Nonlinear System Analysis
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Lecture - 10
Existence and Uniqueness Theorem of ODE-Part 03

This is the good moment to see closely related topic.

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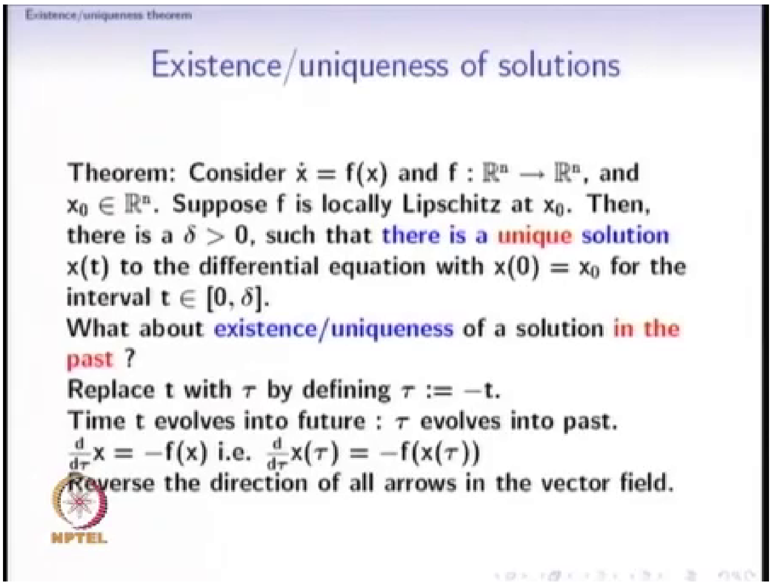

So, we saw the Existence and Uniqueness of Solutions Theorem. So, let us have a quick relook. So, consider the differential equation x dot is equal to f of x where f is the map from R n to R n and at the initial condition x naught. Suppose f is locally Lipschitz. Then there is a delta greater than 0 such that there is a solution and there is a unique solution. In fact, x of t to the differential equation x 0 is equal to x naught for the interval 0 to delta.

So, please note that we are starting from t equal to 0 to some delta greater than 0. So, this is an interval in positive time for the future, there is a solution a unique solution for some time in the future. An important question is there a unique trajectory in the past, so what about existence and uniqueness of a solution in the past. So, for this particular issue, we can easily modify our theorem. So, replace t with tau with defined by defining tau equal to minus t. So, as t evolves into the future tau evolves into the past.

So, the differential equation x dot is equal to f of x becomes d by d tau x d by d tau of x equal to minus f of x. In other words d by d tau of x of tau x is a function of tau, now is equal to minus f at x of tau. So, how do how does one obtain the vector field for this dynamical system? We just reverse the direction of all the arrows in the vector field of the differential equation x dot is equal to f of x. why because each arrow is not f of x, but minus f of x.

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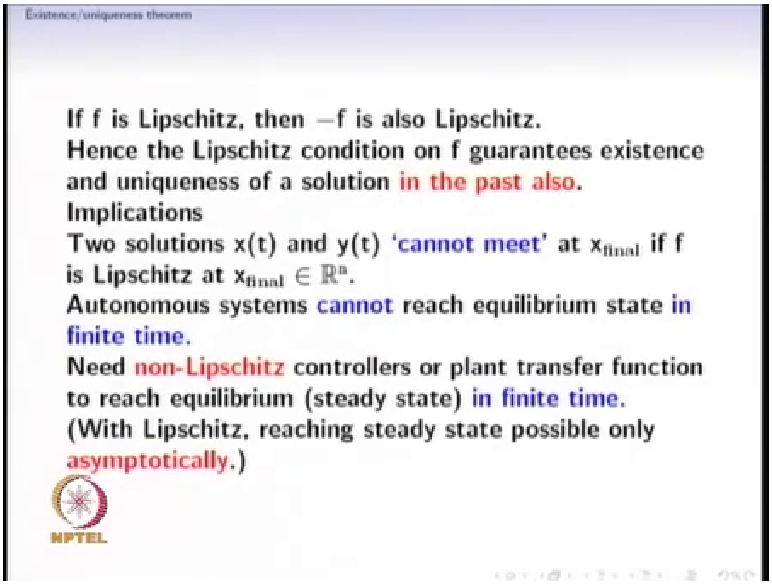

So, if f is Lipschitz notice that minus f is also Lipschitz, hence the Lipschitz condition on f guarantees existence and uniqueness of a solution in the past also. So, what are the implications of this particular observation? So, two solutions x of t and y of t cannot meet at x final, if at a point x final. If f is Lipschitz at that point x final if f is locally Lipschitz at x final, then it is not possible that there are two past trajectories x of t and y of t which have the same final condition x final. Similarly autonomous systems everything that we have been doing so far is for autonomous systems. One of the properties that we can claim about autonomous systems is that the autonomous systems cannot reach the equilibrium point the equilibrium state in finite time.

Why because whenever it reaches an equilibrium state, that equilibrium state already had one past which was the same point for all time. But there cannot be another trajectory that comes and meets this equilibrium state, if f is locally Lipschitz at this equilibrium state. So, if you want to have a particular system, if you want to design a controller in which you reach the steady state in finite time and remain there. Then one would require non Lipschitz controllers or plant transfer function to reach the equilibrium. In this case, we interpret the equilibrium as the steady state if you want to reach the steady state in finite time, then one would need either non Lipschitz controller or non Lipschitz plant transfer function. Why is that? Because with Lipschitz we can reach the steady state only asymptotically; it is not possible to reach in finite time ok.

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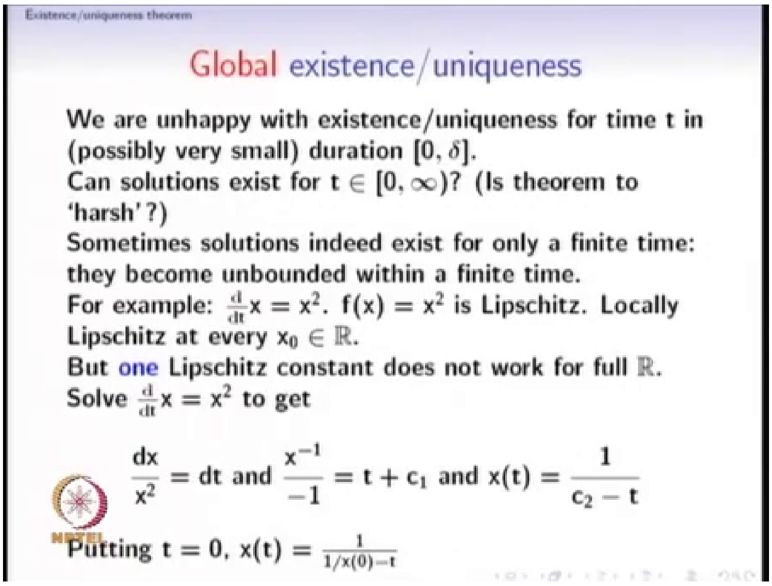

So, another important topic is we have been seen only local existence and uniqueness condition. What is local about it? We saw that there exists a solution and it is a unique, only for an interval 0 to delta. Even existence could not be guaranteed for large enough time, but it could be guaranteed only for a time interval 0 to delta. All that was guaranteed was the delta is greater than 0, but it is possible that this delta is a very small value and we are unhappy with this result about the existence and uniqueness for so small and interval possibly.

So, it is possible that can solutions exist over the interval 0 to infinity. Is it that the solutions indeed exist and their unique, but our theorem is not able to guarantee it is the theorem too harsh, is it that it is assuming locally Lipschitz property on f, because of which we are able to guarantee existence and uniqueness only for a small interval 0 to delta. But there might be some other results some other way of proving that the solution exists from 0 to infinity. So, are the conditions assumed in our the theorem too harsh, because of which we are able to prove only local existence and uniqueness. For this, we will see one small example.

So, it is indeed to that sometimes solutions indeed exist for only a finite time. So, our theorem can also accordingly claim existence and uniqueness, only for a short interval. Why would the exist for only finite time, because it is possible that the solution becomes unbounded in finite time. So, for consider the differential equation x dot is equal to x square where. So, x dot is equal to x square means that f of x is equal to x square. So, notice that this is Lipschitz yeah. In fact, it is locally Lipschitz at every x naught in R.

So, please note that this dot here does not mean it is multiplication of x square and f of x. It is the end of a sentence, x dot is equal to x square is the differential equation and for this differential equation f of x is equal to x square and this particular function f is locally Lipschitz at every point x naught. But notice that one Lipschitz constant does not work for the full R yeah. So, solve we can explicitly solve this differential equation x dot is equal to x square to get dx by x square is equal to dt and upon integrating both sides, we get x to the power minus 1 divide by minus 1 equal to t plus some constant c 1 and upon rearranging this minus 1 and x of t, we will call minus c 1 equal to c 2 and we get x of t equal to 1 over c 2 minus t.

So, when we put the initial condition at t equal to 0. Suppose it was a x naught x 0, then when we substitute we get x of t equal to 1 over 1 over x naught minus t. So, let us just make this so our differential equation solution. This is how the solution to our differential equation looks ok.

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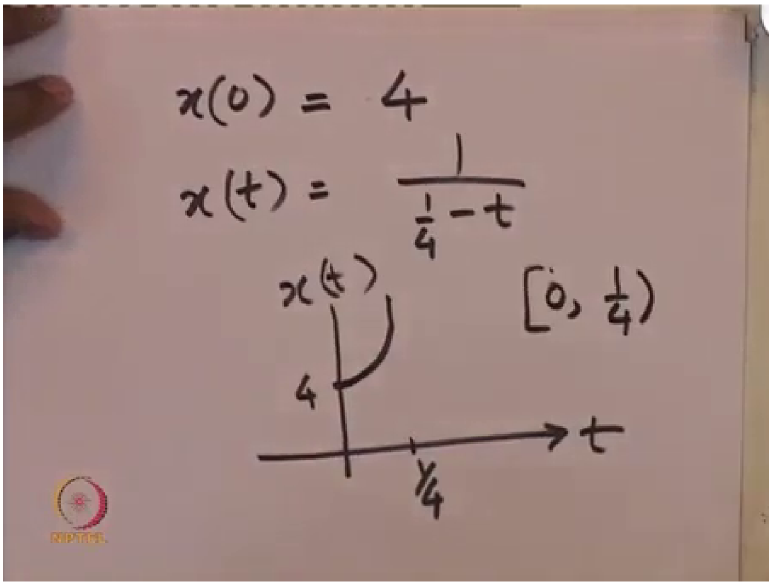

Let us see what this means if x of 0 is equal to some number let say 4, then we see that x of t equal to 1 over quarter minus t. So, we see that for t equal to 0 of course, it is equal to 4 and as t tends to 1 by 4 this quantity becomes unbounded. So, a graph of x versus t, it starts from 4 and it becomes unbounded. So, within a small interval up to 1 by 4 already it is so large that it is unbounded. So, we have solutions defined only over for this particular initial condition, we are able to define existence of a solution only from 0 to 1 by 4. While it is a closed interval on this side, it is an open interval for t equal to 1 by 4, we do not have a solution to this solution does not exist.

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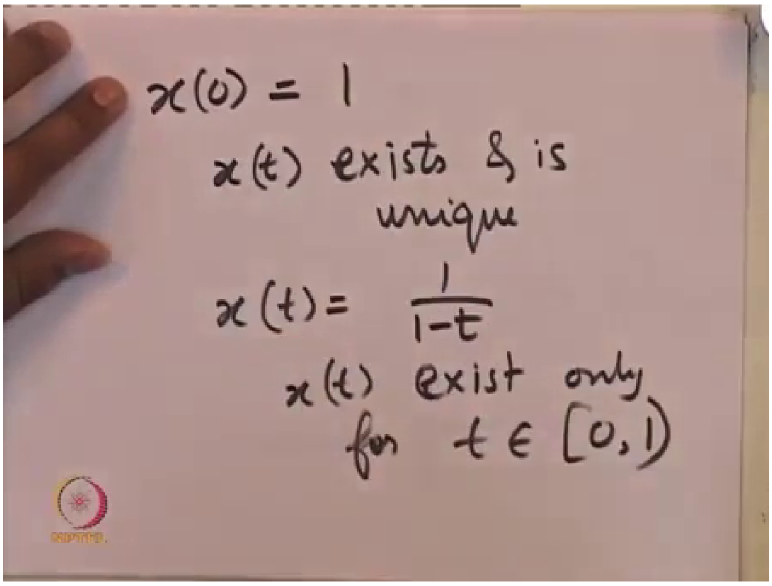

So, what we have seen is when the initial condition is equal to 4. We had a solution only up to 1 by 4, suppose the initial condition was equal to 1 then we have a unique solution for some delta. But when we try to increase this delta we see that x of t is exists and is unique.

How long can we extend this? We see that by explicitly solving this differential equation we get x of t equal to 1 over 1 minus t. So, x t is defined exist only for t in the interval 0 to 1; 0 to 1 for this particular initial condition. So, for each initial condition it becomes unbounded in a finite time in how much time it becomes unbounded that time depends on the initial condition.

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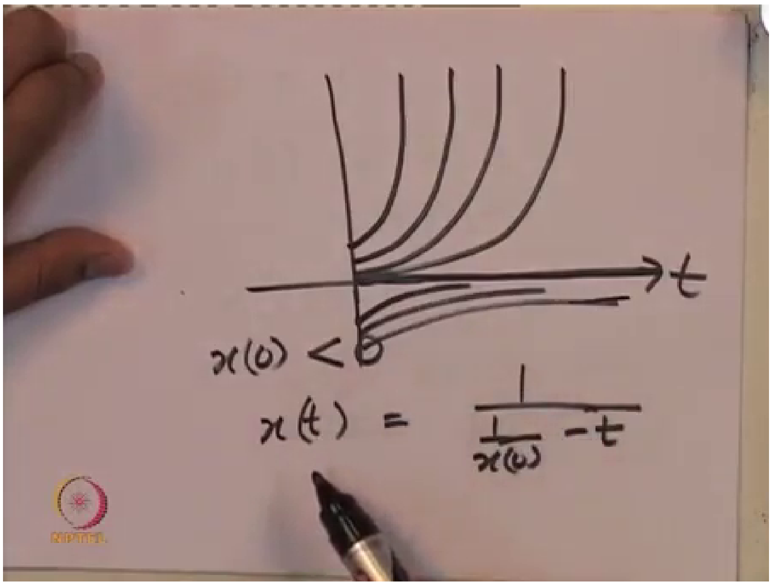

So, a solution so a set of solutions to this differential equation. If it starts below, then the solution exist for some more time. If it is at 0 of course it remains 0 for all future time, because it is locally Lipschitz at 0 this solution cannot emanate out of the equilibrium point, there is a unique trajectory and hence it remains always at 0.

But if x of 0 is negative then what happens x of t is equal to some number some number 1 over x naught which is negative minus t, so the solution always exist. So, when it is negative, then we see that the solutions are coming close by. So, we see that if x of 0 is negative, then the solutions exists for all future time, they are not becoming unbounded in finite time and they all approaching 0. But if x of 0 is positive, then the solution grows and becomes and unbounded in a very short time in finite amount of time and hence we cannot have global existence of solutions when initial condition is positive. But we it appears we can have global existence of a solution when x of 0 is negative.

So, it appears like for certain situations there do exist solutions from t equal to 0 to infinity. While there are other situations for the same differential equation there are certain other initial condition for which the solutions exist for only a finite amount of time. In which case, we cannot have global existence of the solution let alone global uniqueness. So, for this particular differential equation we might have some additional assumptions, under which we have a unique solution from 0 to plus infinity. And it is possible that for certain initial conditions those conditions of the theorem do not hold. In which case, we do not have global existence. So, those additional conditions how to formulate them is the topic we will see in the following lecture ok.

Thank you.