axis like small y that is a distance of small y, we can take that M y and then I is same. So, M y by I square into dV is dA d x right.

Now this part is same, I keep it like that and this part 1 by 2 Y, this volume integral I can break it into two part; one is line integral 0 to L and another is surface integral 0 to A ok. And then we have M square by I square y square dA into dx.

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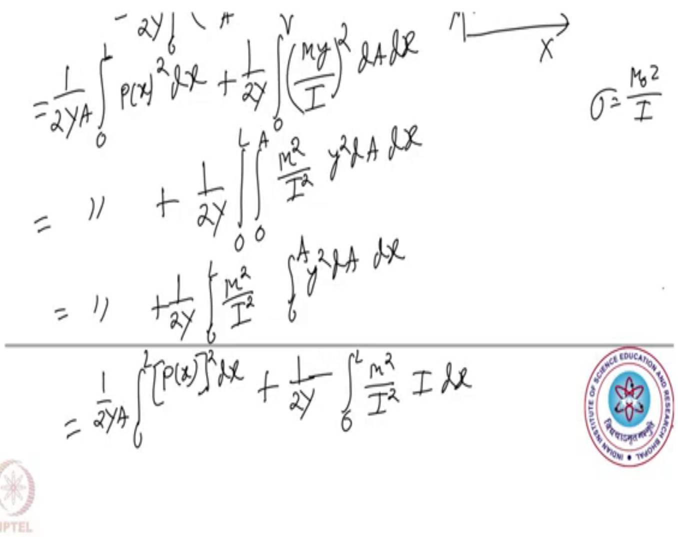

Now, this part is again same and this part is 1 by 2Y again 0 to L, I can write the M square by I square M square by I square I take it outside and then we can write that 0 to A y square dA into dx. This 0 to A, y square dA integration is the area movement of inertia right.

So, this will give me I. So, equals to then again the first term as it is I wrote 1 by 2YA in to 0 to L P x Px whole square d x plus 1 by 2Y, Y is the here the first capital Y is the Young's modulus into integral to L M square by I square into this is I d x. And so, this I will cancel from numerator and denominator and then we will get 1 by 2 Y A integral to L E x and dx plus.

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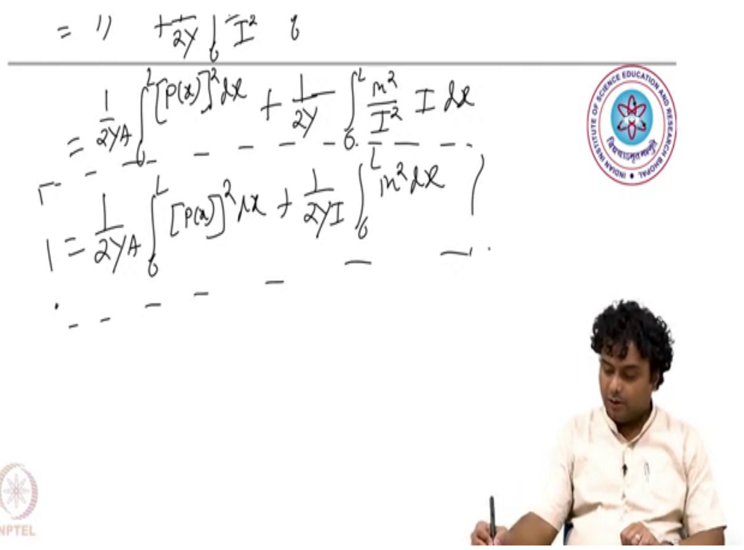

So, I can take the I out because this is a constraint 1 y 2 Y I 0 to L M square dx. Now this expression is important and we will just calculate here what are the values of P x and M for different xs and then we can calculate the total strain energy. So, how we know P x from the free body diagram we can get. So, this is the free body diagram you see that the R 1 x R 1 y and R 2 x if everything is drawn here

So, as per the our first condition that sigma Fx equal to 0. So, R 1 x minus R 2 x equal to 0; that means, my P x is equal to ultimately R 1 x equals to R 2 x because there is no other force in x direction and it is the same force at every point ok. Then mx or moment M x or moment at any point x will be equals to minus M 2 minus F into L minus x right.

Because if I take moment with respect to this point with respect to the left most point which is the fixed point, then M 2 will be there and along with that and like let us say the point is at a distance x at a distance x from the left end. So, that from that point this force F is at a distance x minus L sorry L minus x. From that point this force F is at a distance L minus x so, the that moment for that force will be L minus x.

So, the M x which is actually in this direction. So, it is it will come as negative right M x equal to minus of M2 minus F into L minus x.

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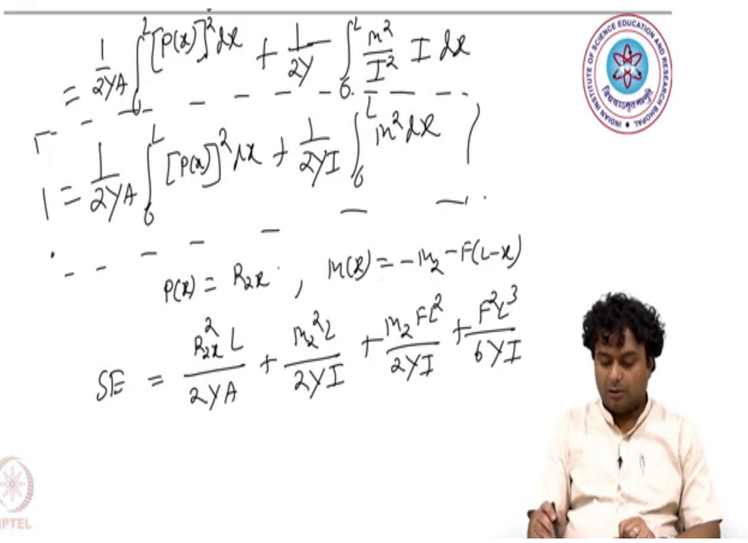

Now putting these two equation that is P x equals to R 2 x and M x equals to minus of M 2 minus F into L minus x. We put these two equations on in the strain energy expression to guess strain energy equals to R 2 x square into L by 2YA plus M 2 square L 2YA 2YI; 2YI plus M 2 2F of L square divided by 2YI plus F square L cube divided by 6YI. So, this you can get by just putting the M values and the r R 2 x and integrating it.

Now we need to calculate the partial derivatives of strain energy. So, first is let us say we derive it with respect to we derivative with respect to R 2 x and M 2 these are the two terms. We need to do right you; if you go back then you see that this is the form of boundary condition and strain energy expression, we have got that u x equal to del del Fx of strain energy and which is equals to 0. And dw d x I s equal to del del M 2 of strain energy which is also 0.

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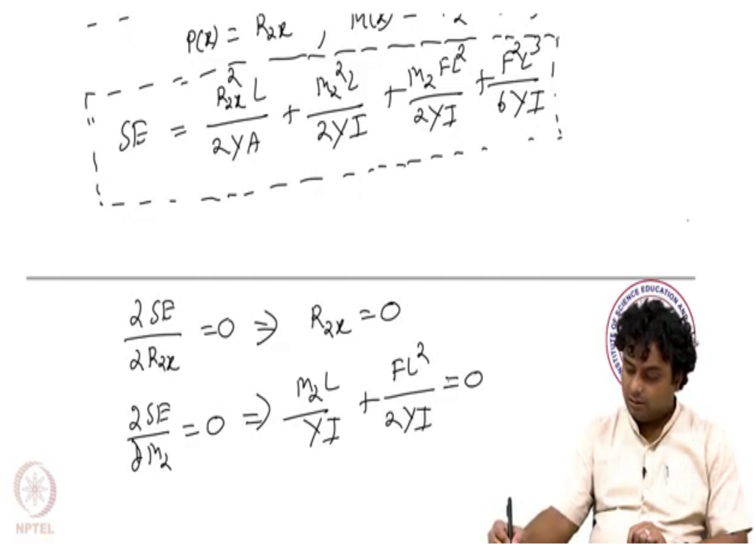

So, delta SE by delta R 2 x equals to 0 lies that R 2 x equal to 0 because in this strain energy team, this energy term the only term which depends on R 2 x is R 2. If the first time R 2 x square L into L divided by 2, I A the other terms are independent of R 2 x. So, that will become directly 0 ok.

Now delta SE divided by delta M 2 equals to 0 and again if we partial derivative it with respect to M 2, then first term will become 0 and the second term will give me 2 M 2 L A 2I. So, it is M2 L into M2 L divided by only YI, 2 2 cancel and then plus F L square divided by 2YI. So, FL square plus x YI this term will also become 0 with respect to partial derivative or partial derivative with respect to M 2.

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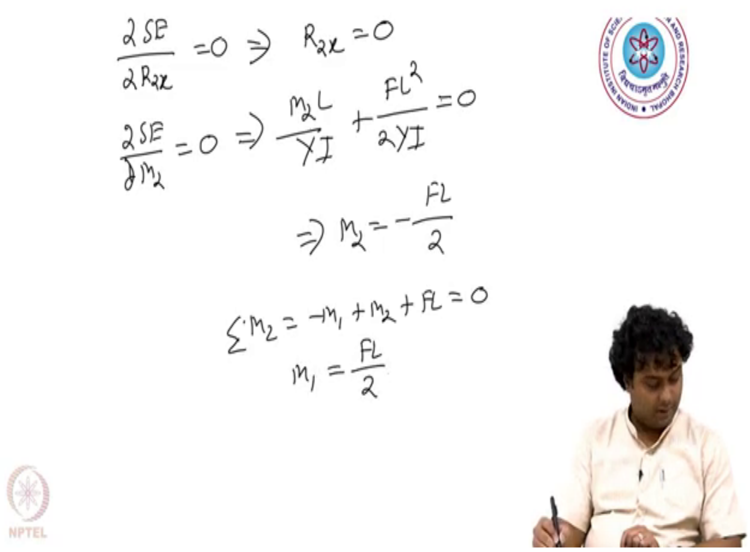

So, this is 0. So, then on this expression we can get that M2 equals to minus of FL by 2. Again from our another condition was there that is sigma M z equals to minus of M 1 plus M 2 plus FL equals to 0. So, now, M 2 is equal to minus FL by 2.

So, now you can write that M 1 equals to plus FL by 2. Now once we got the M 2 and M 1 the M 1 is not required ah, but anyway for continuity we just calculated M 1 also. Now we got a M 2 and M 1 both. So, we calculate the strain energy again in terms of force that.

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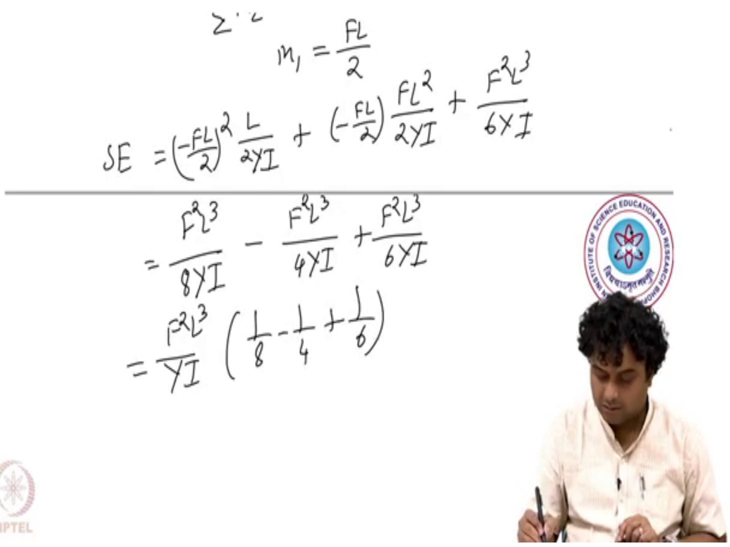

So, strain energy is equal to R 2 x is there right R 2 x is 0. So, it becomes 0 and then we have M 2; M 2 is FL by 2; M 2 is right M 2 is minus FL by 2. So, strain energy is equal to M 2 means minus FL by 2 whole square into L by 2YI plus minus FL by 2 into FL square divided by 2YI plus F square L cube divided by 6YI. And in all the cases, I get this is 8YI minus square L cube divided by 4YI plus square L cube divided by 6YI right.

So, if we take it common, then F square L cube by YI equals to 1 by 8, this 1 by 4 plus 1 by 6. So, calculating we get that a total strain energy is now in terms of F; F square L cube divided by 24YI. Now we need to calculate the deflection actually. So, that is the w right.

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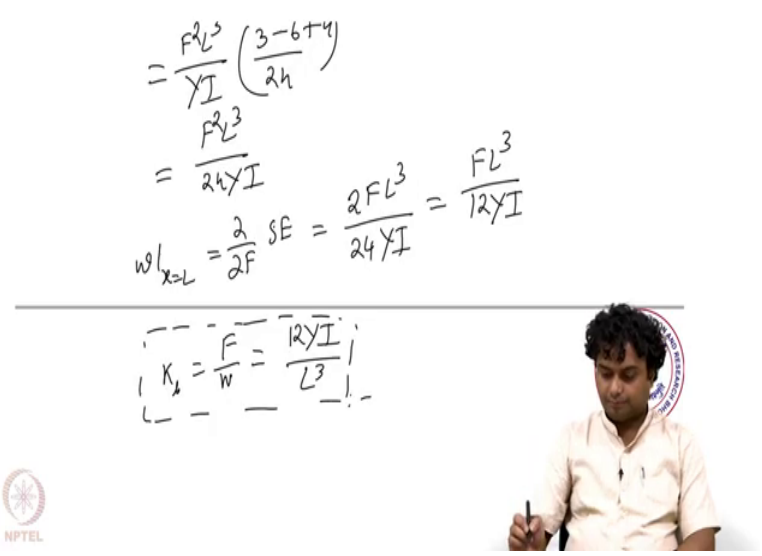

So, w deflection is because of which force F right. So, w at x equal to L is equal to del del of force F because F force is acting at the tip of the cantilever that is at x equal to L and that that is causing the deflection of the tip that is the w So, w or the deflection equals to del del F of the strain energy as per Castigliano's theorem.

So, that is then 2FL cube divided by 24YI. So, that is equals to F L cube divided by 12YI. So, now, we know the deflection. So, force by deflection or the stiffness of this guided beam, let us call it kb or the stiffness of the beam is equal to F by w equal to 12YI divided by L cube ok. So, this is one of the important relation.