Optical Engineering
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Lecture – 29
Transformation of a Gaussian beam due to a lens and a mirror

Good morning, today we continue with the Gaussian beam. So, in the last class we had taken a small deviation, we wanted to look at how a Gaussian beam is affected by a lens. And so, in the last few classes we were looking at how any wave gets affected by an element with a certain phase transmittance.

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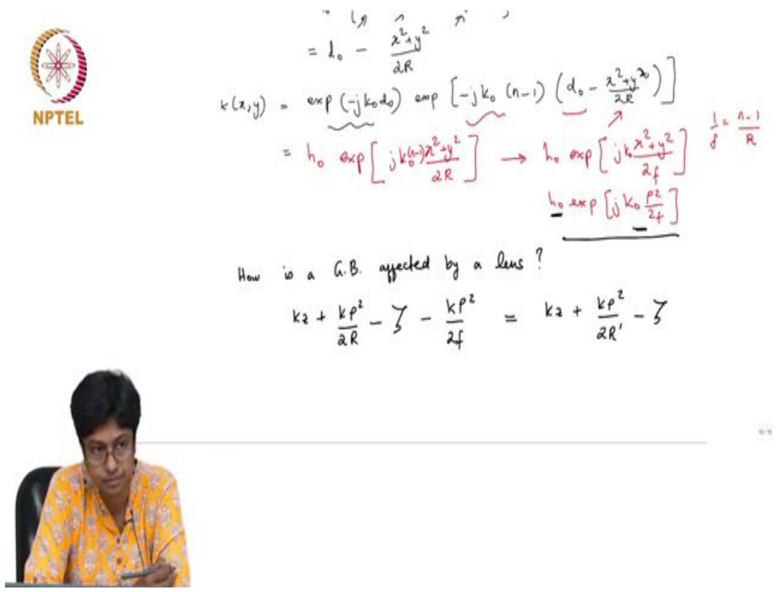

So the equation that you can see over here is the transmittance of a plano-convex lens and I had said that you will have a similar expression for any lens. And what you can see is the transmittance is nothing but an exponential function.

In other words, the transmittance is going to affect the phase of the wave passing through it. How does it affect the phase? It affects the phase by this expression. So now, let us go back to our original problem. We wanted to see how the phase of a Gaussian beam is affected by a lens.

And since the transmittance of the lens is only in terms of an exponential, this term if you remember was just a constant, it affected the phase throughout in a similar way. So, we put it under a constant.

Since only the phase is affected, to answer this question of how a Gaussian beam is affected by a lens, I will not write out the entire Gaussian beam expression which consisted of 2 parts; the amplitude part and the phase part. But I will only pull out the argument of the exponential term.

So, I have 3 terms $kz + k\rho^2/2R$ where R is the radius of curvature of the beam minus the (Refer Time: 02:32) phase term ς. This is now an incident on the lens, right. So, it is going to be affected by this term over here.

So, I am going to add this $k\rho^2/2f$,, and this is going to give me the new phase front and I can write that again in terms of $k\rho^2$, but now the radius of curvature has changed $-\varsigma$.

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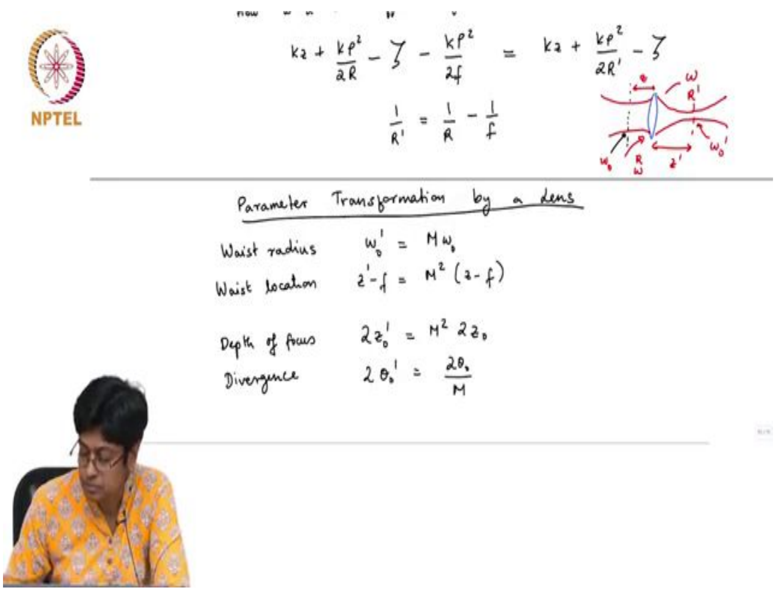

So in other words, the new radius of curvature is related to the original radius of curvature and the focal length of the lens. So, if you had a lens and a Gaussian beam incident on this lens, immediately after the lens and we are assuming a thin lens in this case.

Immediately after the lens, the beam is going to change. What is this equation saying? Before the lens you have a radius of curvature R of the wavefront, you have a spot size omega and immediately after the lens omega does not change, immediately after the lens it does not change right, assuming a thin lens. But your radius of curvature has changed, and how has it changed? It is a function of the focal length of the lens that you are using, right.

Now, if you work out based on this information, how all the parameters change, you will get this set of equations. So, this is how the parameters transform, ok. So, I am just going to give you the equations right. So, the waist radius is ω_0' and it is nothing but, the old waist radius multiplied by the magnification caused by the lens that you are using.

So $M\omega_0$, the location of the waist. What do I mean by z and z'? Well, z is if this is the waist here, so, this is omega naught before the lens, this distance is z and if this is the new waist location, this is omega naught dash and this will be z', ok; always measured from the lens. So the waist location has it changed? It is also changed based on the magnification, but it is now M^2 as well as, of course, the focal length of the lens that you are using. The depth of focus is the Rayleigh range or twice the Rayleigh range, $2z_0'$. And it is again a function of the magnification, the divergence, 2 theta dash you can see the magnification is playing a role everywhere.

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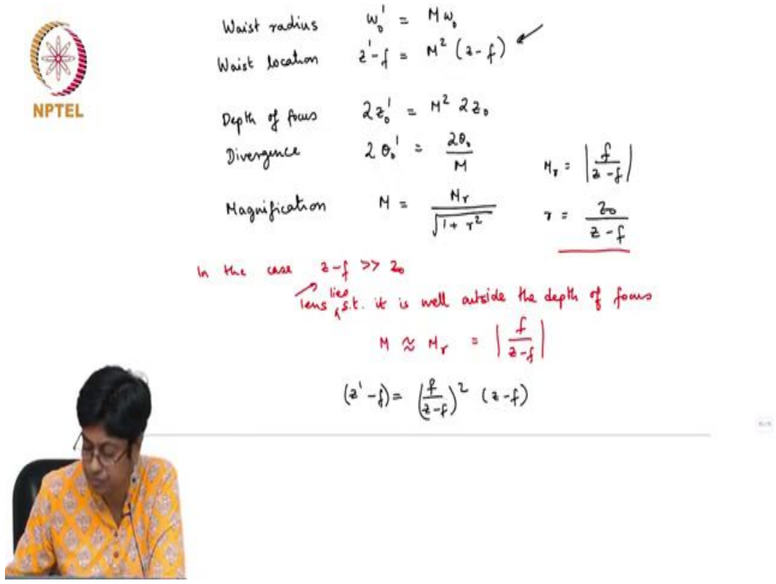


And this magnification M is not a simple parameter anymore, it is defined in terms of some other parameters, where Mr is and r is and we will see why these are important or where they arise from.

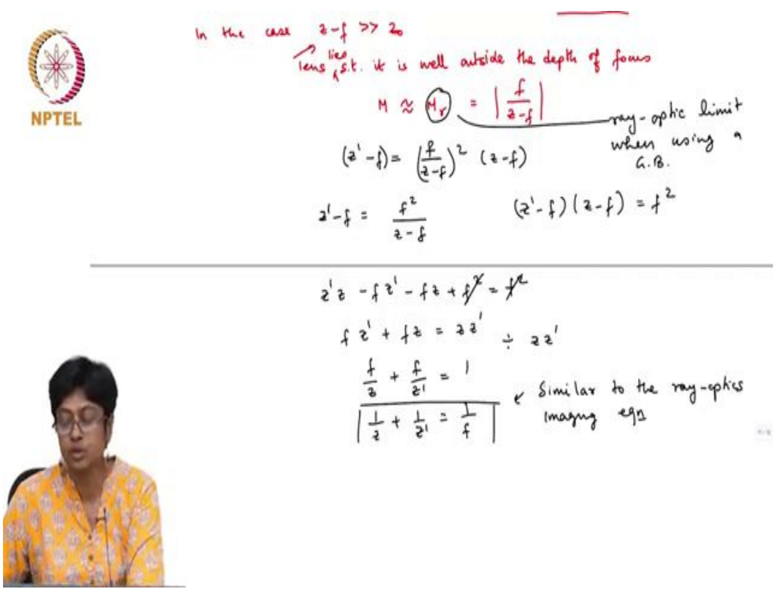
So to understand the importance of the terms that are part of this magnification right, Mr and r. Let us look at r; this expression over here. Now in the case $z-f>z_0$. What does that mean?

If I say, $z-f>z_0$, remember z is the location of the waist from the lens by taking the case where z minus f is much greater than z_0, you are saying you are looking at the case where the lens lies well beyond the Rayleigh range of this system, of this particular setup.

In other words, the lens lies well outside the depth of focus. So, this condition is you are putting the lens such that the lens lies such that it is and the depth of focus of course, is z naught or z_{20} ok.

Now the moment I make this assumption or I look at this case, I am now controlling the values of r. By saying $z-f>z_0$, I am saying r has become much smaller; or in other words I can say M now reduces to this parameter M r.

So, what does that mean for the waist location? Let us substitute that in this equation. So, I am going to take $z'-f$ I am taking this new value of magnification for this particular case. So, I will have$fz-f$ f z (Refer Slide Time: 10:53)



And let us simplify this. So, I have $z' - f = f^2/z - f$ or r I can write this as $z' - f$. So of course, this and this cancel and I have $fz^1 + fz = zz'$, or if I divide throughout by z z dash, I will have $f/z + f/z' = 1$ or $\frac{1}{z} + \frac{1}{z'} = \frac{1}{f}$. This should look somewhat similar; familiar, right? Where have you encountered an expression like this?

Student: (Refer Time: 12:02) By experience.

It is your imaging expression, right where you said 1 over object distance plus 1 over image distance is equal to 1 over focal length. So, maybe a point has not been made obvious all this while that I have been talking about Gaussian beams.

We said that we were dealing with ray optics in the beginning, and you could consider certain postulates and use those postulates and you could work and explain and design a whole set of optical systems using that.

And then we moved on to Gaussian beams because we said the source of light as the source like a laser actually generates a Gaussian beam and I cannot ignore the wave nature of the light.

Maybe it was not very obvious to you, why you could not ignore it or how by looking at ray optics we were ignoring it? We got hints of that because we said in ray optics when we focus we can imagine ideally we are focusing light to a spot of infinitesimal size, and that is a condition where you imagine or you assume lambda tends to 0.

So, with respect to the other dimensions in your system, the wavelength can be considered negligibly small and therefore, you were able to make those assumptions. Now what happens when I am dealing with the wave? There you are going to start to see that the equations that you arrived at in ray optics, they are not valid anymore when I am dealing with a Gaussian beam.

So, if you come back to this expression and your first reaction may be well, we know this 1 over object distance plus 1 over image distance is equal to 1/f. What is this telling you now? Is for a Gaussian beam this equation is true only under certain conditions?

So, if you are designing an optical system and you used $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, but your source was a Gaussian beam, the light that was traveling through your system was a Gaussian beam, you would actually be doing an incorrect design. You would be calculating and saying something has to lie at this place and when they said this place, the image is formed over here and so on and so forth.

But all of that is actually going to be incorrect if the light that you are dealing with is a Gaussian beam and not something that falls under the assumption of ray optics. So, coming back to this equation, do not forget we came to this equation using the Gaussian beam equations by assuming something. What did we say? Assume we said in this particular case let us assume the lens lies very far out of the depth of focus of the beam.

Now what happens to the Gaussian beam as you go far away from the waist? It becomes more spherical in nature, right. So, in that zone and in other words, you are saying the condition that we said right, $z-f>z_0$

In that case, this condition is satisfied and this is similar to the ray optics condition, the ray optics imaging equation. Now we arrived at this using the magnification parameter and we simplified the magnification parameter because of the assumption we took, right.

So, this Mr actually relates to a kind of ray optics limit when using the Gaussian beam or when using a Gaussian beam. So, the best or the highest magnification you can get is that either value of M is going to be smaller than this, right. Because, here we said r <11 for this case and so, we neglected it. So, M =M r is the largest value of M r that you can have. And in reality for a Gaussian beam, the magnification you are going to get if this condition is not satisfied is going to be some value less than Mrright.

So, you are going we are study the Gaussian beam. We should see that while I can consider it to be useful for imaging just like we did for ray optics, the exact imaging equations are going to be different compared to that of the ray optics case. Under certain conditions the Gaussian beam equations will reduce to that of the ray optics case and I can consider it similar when those conditions are met.

So, you actually may be the point to get from this discussion is that you have to see now the difference between ray optics and wave optics you did not consider the wave nature, you just considered that light travels in straight lines and all those postulates. And using that you arrived at working imaging conditions that satisfied a lot of cases when your dimensions were much larger than the wavelength of light.

So, we are going to in the rest of the class just look at how we can use that set of equations, I hope you will write all of those beam transformation equations down. Because, you can play around with them and under different circumstances, see how they tell us how a beam transforms and how that is useful for different applications, ok that that is really what we are going to do for the rest of the class.

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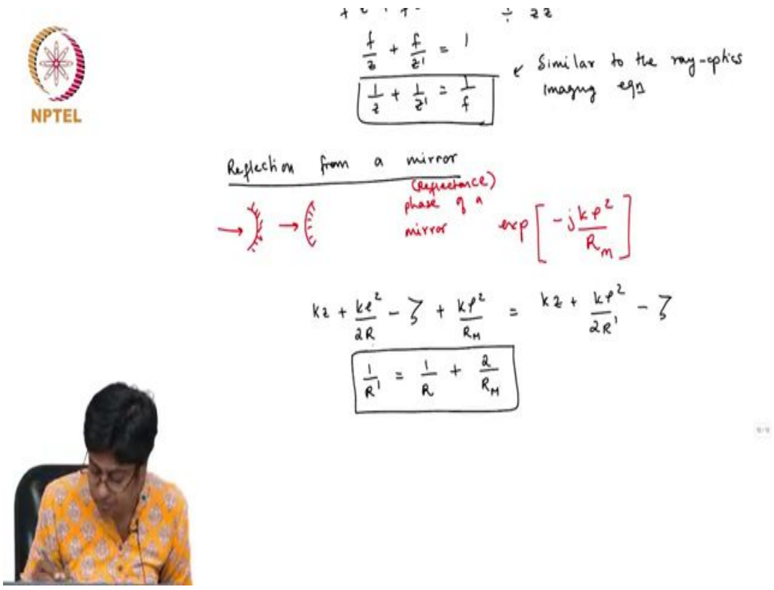

But just for the sake of completion what I will also do once is just look at reflection from a mirror.

So, just as we studied the transmittance of a lens and we saw that we made the assumption that the lens does not affect the amplitude there are no losses. So the only parameter affected by light incident on a lens was its phase and we calculated what is the phase of the lens and we have just seen how that phase affects the phase of an incident Gaussian beam.

Similarly, when I talk about a mirror; so, I can talk about a concave mirror or a convex mirror, right. In either case, again I am going to assume that if I have an beam incident on either of these, there is no absorption by the mirror or the coating of the mirror, and therefore, again the only change to the incident beam is going to be the phase.

So, the phase of the mirror itself, right just like the lens could be its transmittance could be written in terms of a phase. I can write the reflectance in terms of phase, right and it is also of course going to be an $exp(-jk\rho^2/r)$ And to make it clear this is the radius of curvature of the mirror R suffix m ok.

So, again if I take the Gaussian beam phase, where is that? $k\rho^2/2r$ Here, this is the radius of curvature of the beam incident on the mirror plus zeta minus the phase change caused by this mirror.

So, it is minus minus or going to have $k\rho^2 2r$ k rho squared R m and of course, what goes off again is a Gaussian beam with the new radius of curvature. Oh sorry, I have this. This is a negative right. So in other words, the new radius of curvature is related to the old radius of curvature and the radius of curvature of the mirror, ok.

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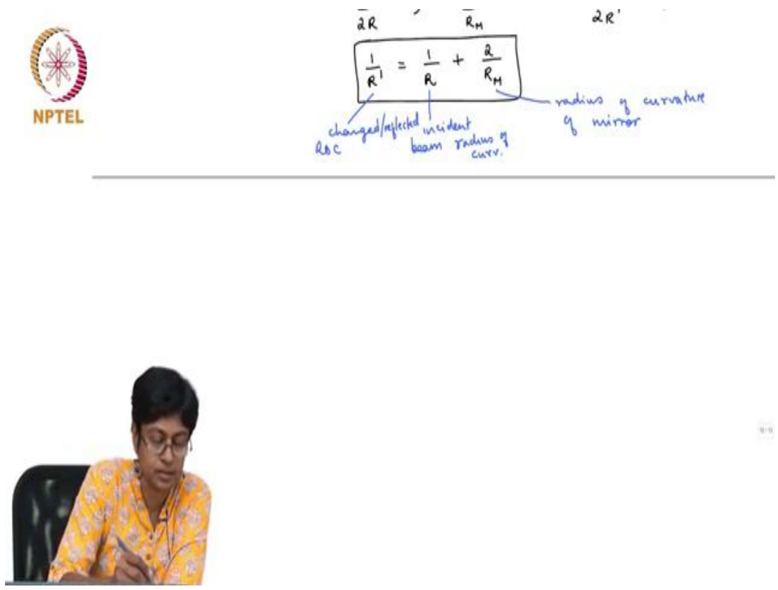

So, just to be clear this is the radius of curvature of the mirror that the Gaussian beam is incident upon, this is the incident beam curvature, the wave front curvature and this is the changed or reflected radius of curvature.

So, how does the mirror effect? It is affected by this equation, let us look at 3 different cases.

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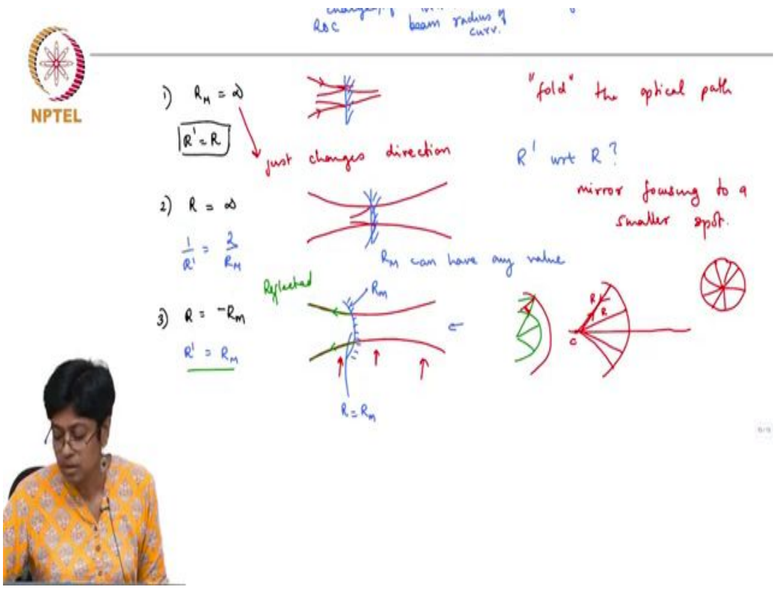

So, let us take the case where the mirror has a radius of curvature R M is infinity. What kind of mirror are we talking about then? We are talking about a planar mirror. So, if I have a mirror that has no curvature and I have a Gaussian beam incident on it, what does that mean?

Well the equation clearly tells me that the new radius of curvature is going to be equal to the earlier radius of curvature. What does that mean for the beam that is incident on the mirror? How do I draw the incident beam, what should I do for the reflected beam?

Student: Same

It is going to act exactly the same as it would if there were no mirror. So, if there were no mirrors let us say the, I have not drawn this. Let us say the beam was doing this, right. If there had been no mirror, but now because you have placed a mirror then, let us say it is normally an incident on that mirror, all that is going to happen is this.

In other words, this just changes direction and often that is why we use a mirror right. You often use a mirror to fold the word that is used to fold the optical path. Because, maybe you will need a certain distance and in a practical system, say you needed 1 meter distance. If you did not fold the path; that means, your system has to be at least a meter long. Whereas, you could use a bunch of mirrors and fold it and maybe make your system 25 centimeters long; something shorter, make it more compact.

So, we often fold the optical path, right. When you do an analysis of a design, you can always unfold it right. So, say you wanted to use Oslo to study image quality and you are not worried about whether your actual system has mirrors in it. But let us say you are confident that the mirrors will be put correctly. So, you are not worried about any errors in (Refer Time: 26:03). In Oslo why go through all the problems of working with coordinates when you do not need to work with coordinates. You can just unfold your path, if you have planar mirrors in your system and analyze a system as if it lay on a single unbent optical axis because all the mirror is doing is changing the direction of the optical axis right.

So, you can fold your system in a practical case for analysis. You might unfold and study your system, right, because a planar mirror only changes direction. I could look at the case when I say R 1 or sorry not R 1 the R is infinity. What is R? R is the incident beam radius of curvature.

So, if I say R is ∞ I am now saying let us put the mirror at a specific place in the Gaussian beam right. So, my Gaussian beam doing this standard drawing, now we are saying if I say $R = \infty$, I want to put the mirror at a particular location. What location am I putting it at? Where is $R = \infty$?

At the waist; so, I am saying put the mirror here; whatever, it may not be I have put a planar mirror could be then, I am using a curved mirror. I have not said anything about R M.; R M can have any value. But because I have located the mirror at this particular location what will happen is, 1 over R dash is going to be equal to 2/RM.

In other words, the new radius of curvature and in fact, let us look at the case where R' sorry the radius of curvature of the mirror has some value, it is not infinite. What will R dash be with R' be with respect to R?

You are now while your incident beam is incident on this mirror at the place where the beam acts like a plane wave. Now, the beam is going to become small, right. Because, the mirror is now focusing to a smaller spot, clear and I can look at a third case where I say R is equal to minus R m.

Again, I am taking a Gaussian beam, let us say that is the Gaussian beam, its radius of curvature has different values as it propagates its planar at this region; it has curvature in the other regions. I have a particular mirror with a certain radius of curvature and now I am going to find the place where the beam radius of curvature matches the mirror that I have and place the mirror in that location.

Let us say I place my concave mirror here. The radius of the beam here is R. So, R is equal to R m, this mirror has a radius of curvature R m. What does this imply? What happens if the radius of curvature of the beam and the radius of curvature of the mirror are equal? What does that happen to the beam?

Well, look at my equation we had $\frac{1}{R'} = \frac{1}{R} + \frac{2}{Rm}$. And now I am saying $\frac{1}{R} = \frac{1}{Rm}$ right. So, I have got a concave mirror over here right. So,R' becomes nothing but the new radius of curvature becomes Rm. What does that mean? It is going to retrace the exact path. So, let us look at this a little closer, let us say I had, this is part of a sphere.

We always draw our mirrors or our lenses like this. Let us say this is the center. So, if this is the center, if I draw a line from this point, this is center to any point on this, so, I have only drawn part of this circle. Each one of these is R, is not every one of these lines that I have drawn and all the lines I have not drawn from that point to this circle or part of the circle are not all normals to the circle? Right, all I am drawing is I am taking a circle and I am saying consider these spokes going out from here.

Are they not all normals to the circle? So, what happens if you have a normal to an optical element? If I have light traveling along the normal, what happens to the reflected light? It is going to always be along the normal, then reflected is also going to be along the normal, right.

Now if my mirror curvature was like this, but my beam curvature was like this; if I draw the normals to the beam, they are not their same normals for the mirror. So, now the mirror normal would have been something like this and I have an angle of incidence here and the moment I have an angle of incidence it means that this would reflect. But in this case, the normal of the wavefront which is nothing, but the rays and the normal of the mirror lie exactly one over each other, because the radii of curvature of the wave front and the mirror are the same.

So, if I were to retrace the reflection. So, what I have drawn here is the beam as it would travel if there were no mirror and now I am drawing the reflected so, let us say the reflected light is in green and the reflected light is going to come back this way. And this is exactly the condition that you set up when you are setting up a laser cavity because, if you had a laser cavity.

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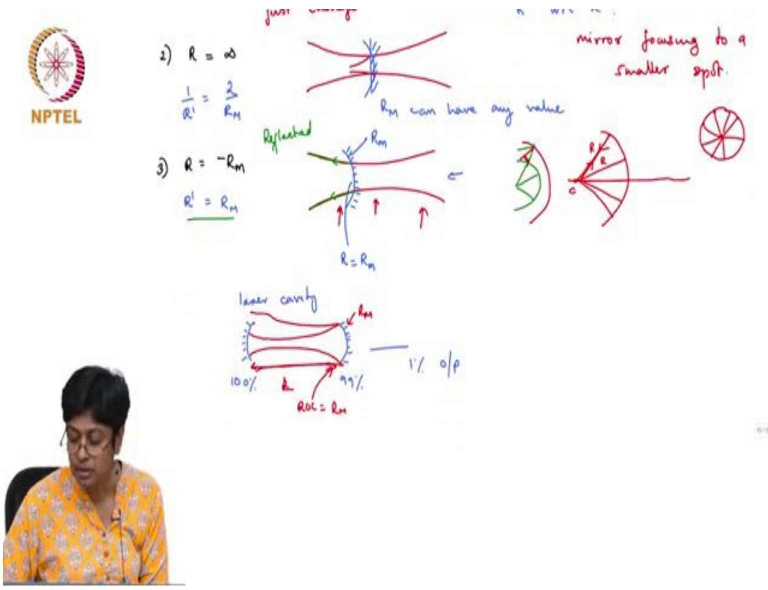

One of these mirrors is maybe 99 percent reflective, this is 100% reflective. Why is it 99 % reflective? Because [Laughter] you need an output of the laser right. So that 1 percent is your output. That is what you are using as a laser, right. And this beam is bouncing back and forth within this cavity, it must be a self-sustaining cavity.

If every time the beam bounced back and forth it grew larger and larger and larger, it would not be guided within this cavity. Eventually, it would just say it did this and then it did this. Well here even it did not make 1 round trip and it is already leaked out of that. Then whatever got reflected here would come back, but it would go on diverging and I am not storing my energy within the cavity.

So, this laser cavity is set up such that for this distance l the beam travels such that the radius of curvature of the beam over here equals the radius of curvature of the mirror. And that way the beam stays within the cavity.

So, we looked at 3 conditions, the first condition was a condition only on the mirror radius of curvature. The second 2 conditions were actually conditions on the radius of curvature of the beam with risk and that gave us a relationship with respect to the mirror radius of curvature.

So, if I put the mirror in such a place where the beam has either R equal to infinity or R equal to minus R M and in this case I am looking at concave mirrors, I will get these conditions. And you can see that in different applications in 1 application I may need to focus the beam to a tighter spot, in an application like a laser cavity I want to ensure that the beam retraces its path. Yeah, I want the beam to retrace its path and then I would choose the location of the mirror with respect to the beam's radius of curvatures such that its radii of curvature are equal.

So, understanding how the phase of a Gaussian beam is affected by an optical element and when I say optical element, I really at this stage we mean either a lens or a mirror.

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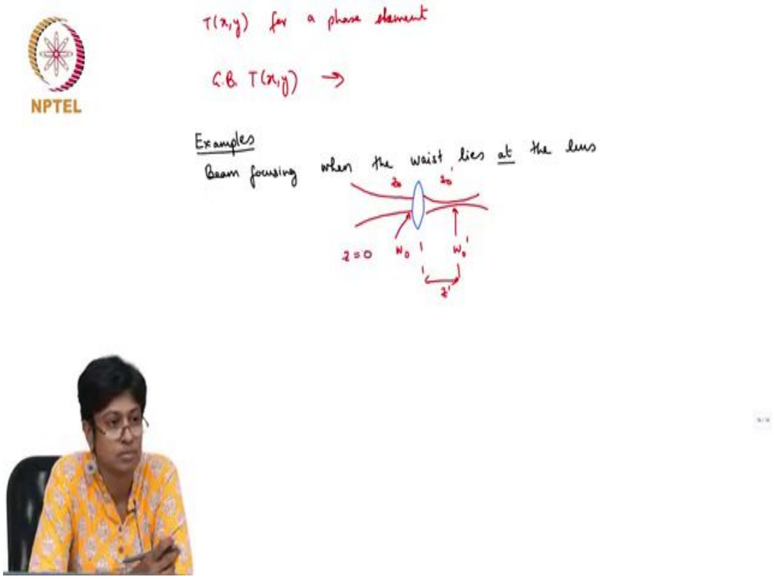

But you can go the step further and you now understand that if a Gaussian beam or any beam passes through a phase element; if you have if you have a description, so, if you have T of x y for a phase element.

So, you have an equation that tells you how the phase varies across the lateral distribution of this element, then you just need to have the Gaussian beam incident on this T of x y and the new phase will tell you how the beam has been altered or affected by this element. We are looking at mirrors and lenses, but there are a whole host of elements that you can use and thereby, manipulate the light and make it do something different, ok. And that is the whole purpose of an optical system.

If your optical system is an imaging system then lenses and mirrors are fine. You want to either focus or collect and redirect it somewhere. But if your system is trying to change a property of the light you want to make the beam twirl about the optical axis, you want to spin about the orbital axis then you need to change some other property and you will change that using the phase of that element.

We are going to look at some examples now. So, let us take the case of beam focusing when the waist lies at the lens. So that means, I have a lens and I have placed it so that this is omega naught right. This is the new omega naught dash and there is of course, az_0 before this

az_0 ' before this and this distance now is the location of the new waist from the lens because I have said the waist lies at the lens. In other words, I am saying z is 0 on this side. The location of the waist from the lens is actually at the waist.

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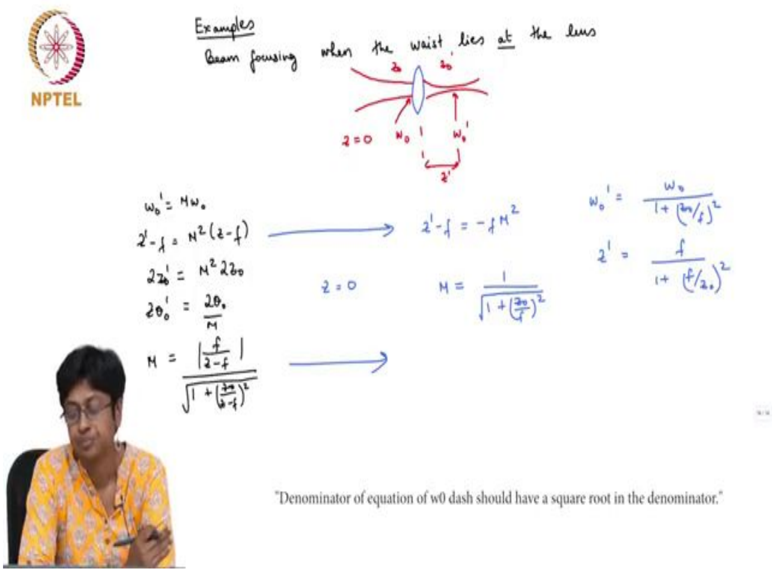

So, if I go back to my equations right, I find it easier to write them down, so, I am gonna do that. $\omega_0' = M\omega$, the location $z - f = M^2 z - f 2z_0'$=M. That is the Rayleigh range, the divergence is related to the old divergence and of course, the magnification.

But, we have taken the condition where the lens is at the waist. So, $z_0' = 0$. So, what is changing, what is important of course, is this and M. How these are going to change?. So, $zz' - f = fM^2$ and M because $z = 0$ the numerator will become plus 1 and the denominator reduces to this clear. And if I play around with these, I can go back to.

Again, I could stop at this point and say under this condition, I can reduce the equations for ω_0' and z' that are the new waist size and the new waist location. They reduced to these 2 equations over here. But it is useful to again look at another case.

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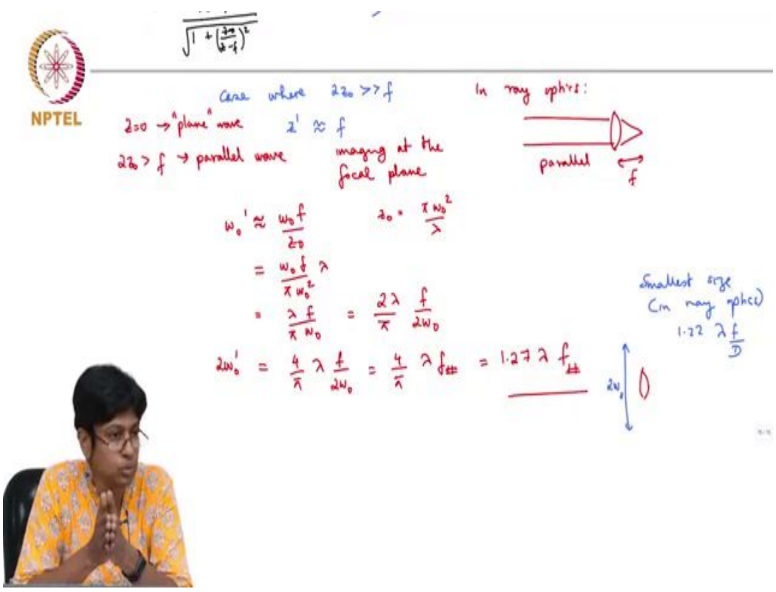

So, if we take the case where $2z_0 > f$. Remember that is the depth of focus. So, you are saying take the case where the depth of focus is larger than the focal length of the lens that you are using, ok; z dash is going to reduce to f. And again, this come should be a result similar to what we had seen in the ray optics case. Because, what does this look like, z' ? You can think of it as the imaging location and we are saying now in this particular case, the imaging location is nothing, but the focal plane.

In ray optics where, when did you get imaging at the focal plane? So, in ray optics if you had a parallel set of rays of light and a lens, then they focused at this distance. So, for a parallel set of rays, you had imaging at the focal plane and this is nothing, but imaging at the focal plane, but it is for a Gaussian beam and there are actually 2 points you have to consider here. So, do not forget this equation was arrived at by saying you have kept the lens at the waist. So, you know that at the waist is when the Gaussian beam acts like a plane wave. So, you have a plane wave so that condition of z equal to 0 is giving the plane wave similarity and 2 z naught greater than f is giving the parallel wave similarity, the wave coming from far away.

And when these 2 conditions are met by the Gaussian beam, you have imaging at the focal plane and again that is not a surprising result for us. So, you can all even use these equations that we have arrived at and bounce between the wave nature and the ray picture by setting the correct conditions in the case of the wave. Because, it has to reduce to the ray optics picture under certain conditions ok, and we can keep confirming that,. Ah you can also look at what happens ok.

So, we have looked at the position or the location, but what happens to the spot size? Note I can go back to my omega naught dash and again with this condition it is going to reduce to this, right. That was the expression for omega naught, right.

We have this expression for $\omega_0 = \omega_0'$. So, I am taking this and applying the condition because z naught is now greater than f. So I can ignore the 1 in the denominator and it reduces to this expression over here, ok. But we know that $z_0 = \pi\omega_0^2/\lambda$. So, I can write this as $\pi\omega_0^2/\lambda$ or $\lambda f/\pi\omega_0$ or if I want to write it in terms of diameters I will write see, this is $2\lambda/f/2\omega_0\pi$.

I am just writing it so in fact, let us write even this side in terms of a diameter. So, I have $4/\pi\lambda f/2\omega_0$ ok. So, I have just made sure that then both the new spot size and the old spot size are written in terms of a diameter. $f/2\omega_0$ can be considered to be the f number of this system and in fact, you might see it written like this, right. And $4/\pi$ is 1.27. So, often you will see in textbooks that they will give you the focal spot, the smaller size you can focus down to is given by this expression. And I do not know if you remember, but when we were doing ray optics we said the smallest size in ray optics was $1.22\lambda f/D$.

So, you might just see some books will tell you the smallest spot size is 1.22 lambda f by D and some books will say its $1.27\lambda f/D$ and again, this difference is arising depending upon whether the conditions are set up for ray optics or for Gaussian beams, right.

And you now know where this 1.27 is coming from for the Gaussian beam, ok. But it is a very similar, it is controlled by the focal length, by the diameter because if I say the beam diameter is this much it means the lens diameter, right. I cannot get this focal spot size if the beam is so large.

This is what I say is 2 omega naught dash oh sorry if its omega naught dash it is a plane wave. So, if this is the size of my beam and then I use a lens of this size, I am not getting this focal spot. We talk about the beam diameter that assumes the lens is at least that large. And if the lens is larger, the diameter of the lens does not matter. It is the size of the beam. So, what have we looked at this in this class? We have started by saying a phase element has a transmittance that affects phase and what we have done in this class is to look at how the phase of a Gaussian beam changes, right.

So, I gave you the equations, all the transformation equations. What are they transforming? The Gaussian beam before the lens to the Gaussian beam after the lens. And to be complete, we also looked at how a mirror will affect a Gaussian beam.

It is very similar, but what is different is I can use a curvature of the mirror to help do some particular operations. And so we looked at special cases and you can either look at cases where the mirror curvature has a special value or you place the mirror with a certain curvature at a specific location where the wave has a certain curvature. And in doing so, you can achieve a certain operation. And then we the last example we have seen is again placing the lens in the specific place and all of these even in the case of the mirror we are saying, if I place the lens or if I place the element at this location.

So the Gaussian beam in ray optics or we consider a plane wave, we never bothered where we place the element we just assume that the wave has the same property everywhere. And you can see now why a design you would do in such a system and a design you would do with Gaussian beams is going to be very different ; because you have to take the nature of the beam into account where you place that element and what happens subsequently is very linked to what was the curvature of the beam or was that beam the Rayleigh range such that you can consider you are in the ray optics limit or not, right. And you have to take all those parameters into account. So, we look at some more of these examples in the class tomorrow.