Optical Engineering
Prof. Shanti Bhattacharya
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture – 28
Transformation of a Gaussian beam

Good morning sorry about the delay. So, we are looking at Gaussian beams now. And we were looking at various properties of the Gaussian beam. And we had come to the last property that I wanted to introduce which was looking at the phase of the beam.

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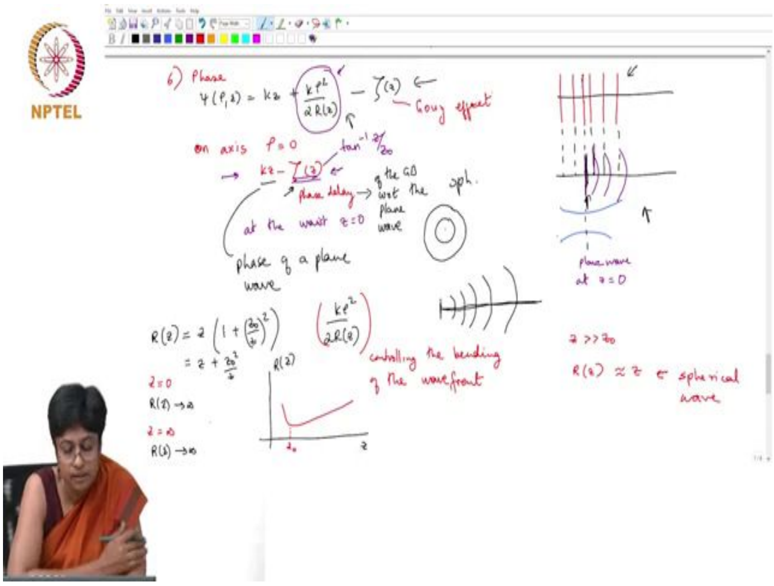

Now, the phase of any beam is important because we know that as light travels the phases change. So, propagation changes phase. And while I might think of phase changing as the beam propagates, one also has to understand that if you have a beam whose phase changes in a particular way, that is actually the very definition of that beam. So, if I say this is a Gaussian beam or if this is a (Refer Time: 01:11) Gaussian beam or this is a beam with orbital angular momentum.

How does it beam get defined as belonging to some category? It turns out actually you are defining that by looking at a way its phase changes as it propagates. So, it should be fairly clear that propagation causes a phase change. And as a beam propagates the phase is changed, but every beam does not change phase in the same way when that beam travels through a distance z.

So, you could have different beams traveling through the same medium. So, the medium has a refractive index n each beam travels a distance z; in that medium maybe even in the same seemingly same direction. And yet each beam will behave completely differently depending on the phase change required by the beam. So, you can decide if you want to say, its because its phase nature propagates in a certain way or because it propagates in that way its phase has changed ok.

So, it's really important to look at the phase of the Gaussian beam. And over here we have written out just the terms of the exponential that indicate the phase is there is actually a minus sign here right in front of this is. If you look at the expression it's exponential minus j k z minus k rho square by 2 capital R plus zeta of z ok. But the relative phase is shown correctly here because the signs are relatively correct.

Now, what we ended the last class with was, simplifying this expression it has three terms in it and we said let us start by looking at when $\rho = 0$ and since $\rho = 0$ is the $\sqrt{x^2 + y^2}$. 0 means you are looking on the axis ok. And if that term goes to 0 you are left only with $k(z) - \varsigma(z)$ but the $tan^{-1}\left(\frac{x}{z0}\right)$. So, these two terms are what control the way the phase of this Gaussian beam changes as it travels, but it's the way the phase changes on the axis ok.

So, we know if we take a spherical wave for example, we know that if this is the origin the wave fronts go out radially in all directions right. For a Gaussian beam let us say this was the beam waist and we will see how I know how to draw these images. Initially the plane front wave front is a planar wave front. And then as the beam propagates it becomes more and more spherically changed.

And what is controlling this behavior is the phase, but by putting $\rho = 0$; we were saying this is a certain phase change that's happening on the axis this term indicates that the phase change that happens to the beam is slightly different when you go off the axis. And what controls are finally, determines the nature of the shape of a wave front is going to be how the phase changes all through the beam.

So, what happens at the axis, what happens away from the axis that combines change is what gives you your resultant wave front. So, right now just looking at what happens on the axis I can say that on the axis I see that kz is nothing, but the phase of a plane waves. So, on the axis the Gaussian beam phase is delayed from that of a plane wave by an amount $\varsigma(z)$ ok.

So, this is nothing but a phase delay of the Gaussian beam with respect to the plane wave. And that is the figure that I have drawn over here in this, where the red lines are not drawn this correctly the plane waves are going to be lambda apart ok. So, its equidistant of these lines should be considered them at equidistant ok; indicating the wave fronts of a plane wave. And you can see that if this were the waist of the Gaussian beam here it's in phase, but as the wave fronts move there is an extra delay that it's going to be one increasing and that is because of this delay; that is what happens on the axis.

Now, it's useful to look at these two terms. So, we know that on the axis the phase of the wave front changes in a certain way, but what is happening to the wave front is not of course, only what is happening on the axis; it's happening also at regions away from the axis.

So, its instructive to go back then to this expression. I do not know if you remember, but we had written this out. So, we have this capital $R(z)$ that appears in the phase term and I had defined it as $z(1 + z_o/z)$

So, let us look at this expression ok. Now it appears in the phase argument as part of this term and if I wanted to understand this term, I really need to look at it a little closer and see it is rather similar to the phase change that I would get. If I had a spherical wave this term is what is controlling the bending of the wave front.

So, if I had only $kz - \varsigma(z)$ if this term did not exist, the only difference between the Gaussian beam and the plane wave would be that the Gaussian beam had a phase delay compared to the plane wave. But it would be more or less plane in nature right except the wave fronts would have had a phase delay.

What makes the Gaussian beam lie someway between the ideal plane wave and the ideal spherical wave is this term that moves it from the plane wave and gives it the bending of the beam ok. Now the radius of curvature is a function of z. So, as the wave travels the curvatures taking on a different size right, it is also a function of z_0 which is an inherent parameter of your beam how does this R of z way?

So, let us look at two different cases. Let us say I take the case when $z = 0$ $z = 0$?

Student: At the waist.

At the waist. So, at $z = 0$ what happens to my expression for $R(z)$, I can always write it like this right. So, at $z = 0$ the first term vanishes, but the second term goes to infinity. So, at $z = 0$ $R(z)$ will go to infinity right. What happens at $z = \infty$? Very far away from the waist. in this case the first term blows up whereas, the second term vanishes. So, the result is the same $R(z) \to \infty$ y ok. So, and what is $R(z) \to \infty$ inclined, what kind of beam is that?

Student: It's a plane wave.

It's a plane wave right, it means a radius of curvature is infinite its nothing, but a plane wave. So, now, you get the sense the Gaussian beam at the waist behaves like a plane wave and very far away from the waist again behaves like a plane wave and in between that has a spherical nature. So, it's very very different from that ideal mathematical plane wave or spherical wave, it's going to look at this graph when you plot $R(z) vs\ z$ right.

And if you actually plot it it's going to look something like this. So, it's if you at this z_0 right. So, what does this graph telling you below z_0 the lower you go of course, with $z \to 0$ the radius of curvature shoots up and in actually will hit infinity. As you move further and further away from the waist the radius of curvature is going to hit infinity, but of course, that is a slower process. So, you have a less of a slope there and you have the smallest radius of curvature actually at the Rayleigh range.

So, again another point to reinforce the importance of that Rayleigh range ok. If z in fact, if I look at this expression if I take $z > z_0 \neq \infty$, but greater than z_0, then what will I do? How does this term reduceR of z? How will $R(z)$ reduce? It's going to reduce back down to so, $R(z)$ would reduce back down to just z alright.

So, again if I am at z_0 that is where I have the largest curvature. If I go beyond that this starts becoming like a spherical wave ok; because I am going to have $k(\rho^2/2z)$. So, this phase delay that arises from this ς, you will often see it called the Gouy effect or the Gouy phenomenal ok. And what $\varsigma(z)$ is telling us, as the wave travels it is acquiring a phase delay compared to a plane wave. And actually the phase delay as a Gaussian beam travels from minus infinity to plus infinity is a full πr ok.

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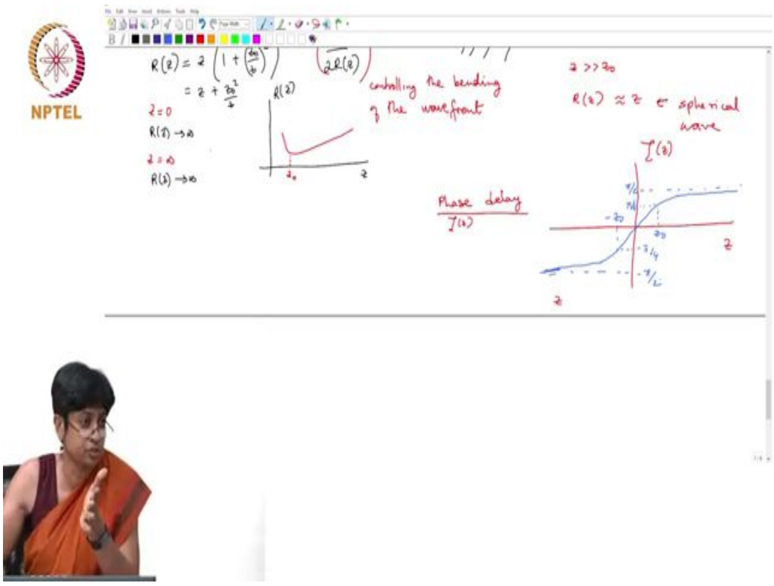

So, you can plot that phase delay verses z and what this curve is going to look like something like this. And symmetrically on the other side slightly this is this is z_0 the same distance over here $-z_0$; this is $\pi/4$ this is minus $\pi/4$ and this aesthetically tends to $\pi/2$. What does that mean? As the wave travels and if you consider it traveling from minus infinity to plus infinity, compared to a plane wave that travels that same distance it could have been delayed by a full pi radians ok.

And again another data point for the Rayleigh range over the Rayleigh range; however, the phase change. So, as the wave travels through the length of $2z_0$, its phase changes by $\pi/2$ ok. So, you can see that the rate of change of phase is also different, as the wave propagates ok. So, keep in mind these properties of a Gaussian beam ok, we have used the concepts of a plane wave and a spherical wave to understand the Gaussian beam.

And we can see that the Gaussian beam has the nature of the plane wave in some places and nature of a spherical wave in other places ok, but it is of course, is very own unique beam ok.

The other key point you should have got from this is the phase is crucial; because phase defines this beam I call it a Gaussian beam because of the way it behaves and I describe that behavior using the phase.

Now, that is one under parameter, it's not a Gaussian beam parameter, it's not always defined as part of Gaussian beam parameters; but it's a very important parameter that people use if they are working with lasers.

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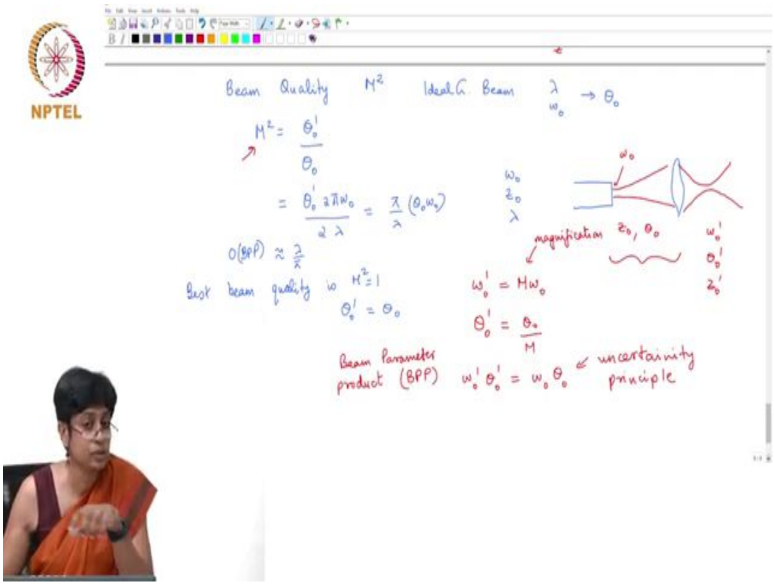

So, I want to mention it here; it's called the beam quality; it's referred to as the M^2 parameter. And if you are using a laser or buying a laser you would always look at the M^2 parameter ok.

Now, the M^2 parameters simple definition, it is the divergence of the beam emerging from the source, compared to the divergence of an ideal beam, with the same omega naught. So, if I have a beam, a Gaussian beam. So, an ideal Gaussian beam is lambda wavelength, it has it emerges with the spot size of omega naught inherent to it then is the θ_0 or the z_0 right and so, I can say this beam has a $\vee\theta$ naught.

Now, if I take another beam with say of the same wavelength and it emerges with the same omega naught. Ideally if it is the Gaussian beam it should have the same θ right, but in reality when you generate light, the quality of the beam coming up varies a lot depending upon how exactly that beam was generated ok.

Especially when you have higher power sources the nature of the method by which you generate the higher power affects the quality of the beam. After all, why am I getting out a Gaussian beam? It is a mode of the cavity and the mode of that cavity if I change some parameter or if I am using much higher power, that mode quality gets effect.

So, the way to compare is to compare the divergence of the beam you have, with an ideal Gaussian beam of same λ and same ω. So, this ratio is what we call the M^2 ratio. And I can always define it in terms of theta naught because theta naught can be defined of course, in terms of omega naught and lambda.

So, if I do that M^2 is of course, this stays the same, but I will write it as phi omega naught by lambda and often because people want to talk about diameters they may see it written like this. So, that's true . ok Now, what is the best beam quality you can have? If I say you had the best beam; what would its M^2 parameter be?

Student: 1.

1. So, the best beam quality is $M^2 = 1$. In other words you say the divergence of this new source, when compared to the other one are the same ok. So, I can never have $M^2 < 1$ and you always try to create a source that will generate an $M^2 < 1$ 1 as possible. Many sources will achieve close to 1, but again as I said if you are working with very high power sources you may be quite far away from someone.

With the Gaussian beam you have had this concept there is some inherent nature to the beam right. So, we bring referring to omega naught z_0 and always do; I mean always explicitly say λ is also playing a role in that. This means this inherent nature of the beam means and if I have a source and the Gaussian beam emerges from the source. So, this is the way then and it has omega naught; that means, this beam has a fixed z naught and a fixed θ.

Now, I could ask them that I can never change it and stick with this ω_0 forever; you can change the ω_0 Of course, why do we use lenses? You use lenses to change focus font size.

So, I could put in a lens right and do it. So, would then of course, maybe I have a lens that is slightly focused. So, it does this.

Now, in this region the beam came out with an ω_0 a z naught a theta naught, after the lens the beam will have an ω_0'; a_0' and z_0'. But they are not completely independent of what the inputs are used for ok.

So in fact, if I see the lens magnified because of magnification. So, I change the spot size by a factor M here is not this M this M is that is this M is never referred to as M there is no M and M squared its always called the M^2 parameter. So, here I am talking about magnification.

So, let us say the lens magnified and remember magnification could be less than 1. So, I could have a smaller spot size. So, ω_0' has been trans is the transform value of the spot; because of the optical system that I am using with magnification M. But if ω_0' changes in this way it automatically implies the new divergence then is the old divergence divided by the magnification. In other words ω_0' right, θ_0' is equal to ω_0 θ_0. And this if you will is like the uncertainty principle in Gaussian beams right.

Because the certain this more sure I make the spot; that means, I make it smaller and smaller and I am saying the spot is not spread there is no uncertainty in the spot. I make the spot smaller and smaller and smaller and pinpoint its location more and more and more accurately. The moment you do that the beam starts diverging more and more faster; alternatively I can decrease the divergence and; that means, the spot size is going to be very large right.

This way, I can use a 100 lenses and transform a Gaussian beam with these 100 lenses and each time yes, I get a new omega naught and I get a new theta naught, but I will never escape the initial inherent properties. It came up with an omega naught and the theta naught and that relationship that product is going to stay the same for this beam throughout no matter how I manipulate it ok. If I want to manipulate it I have to generate a beam with a different nature altogether ok.

This product is called the beam parameter product, or often this reduces to BPP. And if you look back at this expression, I can write this as $\pi/\lambda\theta_0\omega_0$. So, this M^2 is it can be written in terms of the beam parameter product. And therefore, actually you can see out of the beam parameter product is of lambda parameter because M^2 is equal to 1 ok.

So, with this you should start getting the idea that there is only. So, much control you have over the beam ok. Here I am showing you that it is a function basically of the wavelength and there is only so much I can achieve with a source of a certain wavelength. And this idea we have met before because we have looked at resolution and we have said resolution is also related to wavelength and may want to have a very very small spot size, but it is finally a function of wave.

And this idea is coming through here again in the Gaussian beam; because you say in the order of the beam parameter product is lambda by pi and that product is again related to spot size and divergence ok.

So, you can think of the M^2 if you have an M^2 value that is greater than 1, you can think of it as indicating the non Gaussian nature of that particular beam that you're Gaussian. So, if you say M squared equal to 1 you are looking at a Gaussian beam and for any other value you are seeing and 1 you know 1.1 may still be considered Gaussian. But of course, as you go to higher and higher M squared values you say the beam is going further and further away from its Gaussian nature.

And in fact, if that were the case the expression, we wrote down from the phase of the Gaussian beam; that means, that beam does not exactly follow that expression; it means there is some other term in its phase that accounts for its difference from right. So, we were not saying this is a Gaussian beam it has the same expression, but its behave in different. The very fact that a beam has an M squared greater than 1, means that its phase will be different from that of the Gaussian some other term or terms will exist it a phase.

Now, the next part that we need to move on to really is looking at how the Gaussian beam gets transformed. Because our interest is always how do I use it, ray optics is, how do I focus light down and achieve something, how do I do imaging with aberration correction we are not doing anything very different here always saying is the system that. We are working with then you cannot anymore assume ray of if you have to take the wave nature into account, but again I want to manipulate this wave with optics.

So, the next part that I need to really look at is how does a lens transform my Gaussian beam and how does a mirror transform my Gaussian beam? But before we go to that I want to actually look at when I think of waves how does any optical element transform? So, let us do that in general and then use the results we get from that to look at what happens with a Gaussian beam and optics ok. So, the next section is on transmittance.

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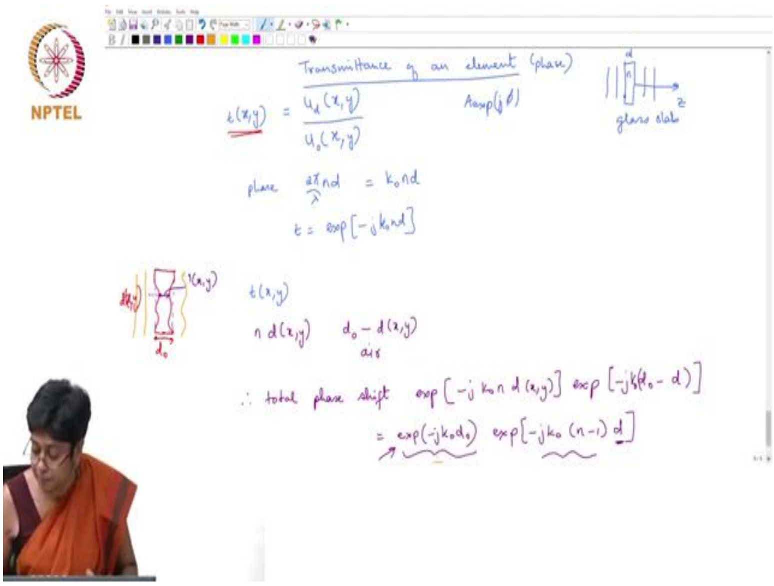

Let us just look at the transmittance of an element ok. Now transmittance we as I as we always normally assume that propagation is along the z direction. So, if you assume you have an element, let us say in this case I have a transparent slab, maybe a glass slab of thickness d naught right and refractive index n. When I say transmittance of this element what I want to find out is, if I have waves incident on this element what happens to them after that.

So, the transmittance is nothing, but the field after the element and I am just calling it d for that. The ratio of the field after the element to the field before (Refer Time: 30:15), we can make the assumption here that this glass is perfectly transparent.

So, there is no absorption of any kind and for the purpose of this discussion we will neglect any reflections that occur. So, we assume there is no change in amplitude as the wave travels through this element ok, my element right now is a glass slab of constant thickness ok. But I am going to assume no amplitude change. What does change as the wave travels through the element? The phase changes ok.

And, by taking the ratio of the fields remember the fields are always $Aexp(j\phi)$. So, enhancing amplitude does not change. So, there is some $j\phi$ before and then there is a new phase after. So, really I am just taking the new phase after as my transmittance ok.

So, I could consider this to be a phase element where there is no change in amplitude. Now if I say that its n refractive index the phase change is going to be $2pi/\lambda$ *n*t, or that $2\pi/\lambda$ we can say k_0nd.

So, the transmittance of this very simple element then is nothing but the exponentially of ok. So, let us go to the next element that was simple you see. Let us say I have an element and this element has variable thickness something like this ok. So, no point is larger than d naught, d naught is the widest it ever is at any point, but it interacts in some places the thickness is less than d naught.

So, I can actually say the thickness here is a function of x and y x and y is the plane perpendicular to the direction of propagation. And so, if I am taking into account the phase change I then here you did not really matter. I wrote $t(x,y)$, but this was the same transmittance everywhere because the thickness was the same everywhere therefore, the phase change was the same ok. Now I cannot ignore the x and y. I really have to take into account the transmittance at every point due to the different phase at every point ok.

Now, let us take some point over here, let us take this point let us see the thickness here is $d(x,y)$. So, what is a ray or the part of the wave that is traveling to that particular section of the element, what is it seeing in terms of the phase change? Well, it's traveling through two different media right, it travels through a thickness d of x y a medium of refractive index n, but in that same line it also travels a thickness d naught minus d of x y through air right.

And that happens at every position of this element. So, the total phase shift when is the phase shift due to so, minus j k naught n d of x y and the phase shift. So, it's a product of the phase shift due to passage through the glass slab and then also through this; I am just writing it as d_0 - d ok, it is d which is a function of x and y ok.

So, if I club related terms together here, I can pull out this $exp(jk_0 d_0)$. And what we found in the remaining terms is $n - 1d$. And if you want to do this at every point, in the slab right you are going to get the same expression of course, with this value or this term d of x y representing the thickness of the slab at that particular position.

So, you have two terms here, product of these two terms, but this actually is a constant and it appears everywhere. So, in principle if I was trying to find out the effects of the relative phase differences, I could actually leave this part out because it appears everywhere what is actually causing the change is this variation. Because some parts of the beam see a thickness of some d x of y of x and y and at other places, it's seeing a different thickness (Refer Time: 36:27) and this is what is going to change the wave front.

So, before this, if I had a plane wave you would have had waves with this wave front coming. And here after this it's going to do something like this and that change in curvature is a reason because of the variation in that thickness. And is a reason because of this; this is like a d c shift everything was shifted and then this curvature.

So, in order to get the shape, I can leave out that dc shift; in order to get the actual phase I have to include everything ok. So, this is what happens, when I have an element with some arbitrary or random variation of things.

Now, why are I doing it step by step like this? Because of course, finally, we are not usually using elements of random thickness, what we are using will be an element of some known shape like a length.

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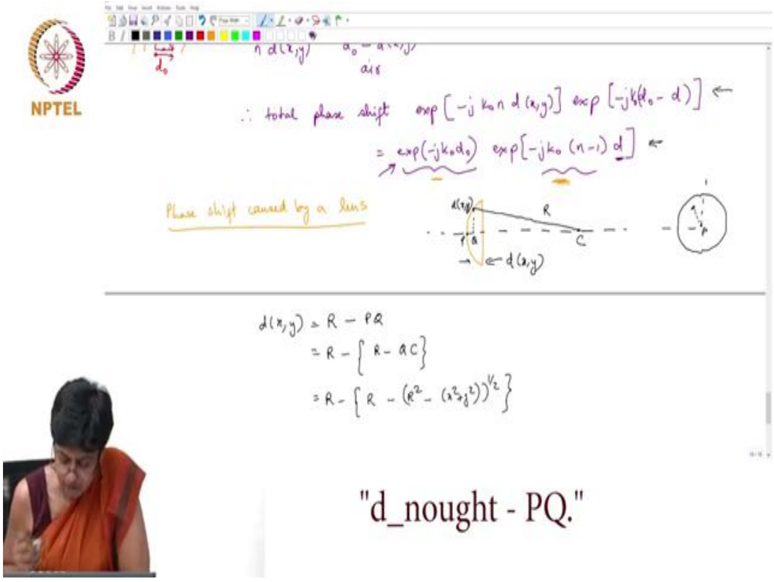

So, I want to know what is the phase shift caused by a lens? And again just to make it simple we will take a Plano convex lens ok, just to make it simple ok. So, its Plano convex its a spherical lens. So, let us say I am considering this point, its goes out to this which is a center of curvature, this is the radius. And if we drop a perpendicular to this I call this point P and this point Q.

Now, again I have varying thickness of d x y across this lens right, it's thickest at the center and symmetrically it reduces in thickness as you go to the edge. So, I want to now find out an expression for d of x y and then substitute that into this image and that will give me the transmittance of a lens right. So, how do I find out an expression for d of x y? So, I can use some simple geometry over here, we have used a Plano convex lens; that means that this is spherical in nature right.

So, d of x y the thickness anywhere is going to be this distance P C which is nothing but the radius of curvature R minus P Q. So, let us say I am looking at the thickness and at this particular yes ok, but what is P Q? P Q can be calculated how?.

It can be calculated by R - QC and to get an idea of how we do that? We have been looking at this image from the side, if I looked at it you know directly this is the optical axis this is the point P, Q lies behind it; I cannot draw it on the optical axis behind it and that point that we are measuring d act well maybe that point is somewhere over here.

So, I want this distance actually right. So, in other words I can write this is R minus R minus what will it be? Now I have a right angle triangle, but it's a tilted right angle triangle that does not matter. Move it back and just treat it as a right angle triangle right. So, I am going to write this as R squared minus x squared plus y squared to the power half its d naught yes go back and look at the figure we had over.

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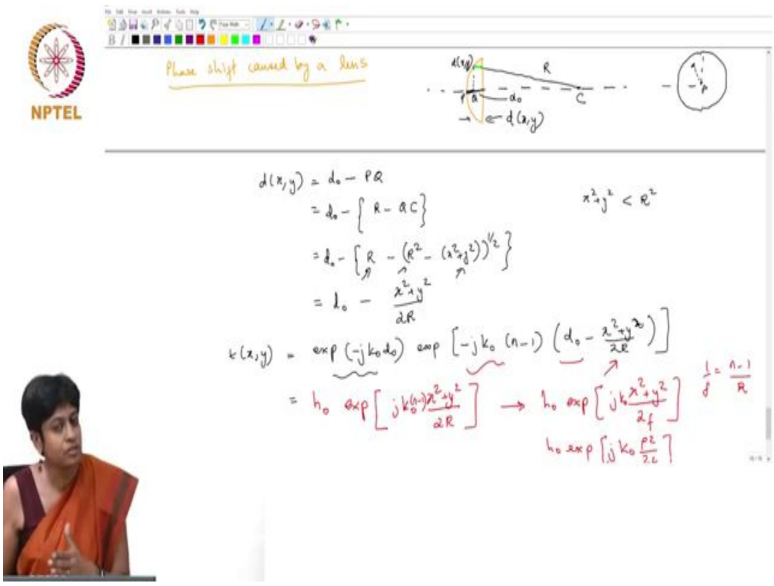

In order to find out the thickness anywhere we took the maximum thickness minus the distance travelled in air or to find the distance traveling and to the maximum thickness minus thickness of the glass apt at that point. So, we are doing exactly the same thing here except now rather than saying we have an arbitrary radiation in thickness we have a specific variation. So, at thickest point the lens has its thickness d naught and with d naught and then if we subtract P Q that may give us this. So, at that position is just the thickness of the glass ok.

So, if I simplify and write this as well as need to account for the fact that $x^2 + y^2 < R$ can be less than R squared why? Because if you think about it the curvature of the lens, it can have a fairly large radius of curvature compared to the lateral sides of it and that is basically what this is saying that your lens is not; it's not as large as the radius of curvature.

So, it's not at all strange to assume this is ok, but if you make this assumption this is then reduced very simply ok. Because this R^2 will come out and get subtracted from this one here and then you have left with this divided by 2 the power half ok.

Now, I can put this into this expression over here, what was that expression? That is the transmittance of a glass plane whose thickness varied according to that d small d and now I am doing exactly the same thing except the same thing the thickness variation has this particular form. So, if I go back and write t of x y this is the transmittance of the lens is $exp(-jk_0d_0)$; $exp(-jk_0d_0 - 1d_0 - (x^2 + y^2)/2r)$.

So, again you see for the lens this is a constant term and. So, this part is also a constant. So, I can write all the constants as sum h naught ok, what are you left with? You are left with the part of the exponential that will contain, because that varies depending on varying R on the lens in other words varying the x and y. So, you will have $-jk_0d_0 - (x^2 + y^2)/2r$ everything else goes into the constant n minus 1 right n minus 1.

So in fact, I can write this as $h_0exp(jx^2 + y^2)$ h from $k_0/2f$; if I say that the focal length of this lens is $n - 1/f_1/f$ right. And you will often see this expression written like this or written $jk_0\rho^2/2f$.

So, in today's class we finished looking at the properties of the Gaussian beam. We saw that the inherent nature of your ideas or concepts of the Rayleigh range z naught should have got reinforced. Because we saw that it very much a part of the inherent nature of the Gaussian beam $\omega_0\lambda\,\theta z_0$ These are related. And for a particular beam it means actual you need to have certain values for this and optics cannot change; may be able to change one parameter and that forces the other parameters to change in a certain way.

We then looked at the M^2 parameter which gave us an idea of how to quantify how Gaussian a beam is? So, I want a Gaussian beam. A Gaussian beam is an easy beam to work with; it gives a good focus spot, but not all sources are very generated Gaussian beams. And the M squared of the parameter allowed us a means of quantifying the Gaussian how Gaussian a beam is.

Then we want to move on to looking at how a Gaussian beam changes or transforms when it interacts with lenses and mirrors. But in order to do that we wanted to see how any beam transforms; because up till now we have been looking at rays not waves and when you look at waves. And I talked about its interaction with an optical element whose phase is going to change. So, will my define this parameter the transmittance for a phase element. And then we went step by step as to how you will calculate the phase of an unknown element like a lens.

We have calculated the transmittance now arrived at the transmittance for a Plano convex lens. But you will see if you do it for any lens it will reduce down to this form where it is a function of the focal length of the lens. So, I will start off in the next class taking into account how the wave and how a Gaussian beam actually changes when it goes through (Refer Time: 48:43) ok.