Transmission lines and electromagnetic waves
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Lecture – 08
Transmission Lines with Losses

We will be beginning with the model of the Transmission Line with some Losses included in the transmission line in the series and in the parallel path ok. So, I will start with the equivalent circuit for the unit cell of the transmission line ok.

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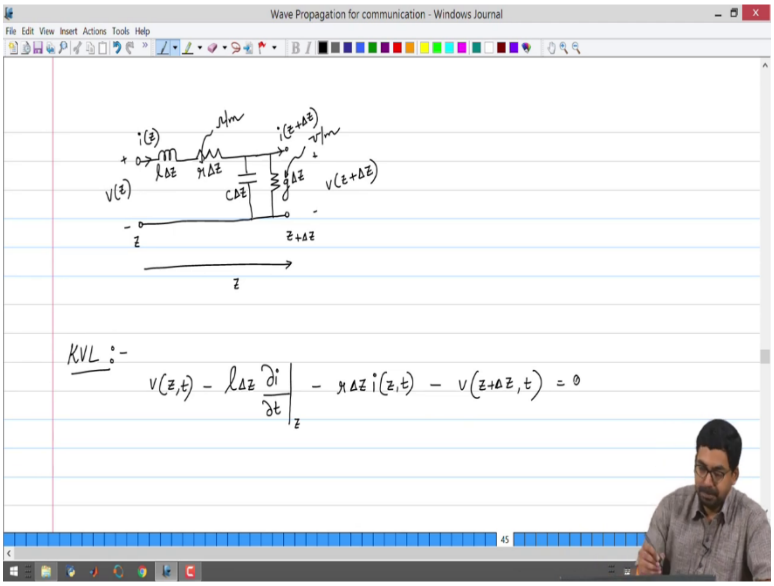

So, I have a series inductor just as we had seen in the prior classes ok and we had a resistor in the series path ok. In the prior lectures in the lossless transformation line, we had connected a parallel capacitor in the model. Assuming that there is a conductive path across the capacitor, a conductor is added in the parallel path also.

So, let the let the inductor be lΔz where Δz is the space step or the dimension of the transmission line unit cell. So, this dimension it is going to be z, the position coordinate on the left hand side can be z and the position coordinate on the right hand side can be z + Δ z alright and the series resistor is denoted by r$*$ Δ z the unit of small r will be in ohms per meter. The parallel capacitor has we had seen earlier is c $*$ Δ z. And to make the calculations simpler instead of considering the parallel arm resistor, we consider this to be a conductor. So, that we can add and subtract I mean we can add the impedances in the parallel arm when we are applying some circuit loss.

So, the conductance it is given by g Δ z. g is in mho or Siemens per meter, r is in ohm per meter. The voltage on the input side is denoted by voltage at position z. The voltage on the right hand side is denoted by voltage at position z + Δ z. The current entering this section on the left hand side is marked by i(z) and the current exiting this section, it is going to be i(z + Δ z).

So, now this becomes the model of the transmission line with non idealities included for the inductor and the capacitor. So, there is a series resistance for the inductor and a parallel conductance for the capacitor alright. And we can begin by applying the circuit loss that we have seen before. You can start by applying Kirchhoff's voltage law around the loop ok. So, we can start with the voltage on the left side v(z) ok.

And to be more precise from the previous lectures now, we can start to include position and time just to denote the level of complexity that we are going to be having right. So, it is going to be v(z, t) right and then we are having a drop on the inductor a drop on the resistor and those can be denoted for example, the drop across the inductor is given by

$$V_l = -(l\Delta z)\frac{di}{dt}$$

And if you want to be precise you can say that

$$V(z,t) - l\Delta z\frac{\partial i}{\partial t}|_z - r\Delta z i(z,t) - V(z+\Delta z,t) = 0$$

Now, the KVL has been written with both the space and the time included right. As the limit Δ z tends to 0, many things will get simplified. But these small details will be important when you write the program alright. You will need to know which quantity is at which spatial or temporal position, however, for now we are interested in finding out the relationship between output current and input current, output voltage with respect to input voltage.

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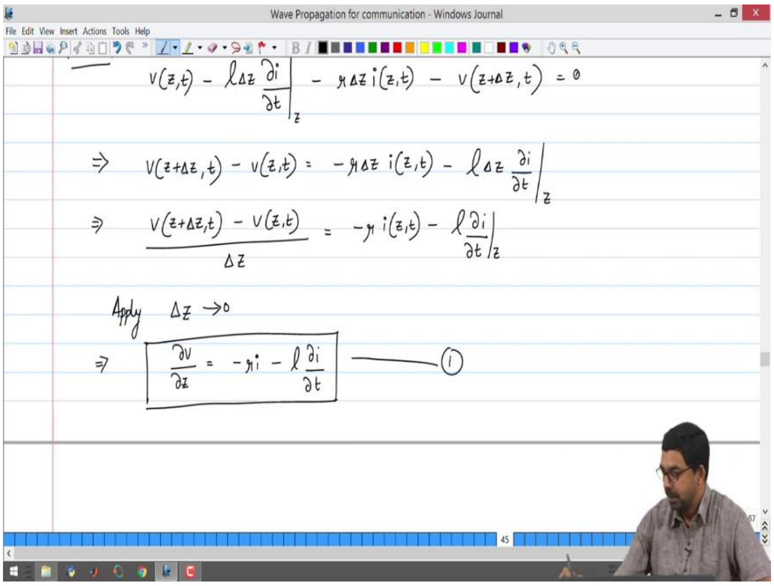

So, we can do a rearrangement of this equation. So, we can say that

$$V(z+\Delta z,t)-V(z,t)=-r\Delta z i(z,t)-l\Delta z\frac{\partial i}{\partial t}|_z$$

$$\frac{V(z+\Delta z,t)-V(z,t)}{\Delta z}=-ri(z,t)-l\frac{\partial i}{\partial t}|_z$$

Now, we can apply the limit, Δ z tending to 0 means that there is an extremely small section of the transmission line with the equivalent circuit that we have drawn, in that case in the left hand side becomes a spatial derivative of voltage. So, we will have

$$\frac{\partial V}{\partial z}=-ri-l\frac{\partial i}{\partial t}$$

Now, this will be the first telegrapher's equation. We notice immediately that the form of the equation remains almost the same. So, previously we had $\frac{\partial V}{\partial z}=-l\frac{\partial i}{\partial t}$, but now due to the inclusion of the series resistor an additional term has popped into the first telegrapher's equation of $-ri$ ok.

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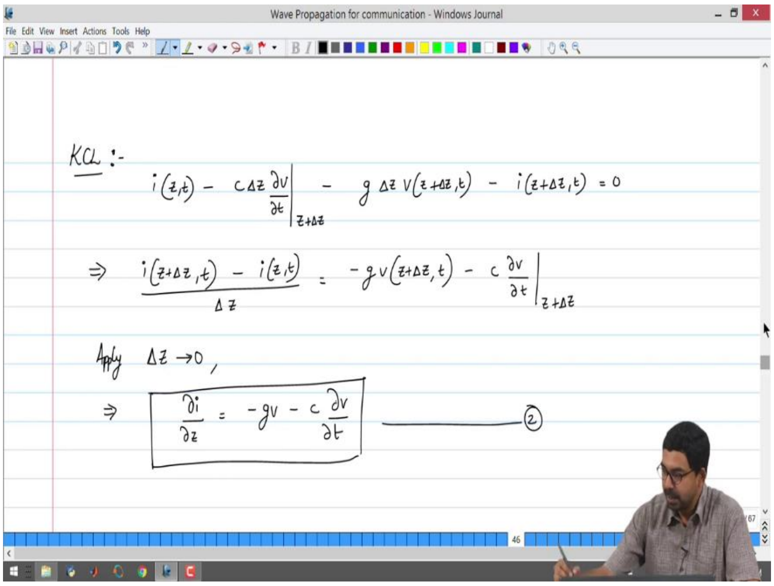

The same way we can apply KCL for the incoming and outgoing currents in the section. If the incoming current is considered as positive and the currents that are leaving this transmission line section are considered to be negative, we can start with

$$i(z,t) - c\Delta z\frac{\partial V}{\partial t}|_{z+\Delta z} - g\Delta zV(z,t) - i(z+\Delta z,t) = 0$$

$$i(z+\Delta z,t) - i(z,t) = -g\Delta zV(z,t) - c\Delta z\frac{\partial V}{\partial t}|_{z+\Delta z}$$

So, we can go ahead and apply Δ z tending to 0. This will mean that the left hand side will become

$$\frac{\partial i}{\partial z} = -gV - c\frac{\partial V}{\partial t}$$

Now this becomes the second telegrapher’s equation. One has to notice that in both these cases, there are some small details that one will have to consider while doing the programming part.

For example when you are taking $-gV$, you are going to be having some different spatial or different time coordinates. You are taking $\frac{\partial V}{\partial t}$ spatial coordinate is present z + Δ z, but the way you take the derivative, if you take v at t + 1 and V(t) and then you divide it by Δt then that derivative will exist at $V\left(\frac{t+\Delta t}{2}\right)$.

So, there can be a detailed analysis on which spatial location and which temporal location each of these quantities is existing and one could then make a decision during the programming as to how to properly code right. I will not go into all the details ok because it is again taught in a higher level course in detail.

But when we are going to be doing the coding in the next class, you will notice that tiny details will start to matter. If you take the equation as such, add or subtract quantities with your existing telegrapher's equation you may end up with some errors. So, some changes in the way it is being programmed will be based on how these terms are existing in space and time, but I leave that exercise to you ok.

So, now we have the two telegrapher's equations ok. We can now write these two equations also in the frequency domain right.

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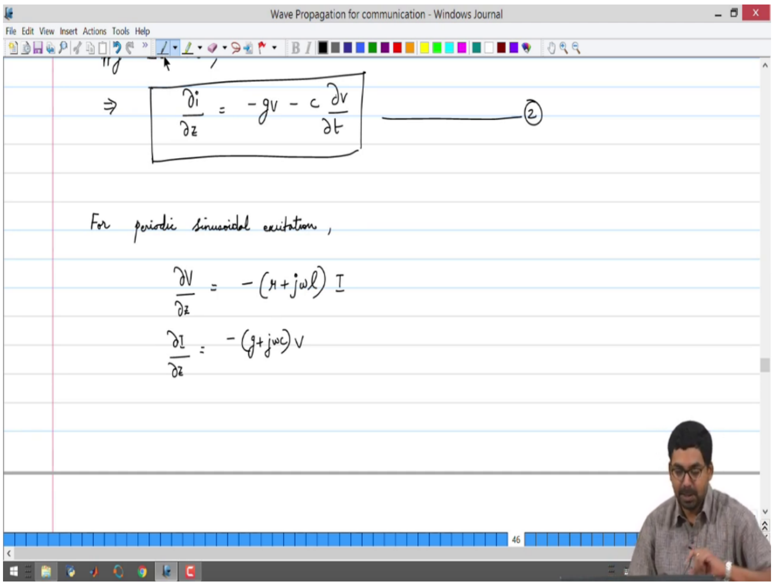

So, we can say that for periodic. So, we can say that if the transmission line is excited by a sinusoidal signal for the lossless case we have already seen how to denote this. So, we could write down the two telegrapher's equations to look like

$$\frac{\partial V}{\partial z} = -(r + j\omega l)I$$

$$\frac{\partial I}{\partial z} = -(g + j\omega c)V$$

This is very very similar to the lossless case that we have seen in the prior class ok.

And also we know that when we have these kinds of coupled partial differential equations, we can always create the wave equation from these two equations by decoupling them right. So, this process we have done twice now. So, now, I am going to directly go ahead and write the decoupled wave equation for the voltage,

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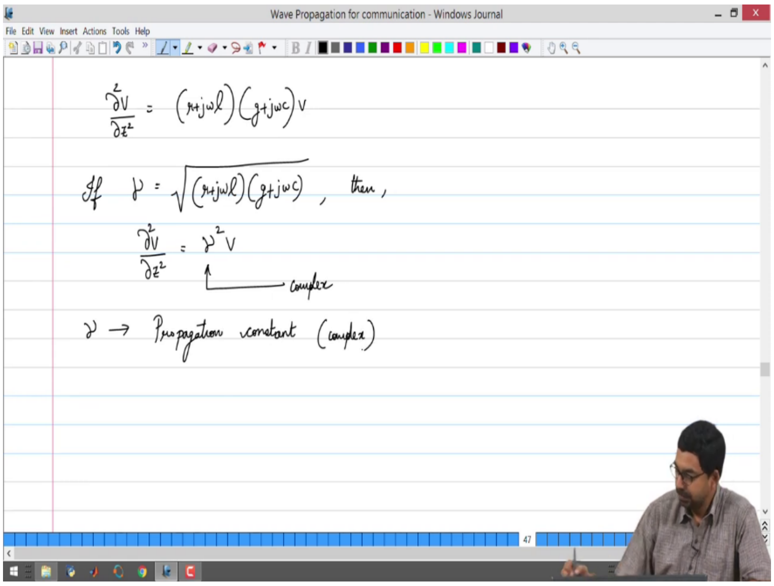

So,

$$\frac{\partial^2 V}{\partial z^2} = (r + j\omega l)(g + j\omega c)V$$

here the assumption is that it is a sinusoidal excitation. So, we can represent this as $Ve^{j\omega t}$ right. If this is the case and this is the wave equation in order to write down the general solution, we can assume the right hand side to be composed of a quantity. So, we will define a new quantity.

So, if

$$\gamma = \sqrt{(r + j\omega l)(g + j\omega c)}$$

then the wave equation can be written as

$$\frac{\partial^2 V}{\partial z^2} = \gamma^2 V$$

The first thing that we notice compare to our prior classes is that in the prior class for the sinusoidal excitation for the a lossless transmission line, we had $\frac{\partial^2 V}{\partial z^2} = \beta^2 V$ where β was not a complex number alright.

But now we are having a complex number alright ok. We are having a complex number, it has a real part and an imaginary part right, it has a real and an imaginary part. So, γ is known as a propagation constant and to be precise, it is a complex propagation constant ok.

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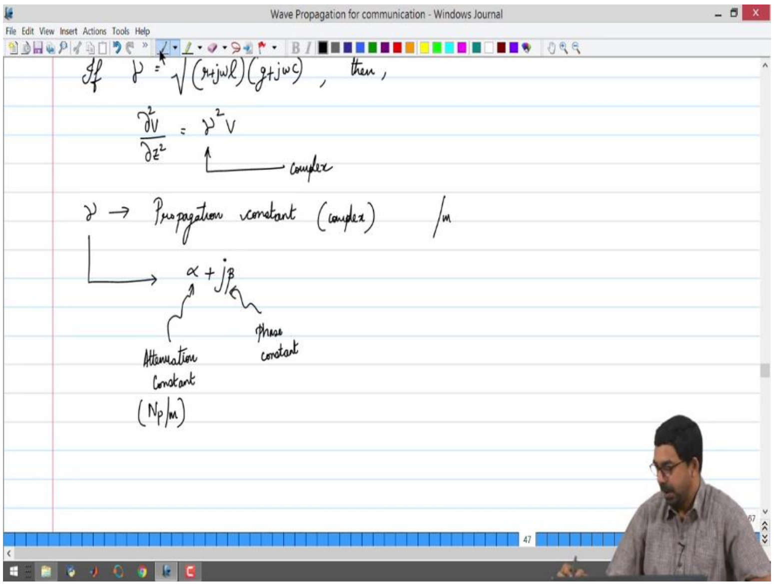

It is composed of a real and an imaginary part and to denote that we write it in the form of a complex number α + j β.

So, the propagation constant is written as α + jβ ok where α is known as attenuation constant. And β from our prior lectures, we already know that this is going to be phase constant ok. So, the complex propagation constant is composed of an attenuation constant which forms the real part and a phase constant in the imaginary part. The unit of β is going to be radians per meter, the unit of α is neper per meter and the meaning of this has to be understood in some slide detail. It is nears per meter ok and the propagation constant of the unit is simply per meter ok.

If the wave equation is written as

$$\frac{\partial^2 V}{\partial z^2} = \gamma^2 V$$

We already know how to write the general solution for this wave equation

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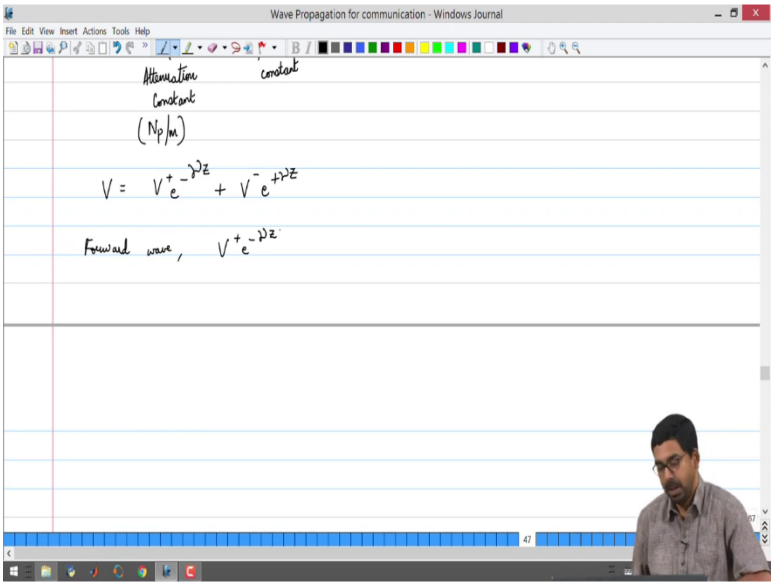

We can write down the solution for the voltage. In this case is going to look like some

$$V(z) = V^{+}e^{-j\gamma z} + V^{-}e^{+j\gamma z}$$

We have already discussed in detail about some forward waves and backward waves. The first term denotes the forward wave with the complex propagation constant γ, the second term denotes the backward wave denoting with the complex propagation constant γ again.

So, the sign change here is just indicating that the first term is forward; the second term is a backward wave ok. So, the general solution is

$$V(z) = V^{+}e^{-j\gamma z} + V^{-}e^{+j\gamma z}$$

Now we can look at only the forward part and then try to see what the meaning of this α could be ok. So, taking only the forward part ok, taking only the forward part, we can substitute for γ to be α + j β ok.

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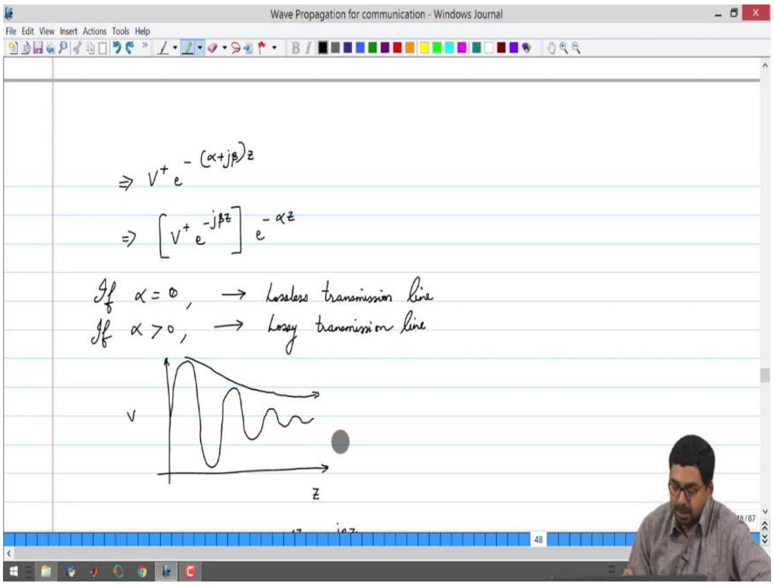

So, we can write down this forward part as $V^{+}e^{-(\alpha+j\beta)z}$.

This becomes $[V^{+}e^{-j\beta z}]e^{-\alpha z}$. Now we notice that the first term in the brackets is similar to the term that we had for the lossless transmission line where the model did not include the series resistance and the parallel conductance ok. So, it is $V^{+}e^{-j\beta z}$. Because of the inclusion of two resistors one in series with the inductor one parallel with your capacitor, we are having an additional term coming into the general solution for the forward wave.

So, it is having an $e^{-\alpha z}$. So, the exponent here is not a complex number, it is not an imaginary number alright. So, this means that there are some details that we have to understand with respect to this alright. First of all we will write down that if α is equal to 0, we call this as a lossless transmission line which is the case that we have seen in prior classes the solution then becomes identical to what we had before. The second term outside of the bracket becomes equal to one alright its exponent of 0. So, it becomes one and your general solution turns to the solution that we had in the prior classes. This is known as lossless transmission line ok.

Now, if α is greater than 0, then we have to notice that α is going to be a positive number. If it is going to be a positive number as z increases, there is going to be an exponential decay of the voltage term that is present in the square bracket. So, this term a dictates what the magnitude of your voltage wave or the amplitude of your voltage wave will be and it denotes that as the wave travels forward, this is a forward travelling wave. As it goes forward it is going to decay exponentially in amplitude right.

So, if α is greater than 0, we just call this as a lossy transmission line ok ok. Diagrammatically speaking previously when we had a done the simulations say for example, we had the position z coordinate in our simulations like this and then we would be having some value of voltage which is coming at you know into your transmission line due to the excitation so, you will be having something like ok.

So, the peak of your voltage will decay exponentially that is what this $e^{-\alpha z}$ means ok. If your propagation constant consists of a real and an imaginary part, the real part will contribute to the loss right and the voltage will continuously decay ok and the imaginary part will just dictate the phase and this part we are already aware of ok.

The question then becomes what if α is less than 0 ok. If α is less than 0, then it is obvious that from the solution as the wave travels forward, the amplitude will keep on increasing ok. Now if the amplitude keeps on increasing, there has to be some physical interpretation for what is happening in the circuit diagram. The only way you can interpret that is you are supplying some external power and that power is being converted into amplification of your existing voltage wave.

In other words it acts like a gain circuit of some kind if α is less than 0. So, if α is less than 0, you will have a transmission line with gain, but we can be sure that in our model that is not going to happen because we have used only passive elements. We have used only resistor inductors and capacitors; none of them are active elements which can provide amplification. So, if you did a want to have α less than 0, the equivalent circuit diagram has to change it has to include external power and use some kind of an amplifier.

An example could be that if you use some operational amplifier of some kind somewhere maybe you are boosting the voltage and then you could have a change in your circuit. Optically it means that you have a gain medium which will increase the amplitude of this a ah you know signals, but α less than zero is not under the purview of this course as of now alright. So, α is strictly positive for most of the cases that we are going to be dealing with and that corresponds to a lossy transmission line ok.

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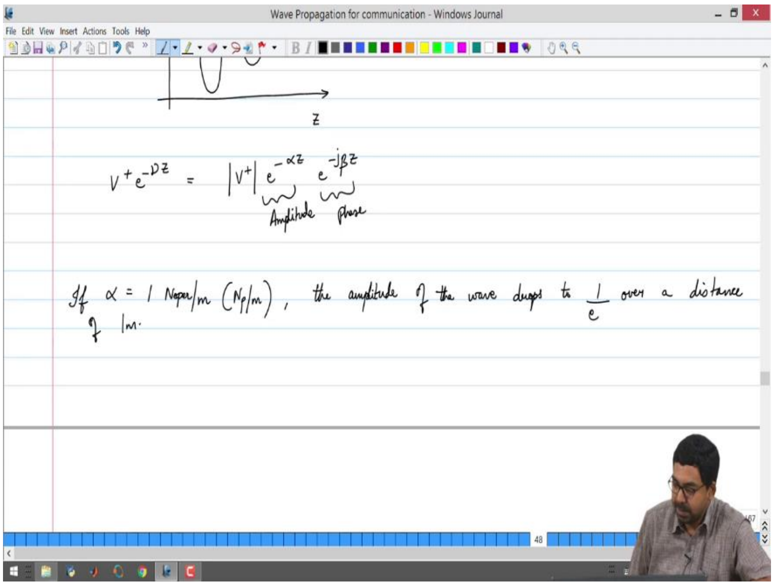

So, just to be very clear, the forward wave is given by say

$$V^{+}e^{-j\gamma z} = |V^{+}|e^{-\alpha z}e^{-j\beta z}$$

So, this term governs the amplitude and this term governs the phase ok. And the unit of α is in nepers per meter and I think we have to understand that you need a little bit more. So, that we are comfortable with problems that we will be solving.

So, if α is equal to say 1 neper per meter. So, in short form we will use this neper per meter ok. If α is equal to 1 neper per meter, then what we are trying to say is the amplitude of a wave drops to 1 over e ok over a distance of 1 meter. This is the meaning of α which is 1 neper per meter.

So, as your wave travels forward if α for your transmission line is 1 neper per meter, if the voltage wave travels 1 meter its amplitude would have to reduce to 1 over e of its input voltage amplitude. That is the meaning of 1 neper per meter ok. So, this also means that we will have to quantify certain other terms in our transmission line which we did not worry about before.

It means that practically many transmission lines are going to be having some value of resistor in the series, some value of series conductance in the parallel path which means that many of the transmission lines will be having some small amount of α. And it is going to lead to some exponential decay of the voltage as the wave is travelling forward which means that there is some limit on the length of the transmission line that one can use to have a voltage wave travelling from one side and also be able to receive it on the other side ok.

If you have a very long transmission line that is lossy, it is given that on the other side you are not going to be having a signal at all. So, this means that some new parameter has to be decided compared to before alright.

And one of these parameters is propagation length ok.

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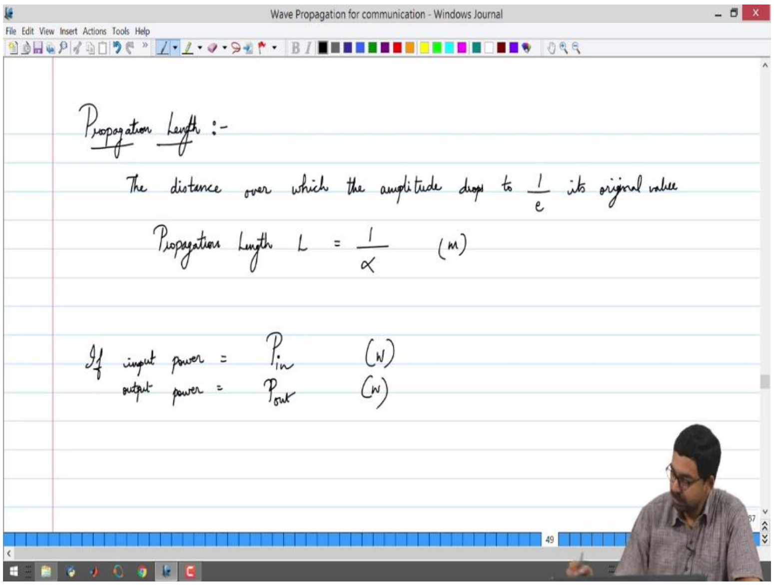

Prior to this lecture where we had not included a resistance and the conductance, we are not worried about propagation length. We knew that the voltage wave that is launched and one side is going to reach the other side with a delay and depending on the boundary condition, there is going to be some reflection extra. But now you are going to be worrying about what could be the length of the transmission line.

So, objectively it is defined as the distance over which the amplitude drops to 1/e, its original value ok. So, the propagation length is defined as the distance over which the amplitude drops to 1 over its original value ok. In short propagation length l is given by

$$L = \frac{1}{\alpha}$$

and the unit, obviously, is becoming meter even though α is a nepers per meter. One has to understand that the unit of the propagation length will become in meters and the propagation constant the unit is per meter ok.

So, the propagation length is 1 over α. So, this means that if you did have a line of a 1 meter and 1 neper per meter is your α that is actually its propagation length itself ok. So, the other thing that we have to consider is the propagation length is signifying the length of the transmission line where the voltage or the amplitude will drop to 1 over e, does not mean it drops to 0 ok.

So, there can be some small value of the voltage wave travelling beyond this right. But for practical purposes we do not consider that to be a transmitted signal, we just say that the voltage way has I mean decayed significantly, but it does not mean that there is no voltage wave existing after this propagation length if your transmission line is longer than the propagation length. It means that you are having very negligible amounts of signals does not mean you have 0 signal ok.

Another way of looking at this problem is people usually launch an input power and they actually try to measure the output power at the end of the transmission line ok.

So, if your input power which I call it as P_{in} ok and the unit will be in watts ok. And if the output power, it is going to be P_{out} in watts right in order to understand how fraction of your input power is coming at the output. One could simply take a ratio of P_{out} to P_{in} alright. we will know what a proportion of your input power is coming out.

Since we are considering a transmission line with no active components, the output power here is going to be always less than equal to your input power. And since we know that most of the conventional transmission lines will have some amount of resistance and some amount of conductivity in the parallel path, it is safe to say that in most of this cases P_{out} will be less than P_{in} ok. And sometimes instead of taking the linear scale just the ratio of P_{out} to P_{in} people also make use of the ratio in decibels.

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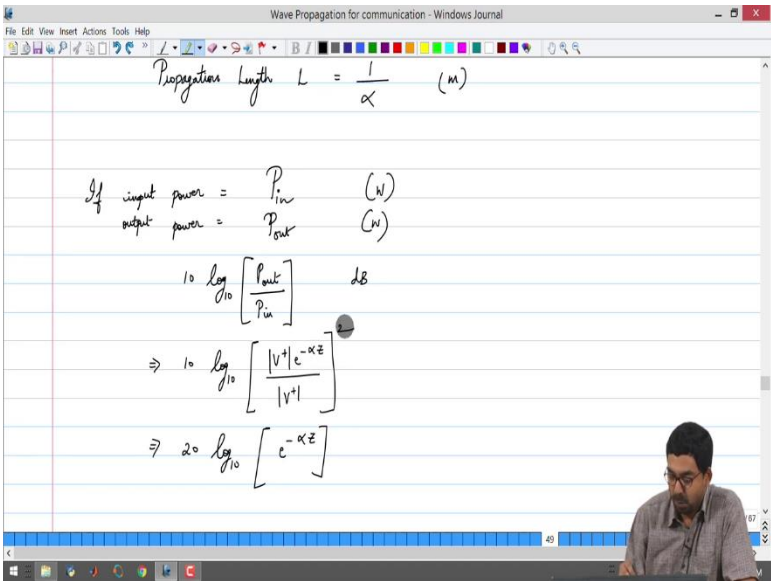

So,

$$10\frac{P_{out}}{P_{in}}]\,dB$$

and this is the definition of say decibels ok. So, with respect to the input power, how much output power I am getting, this is also measured in terms of decibels.

So, since the power is proportional to voltage square right, you can always right this as

10 log to the base 10 alright. So, you can say that. So, it is going to be proportional to the voltage square if the length of your transmission line is going to be some denoted by z, α is the attenuation constant and the launched voltage.

It is going to be

$$=>\ 10\frac{|V^{+}|e^{-\alpha z}}{|V^{+}|}]^{2}$$

$$=>\ 20\frac{|V^{+}|e^{-\alpha z}}{|V^{+}|}]$$

.

So, if you know the attenuation constant of your transmission line, you can estimate the relative output power in decibels that you will be measuring your transmission line. So, this is also made use of in practical conditions alright. Output power is expressed in terms of input power in decibels ok. And what does this mean alright? We need to get a feeling for some numbers ok.

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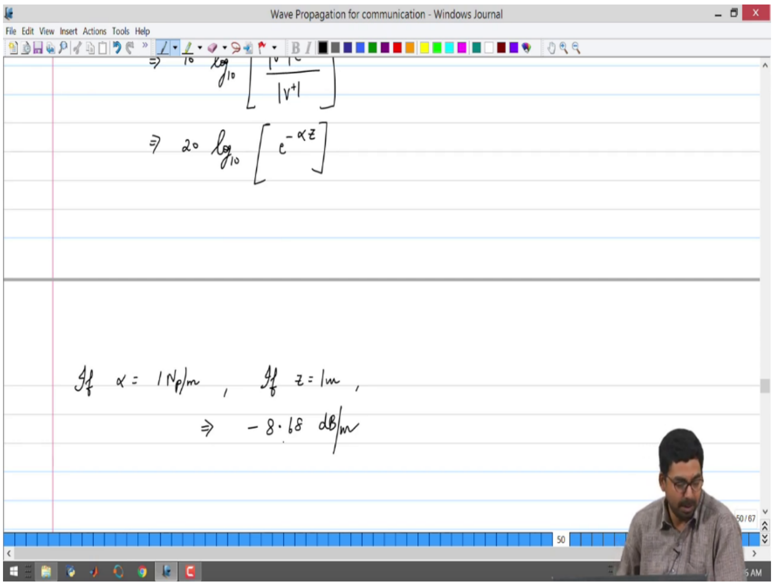

If α is equal to 1 neper per meter and if the length is say 1 meter ok. Substituting this you will get the relative output power to be minus 8.68 dB per meter ok ok. Since the length is 1 meter, you can write down the relative output power as, say , minus 8.68 dB per meter. So, if you are having a transmission line of length which is larger than this, then you can always use this as an indicator of what is the net relative output power that you will get with respect to your input ok.

So, in summary by including the resistor and the conductor in the transmission line, we are noticing a few effects. The first and foremost thing is that the telegrapher's equations look close to what we had seen before, but have an additional term -ri in the case of the first Telegrapher's equation for the spatial derivative of voltage and -gv for the spatial derivative of the current.

Now, even though we have written it in a simple form as Δz tends to 0 while making programming programmed versions of these alright, just we have done like we have done in the prior classes each and every term will matter alright. So, where you take this voltage at which instant of time extra, everything matters. So, small details will cause you to know big changes in your program alright. And we will be seeing that in the next class, I will not be going into too much detail because that is covered in another course at a higher level, however, you still need to have a starting code that actually works.

So, we will be noticing that the place where you add or subtract terms in your existing a finite difference code will actually make a big difference. We are considering an ac excitation or a sinusoidal excitation in this particular case

$$\frac{\partial V}{\partial z} = -(r + j\omega l) * I$$

where this I and V are actually written as a time harmonic waves and these are actually exponentials right.

In order to decouple these equations, we can write down the wave equation as

$$\frac{\partial^2 V}{\partial z^2} = (r + j\omega l)(g + j\omega c)V$$

The procedure for obtaining this is the same as what we have done a couple of times before alright. All we do is take the derivative of one of the equations and then substitute from the other equation from the Telegrapher's equations.

So, in order to find out the general solution, we need to make use of this term $(r + j\omega l)(g + j\omega c)$. If we create a new term now called

$$\gamma = \sqrt{(r + j\omega l)(g + j\omega c)}$$

The wave equation will look like

$$\frac{\partial^2 V}{\partial z^2} = \gamma^2 V$$

And this γ is going to be complex comma square is also going to be complex and γ is known as a propagation constant. It is a complex number and the unit of that is per meter.

Since it is a complex number, it has a real part α and an imaginary part β. The real part is known as the attenuation constant, the imaginary part is known as the phase constant. The unit of α is nepers per meter, unit of β is radians per meter ok. Consequently the solution for the forward wave will look like $V^+e^{-j\gamma z}$ and one can substitute for γ to be α + j β. The result will look like $V^+e^{-j\gamma z}$ which is the same expression that we had for the lossless transmission line case with the ac excitation multiplied with e to the minus α z.

The e to the minus α z is going to dictate the amplitude of your voltage wave that is travelling forward in this case, we are assuming the forward voltage. If α is equal to 0, we will say that your transmission line is lossless, that means, that it is equivalent to not having the resistor and the conductor in the series into the parallel paths which is the case that we had started with. However, if α is greater than 0 we call this to be a lossy transmission line and the interpretation of α greater than 0 is as your wave progresses in the z direction near the transmission line, the amplitude will decay exponentially ok.

Now, the other question that normally one would have is what would happen if α is less than 0. α less than 0 though mathematically is possible here, it will denote a transmission line with the gain. So, as it propagates in the z direction it may have higher and higher amplitudes that are also going exponential. However, the physics of the a equivalent circuit that we have drawn prevents α from becoming negative ok because we are using only passive components. If you did want to make α less than 0 or provide a gain, you will have to use active components which is not the case over here ok.

So, α is going to be greater than equal to 0. The unit of α was nepers per meter and just to understand that if α is going to be equal to 1 neper per meter, the amplitude of the wave drops to 1 over e over a distance of 1 meter ok.

This means that we have to define another term that we have not considered before which means that we have to denote something known as a propagation length, which means that for a given transmission line design, you will communicate with other engineers with an objective term that is called propagation length ok. If your transmission line has r, l, g, c parameters, you will say that the distance over which the amplitude drops to 1 over its original value is denoted by l and that is equal to 1 by α in meters.

Generally, this is the practical length of the transmission line that one could have. But nothing prevents us from making a transmission line longer than this, but the propagation length for objective purposes is 1 over alpha, it still means that you will have a small amount of voltage even after this length. But in the practical scenario, we do not consider the length of the transmission line to be longer than the propagation length because the value of the signal is going to be too low.

Another way of measuring the decay alright due to the introduction of resistance and capacitance is by making use of the ratio of the input, output to the input power. If the input power is P_{in} in watts, the output power is P_{out} in watts. A simple way of a finding out the relative output power is $10\frac{P_{out}}{P_{in}}$] and this is denoted in units of decibels ok.

Now, since this is denoted in decibels right and this is in power, the power is proportional to the voltage square. So, we can always write this down to be

$$=> \ 10 \frac{|V^+|e^{-\alpha z}}{|V^+|}]^2$$

$$=> \ 20 \frac{|V^+|e^{-\alpha z}}{|V^+|}]$$

. So, if you know the propagation constant or the attenuation constant of your line and if you know the length, you can estimate the relative output power in your transmission line. And if it is going to be very low, then you can actually do something about it or cut the transmission line or make design modifications to reduce r and g extra.

Just to understand this decibels very well if α is one nepers per meter and z is 1 meter, then you have the relative output power to be minus 8.68 dB per meter. Now there is one additional detail that you will have to notice that is there is a negative sign. The negative sign here denotes that your output power is less than your input power ok. If your output power is equal to the input power, then your, I mean then your relative output power measured in decibels will be 0 dB ok.

If you get a positive sign for this equivalent circuit, that means that something is wrong alright. If you have a positive sign at the relative output power in decibels, it means that your power at the output is actually increased compared to the input. So, there are some small details that you will have to keep in mind. When you are drawing an equivalent circuit with all passive elements, you should end up with a negative relative output power in decibels ok. So, this concludes this lecture.