Nonlinear System Analysis
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Lecture – 20
Introduction to Bifurcation Theory – 1

Hi everyone. My name is Ramkrishna Pasumarthy, I am a faculty member at the Department of Electrical Engineering IIT, Madras. This is a course we are running on Non-linear System Analysis. We are using some lectures from the existing database of NPTEL and then adding up couple of other modules to make the series of lectures a lot more coherent.

I will be handling a couple of modules in this course and so far you would have learnt a lot of both qualitative and quantitative analysis of non-linear systems typical distinction between that of linear and non-linear you would have talked about or learnt about existence of multiple equilibria, existence of limit cycles for example. There were things like finite escape time and some other things like existence and uniqueness of solutions where the uniqueness was not always guaranteed and there are some very straightforward examples even in the one-dimensional case that you would that you would look at.

We also saw a and a stability analysis via a linearization of a non-linear system what we also traditionally call as the Jacobi linearization. And then we saw how the local behavior around that equilibrium point is in some way related to the linearization around that or the linear approximation around that equilibrium point. And we characterized equilibrium points according to we gave them certain names depending on where the eigen values of the linearized system was were lying, especially the second order we characterize them as nodes, focus, center, also talking about stable nodes, unstable nodes, and stable focus, unstable focus and so on.

So, what we will learn today is some very useful aspect of non-linear systems which are not very common in linear systems. I will start with motivating with the linear systems, but we kind of know why we can avoid that kind of phenomena. So, that is called the theory of

Bifurcations, right. So, we will typically look at what happens when a system parameter is changed, ok.

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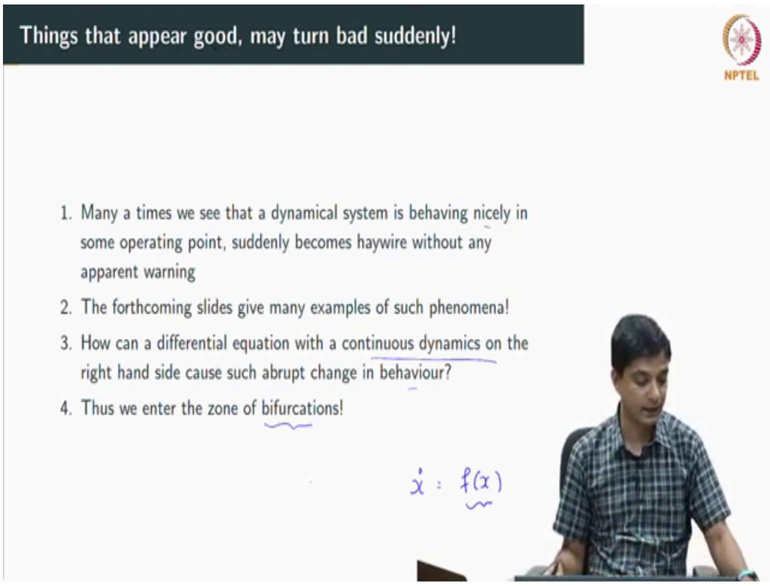

So, what we will eventually learn is that things that appear good may turn bad suddenly and vice versa, what was going well with some small change maybe some environmental condition it can just go pretty bad, something which is not desirable. It also happen the other way. Since we are not going too well and then you can say some site change happens you can see well we are still in some kind of a safe region. We will try to characterize those behavior qualitatively, ok.

So, just to motivate a bit. So, many times we see that system is behaving nicely around some operating point and suddenly it just behaves you know in a way with which we which we would not anticipate it to without and these things typically come without any apparent warning, ok. So, we will give examples of such phenomena. We will try to characterize them in terms of the theory that we have learned so far, right and also try to answer this question of how can a differential equation with a continuous dynamics on the right hand side.

So, which essentially means I am talking of x dot equal to f of x, where the right hand side is smooth, right. So, if this is this discontinuous then you may say oh because the right hand side is discontinuous something strange is expected to happen, would actually happens when you talk of say switch systems where you switch between two stable systems you could actually end up in in a in a resulting unstable system, ok. So, how can a differential equation with a continuous dynamics on the right hand side cause such abrupt change in behavior, and this change in behavior is what we term as bifurcations, and we will see how we can categorize this class of bifurcation points.

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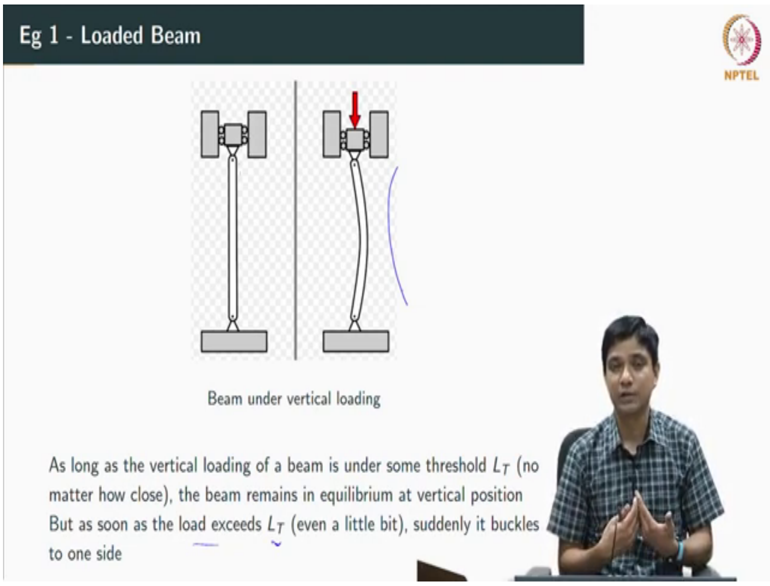

So, one example that we could we would have seen a lot and it is also quite kind of intuitive is you just look at a beam under vertical balancing, right. So, as the load increases where for normal load the beam just behaves as good as it should, but if the load exceeds a certain value it buckles to one side, right. So, you can see it bends either to this side or it also be like bending to this side to the to the opposite side. And we so far have not quantified or even look at looked at it qualitatively why really this happens and what does it actually mean in terms of systems.

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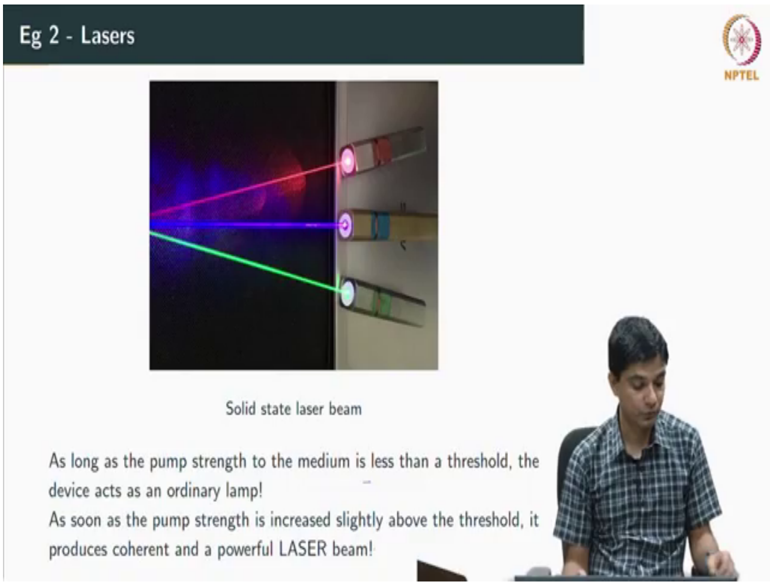

You could actually also look at some other examples of lasers where, as the as the strength is less, it behaves as an ordinary lamp and as the pump strength is increased beyond a certain threshold, it becomes a powerful laser beam.

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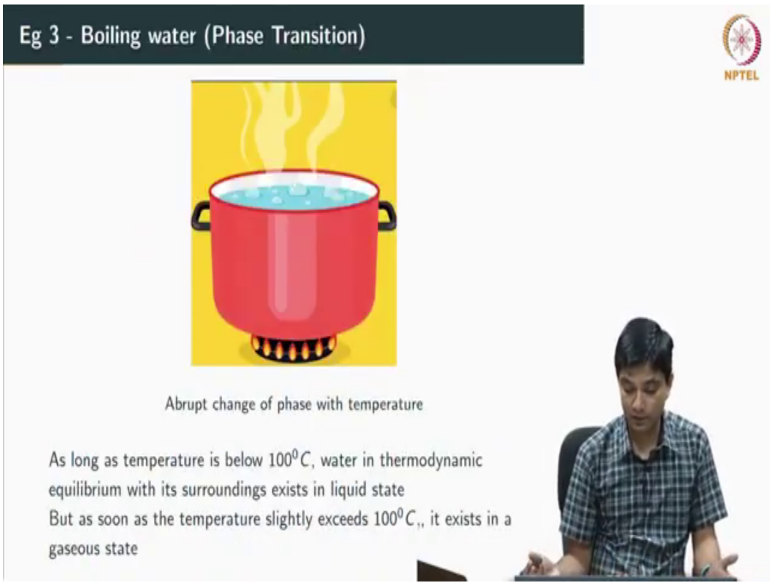

Boiling a water, right. It does not really come with any apparent warning, even though we know that at some level of temperature the state of water changes into a gaseous state, ok.

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And we also look at insect outbreak, right. So, you start with some small set of insects and all of sudden you see that the entire crop or entire forest is infected with pest. Same thing with even you can call about the recent virus outbreak, where you may have initially to begin with few number of people few 10s or few 100s and you see now close to 100,000 people are infected across the globe and it is just keeping on increasing. And these things do not come with any apparent warning or any prior model, right. So, we will try to qualitatively analyze this kind of behavior, ok.

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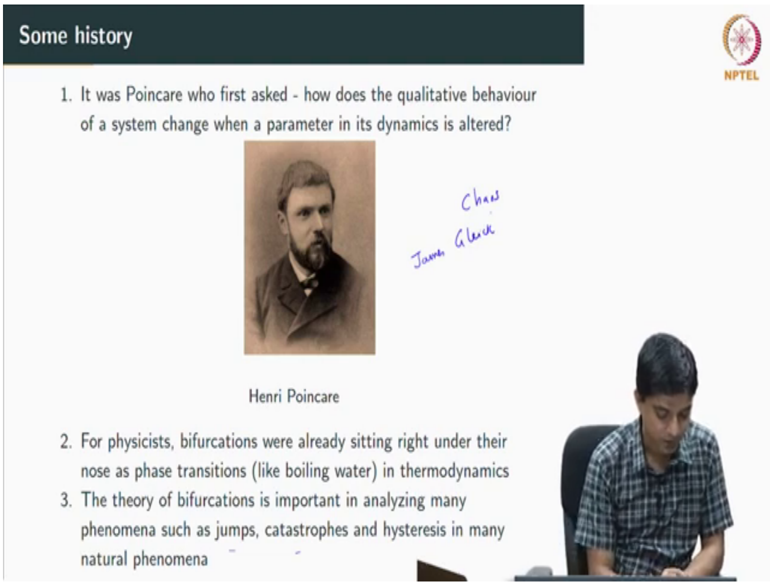

So, where does all this come from? So, this famous French physicist Henri Poincare was first asked how does the qualitative behavior of a system change when a certain parameter is altered, ok.

So, this well this was kind of observed, but never given any formal notion until quite long or until lots of advances in in physics, right, ok. So, this theory is important in analyzing phenomena like jumps and you would have read about catastrophes, hysteresis in in many natural phenomena. And for people who are interested in a lot of this theory there is a book by James Gleick called on chaos theory.

It is a popular read not too much mathematical or academic, but it is a good read. So, so people who are interested beyond what is happening in the textbook or in the course here can actually have a have a look through that. It is a book titled chaos, I do not remember the title completely by James Gleick assuming I have spell it correct, ok.

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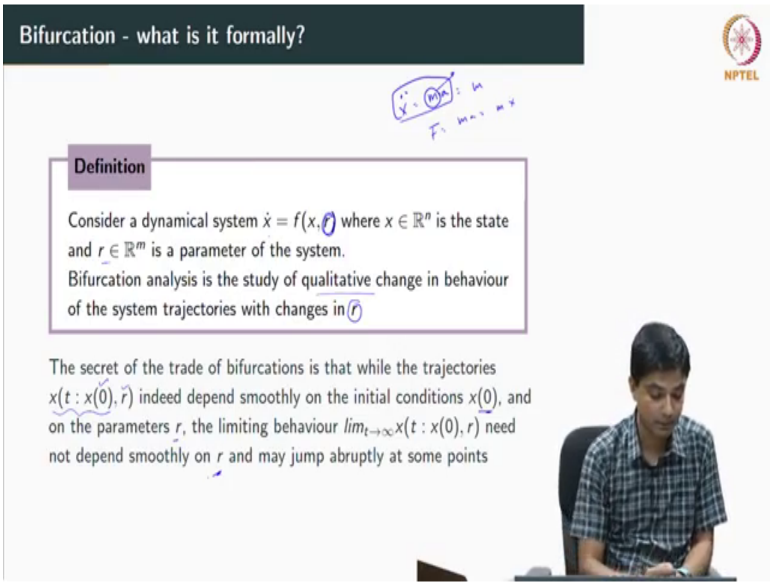

So, what is what is bifurcation? We will first just define it mathematically and then forget about the mathematical definition and then look at systems the way we know. So, we start with a dynamical system x dot is f of x and also r. And what does this actually mean? So, if I write down say the Newton's second law I will have x double dot is mass times acceleration, right or this is m sorry or f equal to m a which is yeah m mx double dot. And if I look at this typical equation this has to do with the state mx double dot whereas, this is a parameter which I usually do not pay much attention to while I am designing systems so far, or what I have whatever I have learned so far, right.

So, we will now pay attention to parameters that that occur here, right. So, that is what we call as r, where x is the state and r is a is a parameter of the system. So, with this study of bifurcation analysis we will see how the qualitative behavior of the system changes or how the solution of the system, how the stability of the system changes with some changes in r, ok.

The secret here is that while these trajectories, right, so you have x at time t starting with some initial condition x naught and some value of parameter is depending smoothly on the initial conditions and the parameters are the limiting behavior need not smoothly depend. And we will see why this could happen, right. It may abruptly jump it may not necessarily depends smoothly on r and may abruptly jump sometimes, ok.

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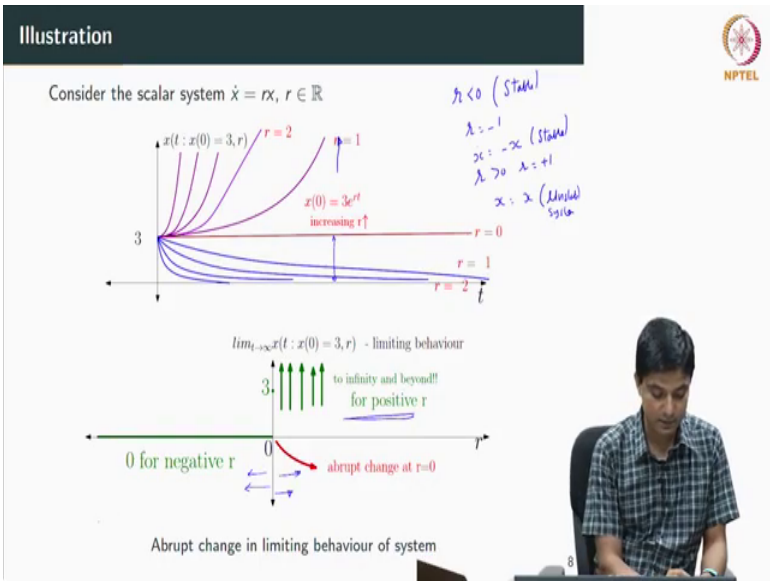

So, let us start with a very simple scalar linear system. So, I start with an x system x dot equal to r, r of x, and for I just if I just plot it the vector field. I see that, I can typically look at r being less than 0, when I call it stable, right x dot let us say r is minus 1. So, x dot is minus of x I know its stable. Then say r going greater than 0, let us say for instance r equal to plus 1. We have x dot equal to x, this is an unstable system, ok

So, r as this value of r changes from negative to positive we have in this green region a stable system, the system behaves as a stable system and all of a sudden as this parameter would increase would change from negative of r would cross 0, you are having an unstable system, right. It goes to infinity and beyond for positive r. For here it is kind of kind of well behaved.

And I can just plot this trajectories x versus time for different values of r and you see here as r increases until this region r equal to 0, your trajectories converge to 0, r equal to 0 your constant and for all r greater than 0 your trajectory just keep going up for arbitrary initial conditions, ok.

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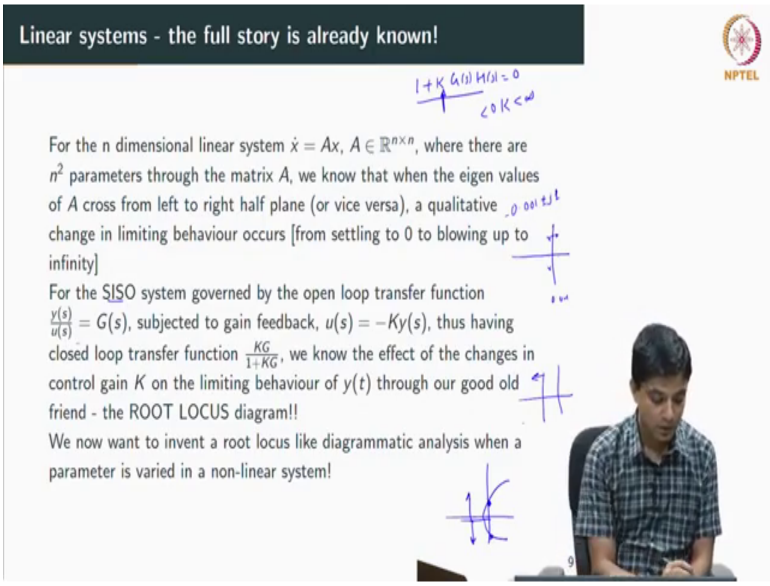

So, in the case of linear systems, let us just start with SISO systems, right. Like for single input single output systems, I have a nice notion of transfer function, I have a nice notion of feedback and I know say for example, what does what does the root locus tell you. If I just say 1 over KG times H equal to 0 is the characteristic equation of the closed loop I am just interested in checking how does the system behave for values of K going from 0 to infinity and I just get some plot sometimes which they just start here, here and they are stable for all K, sometimes they are unstable. So, I can find this value of K for which a system goes from stable to unstable.

Like in this case the for all values of r less than 0 I was stable, all values of r greater than 0 the system was unstable. So, I know this this characterization, in terms of root locus. And therefore, while design you typically as do not want to design a stabilizing controller where the poles of the closed loop system are just slightly to the to the left of the complex plane. They say this for example, 0.001 minus 0.001 plus minus j 1 may not be a very smart design because a slight change in parameter could or slight change in environmental conditions could change the parameter and push it to an unstable region, and you know that 0.001 is actually unstable even though it is very like close to the to the origin, right.

So, we do we unknowingly did this analysis while we were doing the root locus in the case in our first course on control engineering, what are the how does the system behavior change as the value of K varies from 0 to infinity. You can also draw the negative root locus saying what happens from minus infinity to 0 and we find use the Routh Hurwitz criteria to find out what are the values of K for the system is unstable.

It could turn out in many cases or some cases that the system will be stable for all K, for some cases for K slightly greater than 0 you will be unstable and a and a bunch of bunch of other things too. And therefore, we also learn relative stability in Routh Hurwitz where we say, place all my poles to the left of minus 1, ok.

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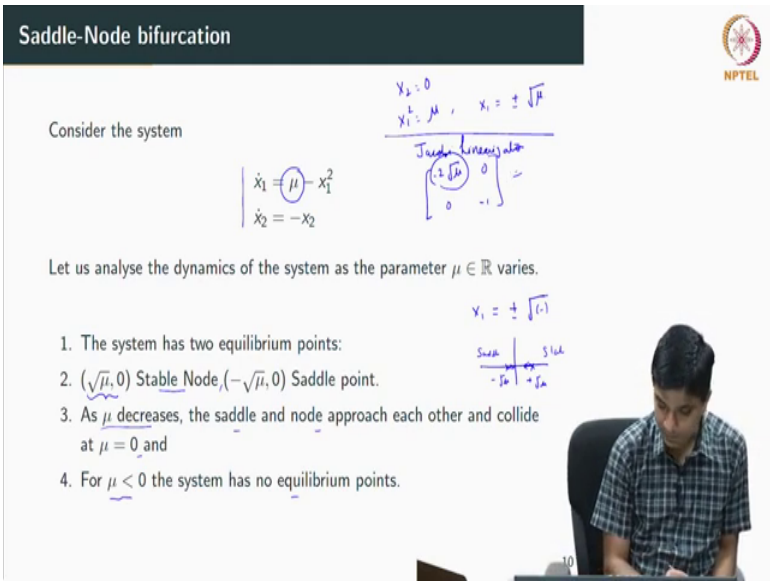

So, let us come to the non-linear case and just start with few examples, ok. So, let us let me start with this second order system x 1 dot is mu minus x 1 square, x 2 dot is minus x 2 and we will check what happens to the behavior of this system as mu varies, ok.

So, first is I will just quickly do it for this example, and the remaining words I will just leave it you to solve. So, what are the equilibrium points? For sure x 2 equal to 0 is the equilibrium point and then another equilibrium point is x 1 square equal to mu which means this is I am looking at x equal to x 1 equal to plus or minus square root of mu, ok. So, I do it the standard way. I just look at the Jacobi linearization, resulting in the Jacobian matrix which in this case looks like this, minus 2 square root of mu, 0, 0, minus 1, ok.

And the other equilibrium, so the system therefore, has two equilibrium points; you have plus mu comma 0 which is from this table from this Jacobian you can find out is a stable node and minus mu comma 0 which is a minus square root of mu comma 0 which is a saddle point, ok. So, what will be interesting is to check what happens as mu decreases, right.

As mu decreases the saddle and the node approach each other collide at mu equal to 0 and for mu less than 0 the system has no equilibrium points. Yes let us check this, right. So, what happens when mu is less than 0 I am looking for solutions is x 1 is plus minus square root of minus mu, all right, ok.

When mu is less than 0, this the guy inside, this is a negative number and I would definitely not deal with imaginary equilibrium points, right. So, as mu, so these are the two equilibrium points, one is stable node, another is a saddle point. As mu decreases from certain values to 0, the saddle node and the saddle and the node, so you are looking at something like this, right. You have a plus square root of mu you have a minus square root of mu, ok, this is stable this is saddle. And as mu decreases, they will keep moving to this guy will keep moving to the right, this guy will keep moving to the left, ok.

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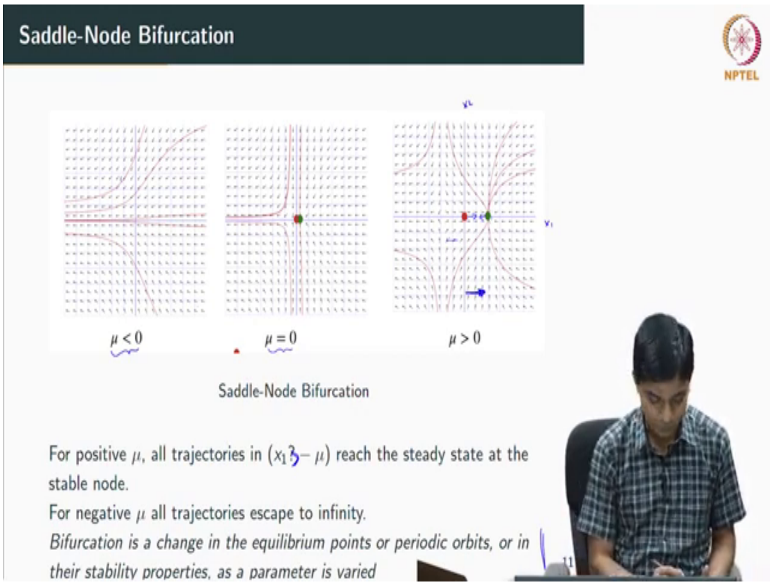

So, let us see this here from this, ok. So, for mu greater than 0 what we have is, so this is this is a stable node, this is this is a saddle point and as mu decreases these two keep moving towards each other, they meet each other at mu equal to 0, ok. Let us not really worry about what is the stability of mu equal to 0 that we will keep for later discussion, and for mu less than 0 the system does not have an equilibrium, ok. So, for positive mu all trajectories, so if I look at here this diagram, for positive mu all trajectories in this region x 1, so I am just drawing the diagrams like this x 1 and x 2 in the standard phase portraits.

For all trajectories for which x 1 is greater than minus mu they reach the steady state which is a stable node in in at asymptotically. For negative mu all trajectories, so it is for negative mu all trajectories no matter wherever you start you escape to infinity along these lines along the arrows. Here at least for mu in which in this region you can converge to this this this green and the green dot is this I just used the green for stability red for unstable, ok.

So, one thing that we observed here is as the as mu changes from positive to negative, you have two equilibrium points and for mu less than 0 the equilibrium points disappear totally, right. So, this is this something very strange, right.

In the linear case you had system only going from stable to unstable region for some certain values of K. Here well, the equilibrium points one was stable, one was unstable as they keep coming close to each other with decreasing mu they just disappear, right. There is no equilibrium point, there is no steady state when mu is less than 0 all trajectories escape to infinity ok.

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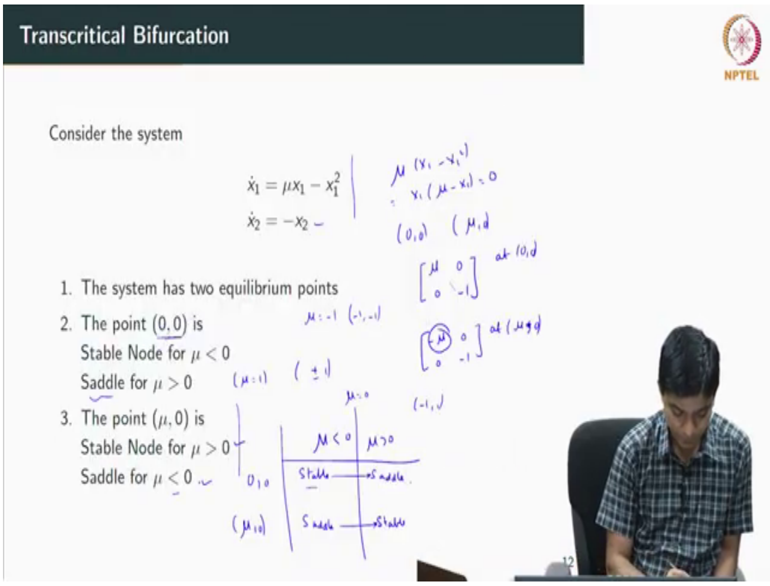

Similarly, I now look at this system where I have x 1 dot is mu x 1, minus x 1 square, x 2 dot is, so this this equation will pretty much remain the same. So, this system will have two equilibrium points. So, if I just look at mu x 1 minus x 1 square is x 1 mu minus x 1 equal to 0. So, the equilibrium points would be of course the origin, another equilibrium point will be mu comma 0, ok.

And Jacobian would be should be easy to compute the Jacobian at, so the Jacobian at 0, 0, ok. I now I will skip some of these steps because you would already have know how to compute this you have 0, minus 1 and the Jacobian at mu sorry mu comma 0 would look something like this minus mu, 0, 0, minus 1, ok.

Now, look at this at 0, 0. At 0, 0 for mu less than 0 we get that all the eigen values are negative, right. So, this will be a stable node for mu less than 0. Ad this will become a saddle, so say for mu is equal to minus 1 my eigen values here would be minus 1 minus 1 this corresponds to a stable node. For mu greater than 0 let us say some arbitrary value say mu equal to 1, my eigen values should be plus or minus 1 that corresponds to a saddle, ok.

Now, look at this equilibrium point at mu 0. Mu comma 0, for mu greater than 0 for mu greater than 0 say mu equal to one my eigen values are minus 1 minus 1. It is a stable node, and it is a saddle for mu less than 0, ok. What is happening here? Let us just check here, right, ok.

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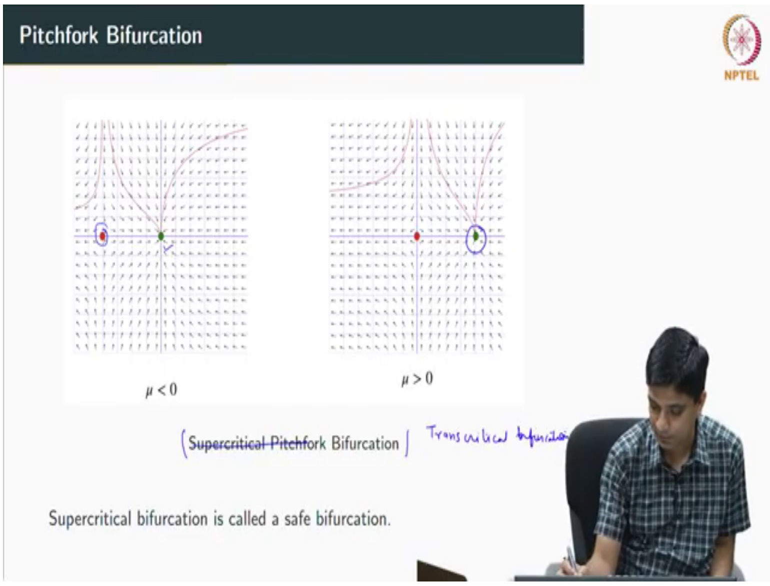

The some pictures got exchanged in in the slide. But this is essentially corresponding to not a super critical, but this is like a transcritical bifurcation, ok. So, so apologies for the for the typo here or for the for mixing up the images, ok.

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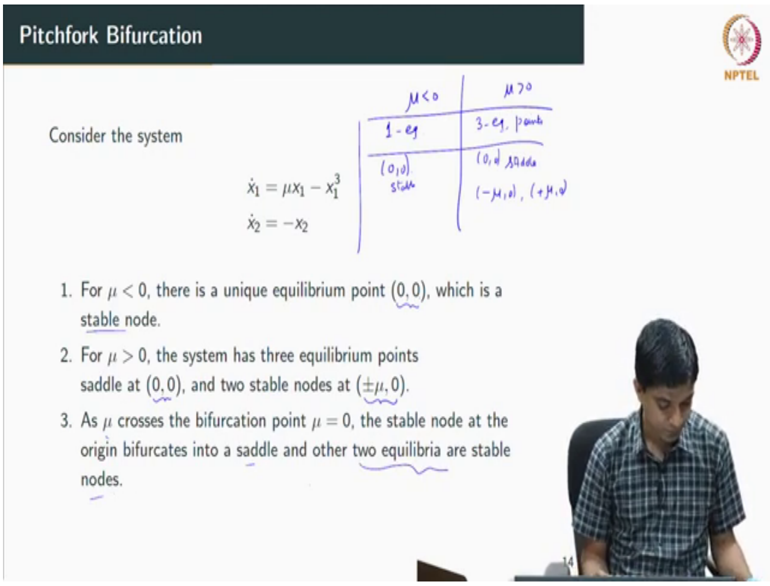

What happens is look at for mu less than 0, ok. And if I look at 0 comma 0 is stable and for mu greater than 0 I will just call it (Refer Time: 21:20) reverse is a saddle, ok. Now, similarly the point mu comma 0 when mu is less than 0 is a saddle and this is stable for mu greater than 0.

So, as we cross the bifurcation points the stable node becomes a saddle and the saddle becomes a stable node. So, they just kind of exchange their properties, so here, right. So, this was for mu less than 0 the mu less than 0, the origin was stable, ok, mu less than 0, origin was stable and then the other equilibrium point well this was this was like a saddle, right. And as you keep increasing mu you cross mu equal to 0, they just change their properties, right. So, what was stable now becomes unstable and then there is another equilibrium point here at this this equilibrium point which becomes a saddle point, for mu less than 0, right.

So, so that is, so two things we have seen, right. In in the first case where I have a saddle and the node they meet together for increasing values, for decreasing values of mu and then they just disappear. Now, I have two equilibrium points where the stable node here it becomes a saddle as I cross the bifurcation point and the saddle becomes a stable node, ok. So,ok this is a text with transcritical bifurcation. I just messed up those pictures, ok. I will I will clear I will correct those slides, right.

So, the equilibrium points they exist through all values of mu, for mu less than 0 I have equilibrium points, mu greater than 0 I have equilibrium points, ok. So, only thing that changes contrast to what happened before, right mu lesson 0 equilibrium point disappear. Equilibrium points persist only they flip their characteristics, right. The point 0, 0 changes from a stable to a saddle and mu 0 changes from a saddle to a stable node, ok, right, ok.

Now, the third kind of bifurcation which we will look at is what is called as the pitchfork bifurcation. So, again I am looking at a system which looks like this. So, I have a system of mu x 1 minus x 1 cube x 2 dot remains the same. For mu less than 0 you can see there is a unique equilibrium point and we can check that this is a stable node, ok.

And for mu greater than 0 the system will have 3 equilibrium points and those will be a saddle at 0, 0 and two stable nodes at mu comma 0, right, ok. Let us quickly check what is happening. First this equilibrium and this equilibrium, so you can just say mu less than 0, mu greater than 0 and if i say the number of equilibriums I have 1 equilibrium point, here I have 3 equilibrium points, ok. Now, here well 0, 0 was stable and this 0 comma 0 the origin becomes an unstable it is becomes a saddle, and then two more equilibrium points appear one at minus mu comma 0 and one at plus mu comma 0.

Look at how interesting, right. So, we had cases where equilibrium was disappearing here all of a sudden two additional equilibriums emerge, ok. So, as mu crosses the bifurcation points, the stable node at the origin it bifurcates into a saddle this the stable node becomes a saddle and two other equilibriums are created which are stable nodes, ok. How does the picture for this look like? Ok.

So, this is, called the transcritical, but the super critical pitchfork, ok. So, for mu less than 0, I had one nice stable point all of a sudden what happens is that this equilibrium point when mu goes to slightly higher value than 0 or even larger values it becomes unstable.

Well, this is, so generally this is called a safe bifurcation that is because even though this system the equilibrium the origin changes from a stable to an unstable for very small values of mu you see here the system does not go to infinity, it just settles at this point and or settles at this point. So, for small values of mu, this actually well this is that is what a disaster that is happening. As my system does not jump from being stable all of a sudden to being unstable, it just shifts and sits up at a point which is which is like close by, right.

So, the trajectories do not like go away to infinity, right, ok. So, and then now I have something called if I just flip the sign of x 1 cube, the second term here I have the reverse behavior, right.

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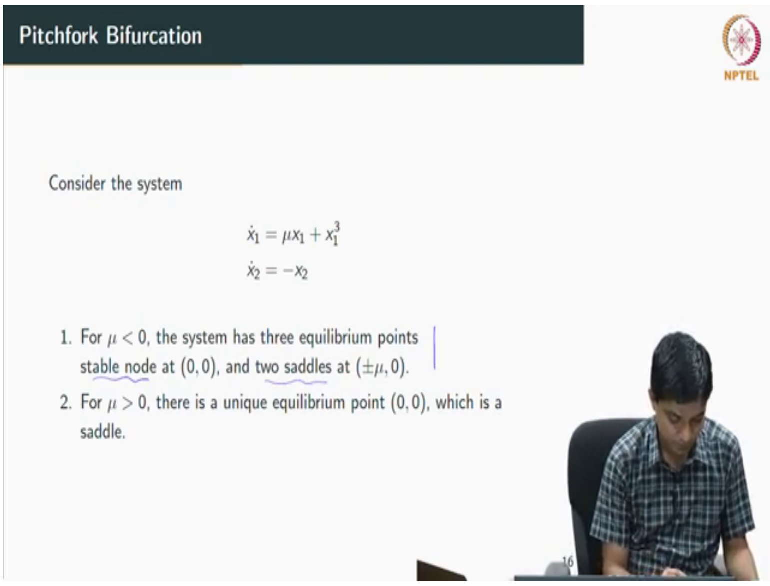

So, for mu less than 0, I will have 3 equilibrium points, a stable node at 0, 0 and two saddles at plus minus mu comma 0, ok.

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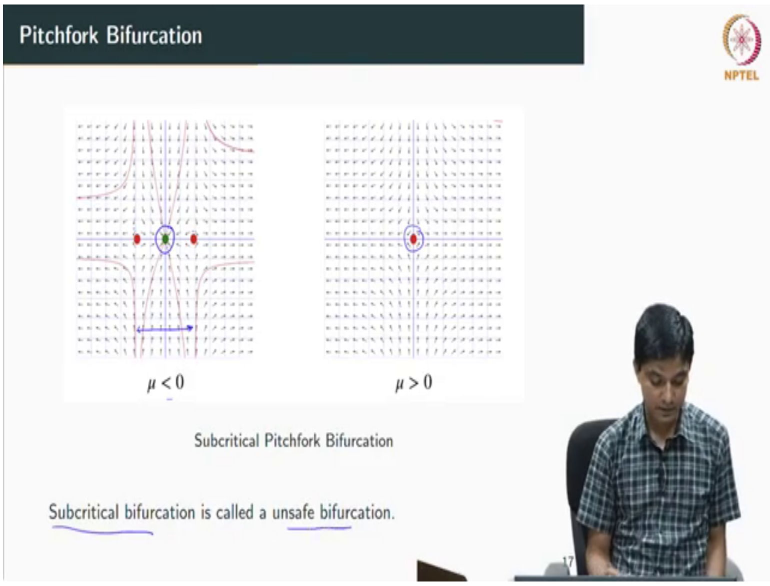

So, I have picture, right, yeah. So, for mu less than 0 this is stable. So, this is all good now these two are unstable, ok. Assume this is my operating point and a sudden change in parameter or sudden change in conditions pushes the parameter from mu less than 0 to mu greater than 0. What you see is that these two red dots which were appearing here they disappear.

So, 3 equilibriums they change to one equilibrium, not only that this one equilibrium is unstable therefore, you would not you would never want this to happen, right because whatever is you are in in in the stable region, so even in this case is if you are within this region here you will be pushed to the to the to the green dot, ok.

Whereas, once you bifurcate into this region then all the trajectories become unstable, right. So, all trajectories go to infinity and therefore, a sub critical bifurcation is called an unsafe bifurcation, ok.

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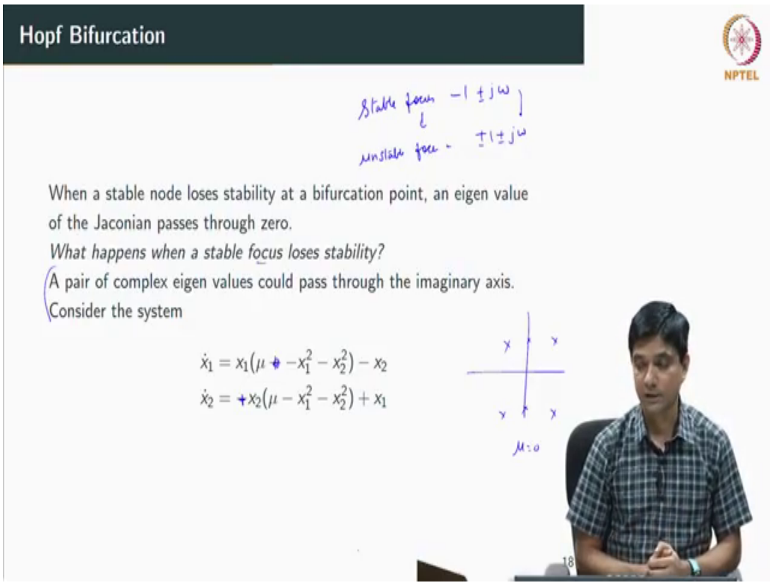

So, to summarize or to look at what really is happening in this in all the all of these examples is that when mu equal to 0 in this case you are encountering a 0 eigen value here, when mu equal to 0, 0 eigen value of the linearization, right and when mu equal to 0 here you still encounter a 0 eigen value, right. Similar things will happen also here. When mu equal to 0 you encounter again a 0 eigen value, right, ok. So, whenever a stable node loses its stability at bifurcation point an eigen value of the Jacobian passes through 0, ok.

So, this, we were just looking at a stable node so far and stability is not just or all stable equilibrium are not just nodes, it could always be or they can always be as a stable focus. So, the last example that we will do for today is for this lecture is what happens when a stable node or when a when a stable focus sorry lose a stability, ok. So, I just do this little example where I have x 1 dot is mu, ok, this plus this is a typo minus x 1 square minus x 2 square minus x 2, x 2 dot is, so this we will again be a plus, plus x 2 of mu minus x 1 square minus x 1 square plus plus x 1, ok. So, just to do this analysis a little rigorously let us do it in a do it little differently.

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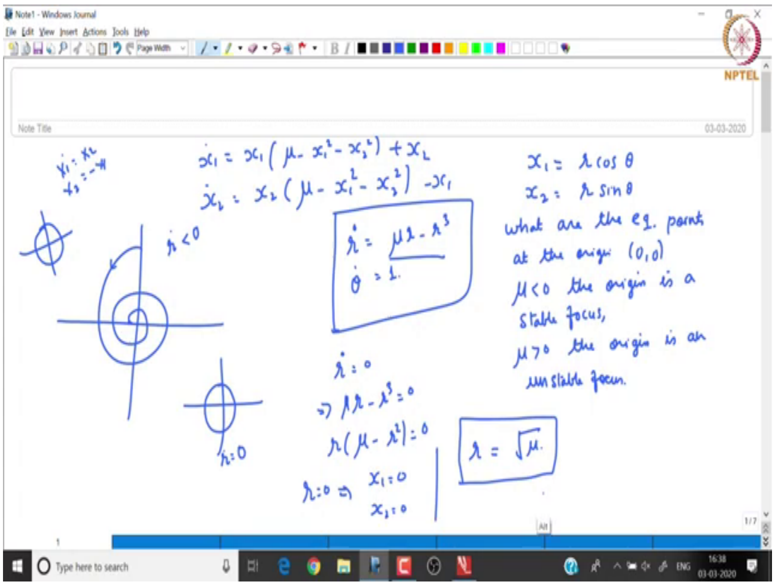

So, I have x 1 dot is x 1 mu minus x 1 square minus x 2 square plus x 2 x 2 dot is x 2 mu minus x 1 square minus x 2 square minus of x 1, ok. This may know look a little complicated to solve, but some of our earlier courses taught us coordinate transformation, right. So, where you just go from rectangular to polar, and in this coordinates I have something nice r dot is mu r minus r cube and theta dot is 1, ok. So, I will just analyze this in the polar coordinates now, ok.

So, what are the equilibrium points? Ok. So, this simple inspection would show that the system has a unique equilibrium point at the origin. Even from here you can find, right that system has a unique equilibrium point at origin and that is the only thing that happens.

So, let us check what happens, it is for and the standard analysis you can find that for mu less than 0 the origin is a stable focus and for mu greater than 0 the origin is an unstable focus, ok. So, let us look at, ok. So, how does a focus look like? So, it is a stable focus would just spiral this way to the origin which means a certain radius r dot is negative is less than 0 and some cos 1 theta. For a periodic orbit you will have that r dot is equal to 0, that the radius does not change, right.

So, if you look at system which is a the simple harmonic oscillator x 1 dot is x 2 dot is minus x 1, it will have a phase portrait which is not just a circle, right, and if r dot is greater than 0 then you will have an unstable trajectory. So, this is a stable one and then r dot will be the opposite sign, ok.

So, just check, let us check there is something interesting happen here when r dot equal to 0. r dot equal to 0 implies mu r minus r cube equal to 0 or r mu minus r square equal to 0, r equal to 0. Well, not very interesting that is r equal to 0 is the origin x 1 equal to 0 x 2 equal to 0, r equal to 0 I have x 1 equal to 0, x 2 equal to 0 whereas, the other equilibrium point r will be square root of mu, ok.

So, something we can infer from here, right, that there is a possibility of a periodic orbit and possibly this could also be a limit cycle, right of radius r equal to square root of mu, ok. So, let us just check what happens with the phase portraits of this.

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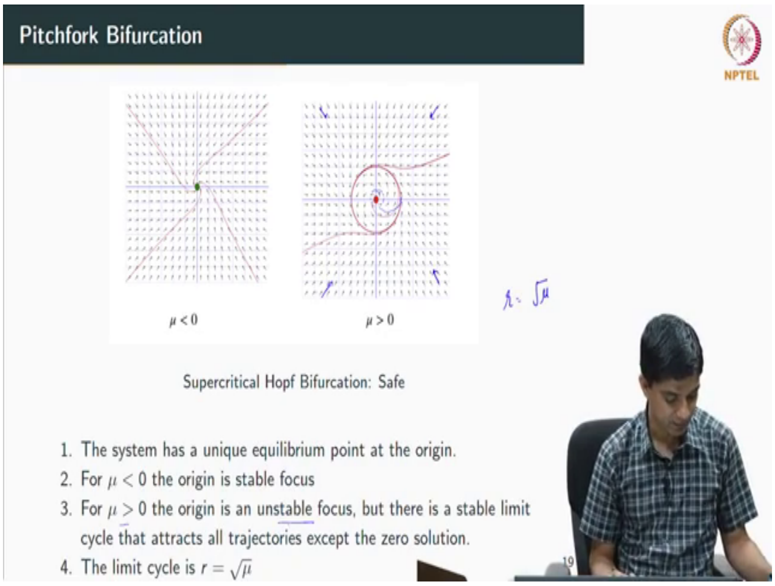

When mu less than 0 what I have is that the origin is a stable focus all trajectory spiral to the origin when mu greater than 0 this becomes an unstable equilibrium. But what happens is all the trajectories go and converge to a limit cycle whose radius is given by r equal to square root of mu, ok. For mu greater than 0 origin is unstable, but there is a stable limit cycle that attracts all trajectories except the 0 solution, right.

So, what happened at the 0 solution? 0 solution is r equal to 0, nothing changes here, right. So, in this case we have a limit cycle and you see all trajectories, trajectories starting from here, from here, from here or from here they will converge to this limit cycle, ok. So, in this case what is essentially happening is when you change from a stable focus, so where eigen values could be minus 1 plus minus j omega to an unstable focus.

Say, with eigen values of plus 1 plus plus 1 plus minus j omega, right. You a pair of complex eigen values actually pass through the imaginary axis. So, you had things here, things here and the bifurcation point at mu equal to 0. Well, your eigen values actually pass through the imaginary axis, right, ok. So, this is a bit of qualitative behavior of how equilibrium points can change with changes in the system parameter.

So, I will end this lecture here. And next time when we meet, we will do a little more analysis on when do these kind of equilibriums occur or we will try and derive certain necessary and sufficient conditions where I can guarantee that a system undergoes a certain kind of equilibrium. And again, we will revisit all the examples that we motivated our lecture with.

Thanks for listening.