Optical Engineering
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Lecture – 10
Ray Tracing Matrix – Part1

Good morning. So, today we continue with Ray Tracing. In the last class, we looked at tracing some specific rays namely the chief ray and the axial ray and we reinforced our understanding of the definitions of aperture stop, entrance pupil, exit pupil by tracing those specific rays through the system. We also had looked at some of the techniques that we as the Optical Engineers or a software like OSLO uses in order to trace a ray through a system. So, we are going to continue along these lines.

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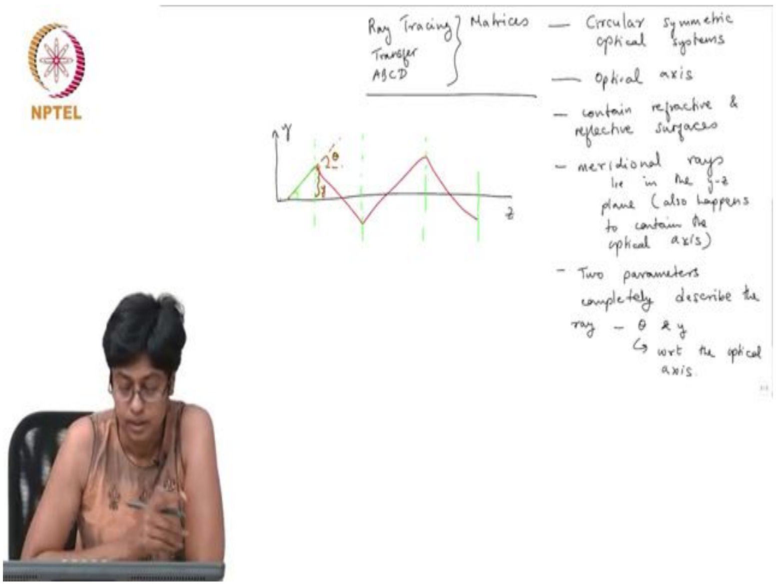
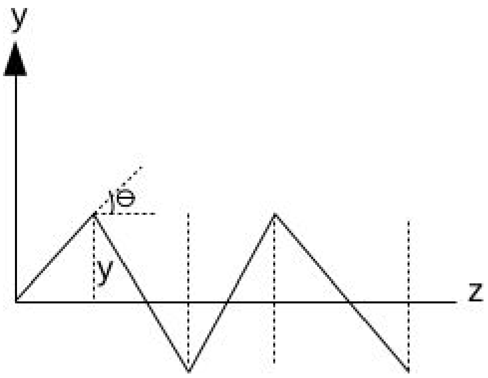


And today we are going to look at what are called ray tracing matrices ok. They are also sometimes called transfer matrices, sorry or ABCD matrices ok. So, we use this technique the ray tracing matrix when we are dealing with systems optical systems and let us see let us be a little more specific we use this when we are dealing with circular symmetric optical systems. Of course, they all will have the optical axis which is the center of symmetry.

We are considering systems that could have a number of elements in them. So, they contain refractive as well as reflective surfaces and we are going to use this technique to trace what we call meridional rays ok. What do I mean by that? Let us say I have this as my optical system or rather this is the optical axis I am talking about rays that lie in this plane ok. What is this plane? The ray is generally traveling in the z direction or in other words the z axis is the optical axis.



And, the ray is I have shown it goes up and down clearly, it has met an interface and therefore, it has bent, but all of this is happening in this plane let us say in this case I call this the y z plane. So, I am talking about rays traveling in a plane this plane contains the optical axis, but I am not talking about rays that get pushed out of this plane you. So, these are meridional rays that lie in the y-z plane and this also happens to contain the optical axis.

You can have rays called skew rays and as the name suggests they do not travel in a plane, but they travel in a skewed fashion. So, you could imagine if you had a hollow cylinder and you sent light into that hollow cylinder and it bounced around and maybe it went to this side and then it went up here and then it went down here. It is no longer in one plane you could not trace, put one plane and capture the entire path of this ray as it traveled through this hollow cylinder right.

You need not use this technique to trace those kinds of rays of course, in an optical system you may have rays that are traveling it depending on your optical system. You may have skew rays in it, but this technique would not be used in that case ok.

So, we are dealing with meridional rays that lie in one plane and the plane can contain the optical axis. Is there only one plane containing the optical axis? There are an infinite number of planes right. So, we are taking one such plane and looking at how the rays travel in that plane. Of course, I can apply this technique to each plane when I will trace rays in that particular plane and those are meridional rays too. In this case I have just defined the y z plane as my plane of interest ok.

Now, the rays, so I have drawn this particular image with these rays bouncing and basically; that means, there is some interface at these points, but the point to notice that at any instant I can specify this ray with just two parameters ok. What are those two parameters? What are?

Student: (Refer Time: 05:57)

The angle, the slope with respect to the optical axis. So, when you say angle you have to be very clear, in an optical system it is always the angle with respect to the optical axis. I might when doing my calculations take into account angles with respect to normal, but even there finally, I reduce that to angle with respect to the optical axis ok.

So, it is the angle with respect to the optical axis. So, in this case for example, here this would be the angle with respect to the optical axis. At this point if I were to specify this ray at this point this is its angle with respect to the optical axis, this is its height with respect to the optical axis. So, if I am looking at this point this is the angle with respect to the optical axis and this is the height with respect to the optical axis. So, the two parameters completely describe the ray theta and y and this theta is with respect to the optical axis of the system ok.

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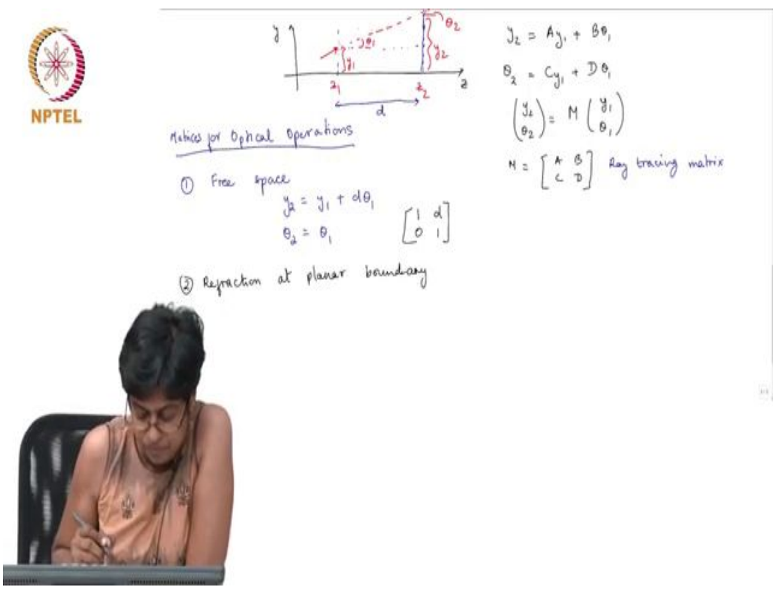
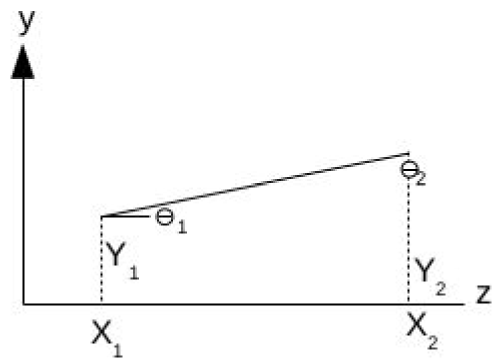




So, let me redraw this diagram. So, this is the optical axis, this is y and let us say I am interested at two different points on the optical axis. So, I have a ray that is coming in here it has height y_1 it is angle is θ_1 and at some other distance z this ray is now traveling and it has now assumed that is a straight line. So, over here it has continued and at this interface maybe something happens. So, this is θ_2 and this is y_2 and this is at θ_2.

So, I know the information of the ray at z_1 it is θ_1 y_1 and then it travels through some optical element and I mean even air is an optical element because just traveling through free space changes y. It means it does not change the angle the beam is traveling at the ray is travelling at, but it does change the height. So, I will consider traveling through a homogeneous medium as also affecting the ray. When do we say the rays affected? If θ and y are changed or either one of them is changed the ray has been affected and we will consider that as an optical element. You should be familiar with this idea because even in OSLO traveling through a medium is a surface in OSLO right to change y or θ you put an interface and something happens ok.

So, our goal in ray tracing is to find out y_2, the new height and the new slope given that you know the old height and the old slope. So, I will say there is some relationship and I am able to do this because I say the relationship is linear and similarly θ_2 will be again functions of the old height and the old angle and now it should be clear where this matrix formulation is coming from because I will say we want to find out y_2 and θ_2 in terms of a matrix M and the original parameters what is this matrix M? M is nothing, but this A B C D matrix or it is called the ray tracing matrix.

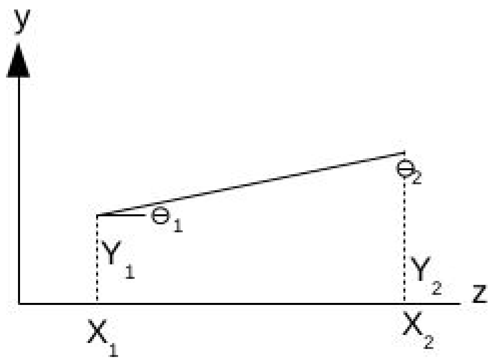
$$y_2 = Ay_1 + B\theta_1$$

$$\theta_2 = Cy_1 + D\theta_1$$

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = M \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

Now, again the convention that I am using here is that y_2 is in the first row and θ_2 is in the lower row; different books, different people use different conventions. They may reverse this they may put y_2 is a θ_1 plus b y_1. So, you need to know what the convention is before you blindly take an A B C D matrix and use it somewhere right. I would say stick to this convention in this course, but if you are looking or reading up something somewhere else do not just blindly look at what is being given to you. Check the convention they are using because it could be some variation of this ok.

So, how do we use this? So, the easiest way is to start looking at ray transfer matrices for different optical operations ok.



So, let us start with the simplest optical operation and that is traveling through free space. So, matrices for optical operations and the first one we are going to do is free space. So, for free space I can just use the figure that I have directly above. When between z_1 and z_2 is nothing, but free space what do I expect the change to be in y_2 and θ_2? Now, let us say the distance between these two planes is d, what do I expect the difference to be? What could I see the new y_2 will be? It is nothing, but Y1.

Student: d theta.

$$y_2 = y_1 + d\theta_1$$

$$\theta_2 = \theta_1 \qquad \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$d\theta$ or let us be explicit use the terms we have used. So, it is d theta 1. The new height is nothing, but the old height plus the change in height that has come because of this extra travel. On the other hand, what has happened to the angle? This is free space; the angle does not change at all. The ray continues to travel along the direction it was traveling if there is nothing to cause a change in direction. So, I know that θ_2 is going to be equal to θ_1. If I write out my matrix now my ray tracing matrix now for free space A is 1 B is the distance traveled, C is 0 and D is 1; capital D is 1. So, this is a ray transfer matrix for a ray that has traveled through a thickness d of free space right ok. Let us do another one.

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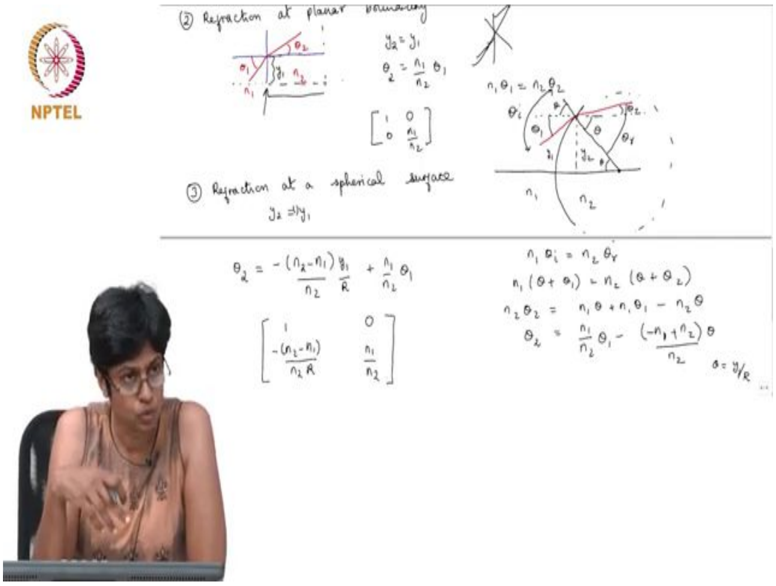
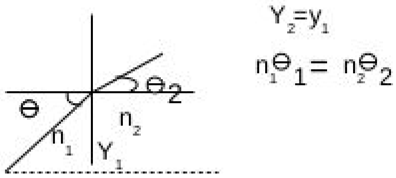


Refraction at plannar boundary



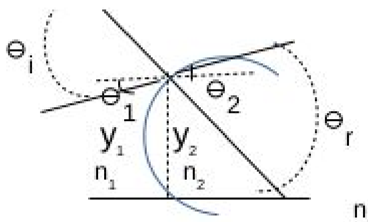
Let us say we do refraction and we will first take up refraction at a planar interface or planar boundary. Always considering paraxial rays so, if I have a ray that is incident with some angle theta1, it is getting refracted at angle θ_2. It is n_1 before this it is n_2 before this and its height is y let us say it is y1 to start with. How does the height of this ray change as it crosses the interface? There is no height change right at the interface; we are only looking at what happens at the interface.

Of course, if I look at the ray at some other distance d there is a height change, but now the matrix that we are trying to arrive at is not to do with the travel through this distance, it is only what happens at this interface. At this interface the height before the interface was y1 and the height after is y2, but they are equal because clearly at the interface it is not that the incident ray does this and the exiting ray does this. This is not what happens right obviously, this is not what happens.

So, y2 is directly equal to y1 here. But, unlike the previous case the angle is going to change because you have refraction at the surface. We are talking about paraxial rays so, I can write my Snell's law as n_1 θ_1 is equal to n_2 θ_2; in other words, θ_2 is n_1 by n_2 θ_1 my ray transform matrix in this case is 1 0 0 n_1 by n_2.

Let us do a third one refraction at a spherical surface. Now, we actually did this derivation in part last time arriving at this equation, but let me just draw it again anyway. That is your optical axis, that is your spherical curved surface right. If we were to continue you would get a circle, the center of that circle is over here. So, if I draw a line out this is going to be normal to the interface because it is coming from the center of that circle ok.

Refraction at plannar boundary



$n_1\Theta_1 = n_2\Theta_2$

$$n_1(\theta + \theta_1) = n_2(\theta + \theta_2)$$

$$n_2\theta_2 = n_1\theta + n_1\theta_1 - n_2\theta$$

$$\theta_2 = \frac{n_1}{n_2}\theta_1 - \frac{(-n_1 + n_2)}{n_2}\theta$$

$$\theta_2 = -\frac{(n_2 - n_1)}{n_2}\frac{y_1}{R} + \frac{n_1}{n_2}\theta_1$$

$$\begin{bmatrix} 1 & 0 \\ \frac{(n_2 - n_1)}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$$

And, then I have an incident ray, I have a refracted ray you can draw my optical axis here again. What are all the angles that we have now? Well, we have this angle θ which is the same as this angle theta with respect to the optical axis. I have θ_1 here and θ_2 here, but I want to use Snell's law at the surface. So, this angle is the incident angle and this angle is the reflected angle right. It is surface with refractive index n_1 here and n_2 here.

And, again what is the height of the ray? Well, it is y 1 before the interface and it is y 2 after the interface because again we are finding the ray transfer matrix for refraction at the curved surface, not after any propagation. So, at that point when the ray hits the interface at that point of hitting the interface the angle changes after the interface, but the height does not change. So, we are again going to write this condition because this is true at this surface.

We need now from this figure to figure out an expression for θ_2 in terms of θ_1 right. So, let me do that here. We start with Snell's law. So, that is n_1 θ_i is equal to n_2 θ_r which is nothing but theta plus θ_1 theta plus θ_2 and n_2 θ_2 is going to be n_1 θ plus n_1 theta 1 minus n_2 θ . So, I can write this and in fact, I am interested in θ_2 .

Now, I can of course just transpose and write this equation over here saying I have θ_1 in terms of θ_1 , but I am going to make change one thing just to put it into parameters which are more measurable. So, I will not use theta directly, but I will say theta is y by R because remember this is theta and since y 1 is equal to y2 I can also go ahead and say this is y 1 by R, yes.

So, now, I will say I have θ_2 in terms of θ_1 . What is the equation I have? I have minus n_2 minus n_1 by n_2 y 1 by R plus n_1 by n_2 θ_1 . And, again I can write out the ABCD matrix for this. In the first case, it is 1 the coefficient here is 1 right and below that the coefficient of y 1 for θ_2 will be minus n_2 minus n_1 by n_2 R there is no θ_1 term in the height. So, there is a 0 there, but θ_1 does appear in the θ_2 expression. So, I have n_1 by n_2 here. So, this is the ABCD matrix for refraction at an interface ok.

So, fairly straightforward I think right, on any doubts or any questions? No, ok. So, instead of me working out the next one, I think it makes sense if you work out the next one.

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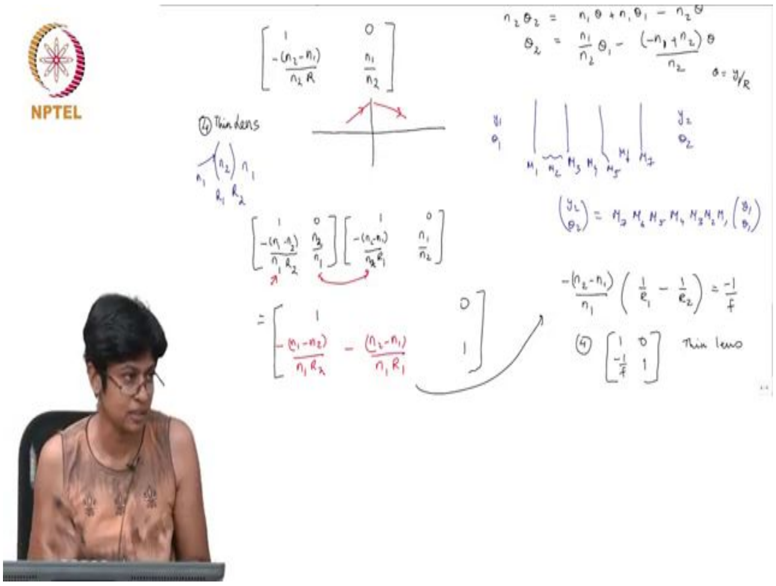
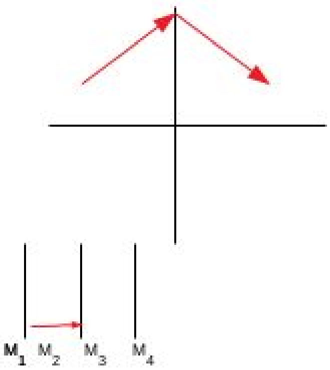


Thin lens



$$\begin{bmatrix} 1 & 0 \\ \frac{-(n_1-n_2)}{n_1 n_2} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-(n_1-n_2)}{n_1 n_2} & \frac{n_2}{n_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \left(\frac{-(n_1-n_2)}{n_1 R_2}\right) - \frac{-(n_2-n_1)}{n_1 R_1} & 1 \end{bmatrix}$$

And I want you to work out now for a lens. So, what is the ABCD matrix for the lens and let us make it really simple. So, I am going to say a thin lens. So, we are talking about paraxial rays hitting a thin lens ok. So, just to allow you to recollect that the way I would draw a thin lens under paraxial optics regime is I would just draw it as a straight light like this right ok, but that does not mean you will apply Snell's law here right. What are you going to do to calculate the lens, what do you think the power of this matrix method did? I have done three cases and we found one matrix and then we found another matrix and then we found another matrix, how do I trace a ray finally? How do I use these different matrices to trace a ray through a system?

Student: Cascading.

You are going to cascade the matrices right. So, if I have ten optical operations and even traveling through free space or traveling through a homogeneous medium will be considered an operation I will have 10 different ABCD matrices and I will cascade them. How will I cascade them? When let us say I had the first operation here the next here the next here the next here this it is. So, this is M 1 this is M 2 this is M 3 traveling through this is M 4, this is M 5, this is M 6, this is M 7.

In order, I know the height of the ray and the angle of the ray before the system and I want to find out those parameters after the system. A simple cascade operation is going to give me this, but I have to be very careful when I do the cascading because the first optical operation seen by the ray is that given by M 1. So, when I write it out mathematically I am going to write out M 7, M 6, M 5, M 4 right.

So, now, I am asking you to find out the ABCD matrix for a lens a thin lens you can use a cascade operation to do that. What are you going to cascade? What matrices will you cascade? You do not have much choice. I am not. I have not given you 7 or 10 matrices yet we have done just 3 all right clearly some combination of those 3 has to be the answer. So, out of those three which ones will you use in order to arrive at a thin lens formula?

Student: Spherical.

Sorry.

Student: Spherical.

Exactly, you will use the last one we did refraction at a spherical surface because a thin lens a rays going to encounter first this surface radius of curvature R_1 and then it is going to encounter this surface of radius of curvature R_2. It has refractive index n_1 before it has refractive index n_2 before it and let us say it has refractive index n_1 again after it ok. I have drawn it like this and you might say, but there is a distance in between well that could be another exercise to say take into account the fact there is some thickness to the lens, but I am making it simpler for you and saying let us assume the distance d is 0 where does that help us it means that I know y 1 and y 2 I will consider them the same.

The ray height after the lens will be the ray height before the lens because it has not travelled through any thickness of the lens ok. So, then it is a simple cascade of these two, it is actually the same matrix, but I have to change the parameters to suit these conditions. So, can you do that and arrive at the answer for me? So, what are they? So, the a term is 1 into 1 plus 0 into this it is nothing, but unity; the next parameter b is 1 into 0 and 0 into this; so it is 0. So, what is of interest now is the C parameter right and that ends up being the sum of this term here this term plus the product of this and this right.

So, if I write that out I will get minus n_1 minus n_2 by n_1 R 2 plus and there is a negative sign there. So, it is going to be n_2 minus n_1, n_2 and n_2 cancel out. So, I have n_1 R_1 over here may erase this. And, then finally, the D parameter is the product of these two ratios of refractive indices and it is you left with one that one comes about because you had refractive index n_1 on either side of the surfaces. If I had n_1, n_2, n 3 I would not have got that 1 over there right ok.

Now, if I look at this term here that has not simplified the term in red I can simplify this term and it should look familiar after you simplified. So, I can take this term and I could write it as minus n_2 minus n_1 divided by n_1 that is common to both the terms and I have 1 by R_1 minus 1 by R_2. It is not surprising what this represents, it is nothing, but one over the focal length of this lens and that was what you set out to prove, right.

So, now, you know that if your lens is not a thin lens it is very easy for you now to arrive at that equation if. Do you remember how we arrived at this equation the first time round? I drew out that whole figure we traced a ray, then we said this is where the object is if the object is here this is where we assumed their images, then we use this is the image point, this image point as the object for the next surface. And, then we calculated and said using this image point as an object for the second surface here is where the second image and final images.

I mean just those two surfaces and it took quite a bit of time and effort and it is easy to make a mistake somewhere and you can see now with this matrix method is so much easier in fact, why two surfaces? I could add thickness so, I forget the thin lens we can make it a thick lens plus I could say then after the first thick lens it travels a certain distance d then I just insert that matrix and then it encounters another lens and I can either just write it in terms of.

So, I can calculate what happens when I encounter a lens using this whole formulation that we just did or I could say the ABCD matrix for a lens for a thin lens is nothing, but 1 0 minus 1 by f sorry 1. This is the ABCD formulation for a thin lens. You should be able to see now how much easier it is how much less error prone it is to just use these matrices and trace any meridional ray through the system and should be clear why it has to be meridional, why it has to lie in the plane because the way we are defining those angles it will not work if the angle is now moving into another plane right. I can use this two-dimensional definition and trace a ray very easily ok.

So, we have looked at four different or what was that actually this was the fifth one right well we will call this 4 b because both are thin lenses. So, one we have calculated and written it out in terms of n_1 n_2 R_1 R_2, but that is the same as writing it in terms of the focal length of the lens.