Transmission lines and electromagnetic waves

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Lecture – 02
Lossless Transmission Lines: Wave Equations

We will get started with a quick review of what we had seen yesterday.

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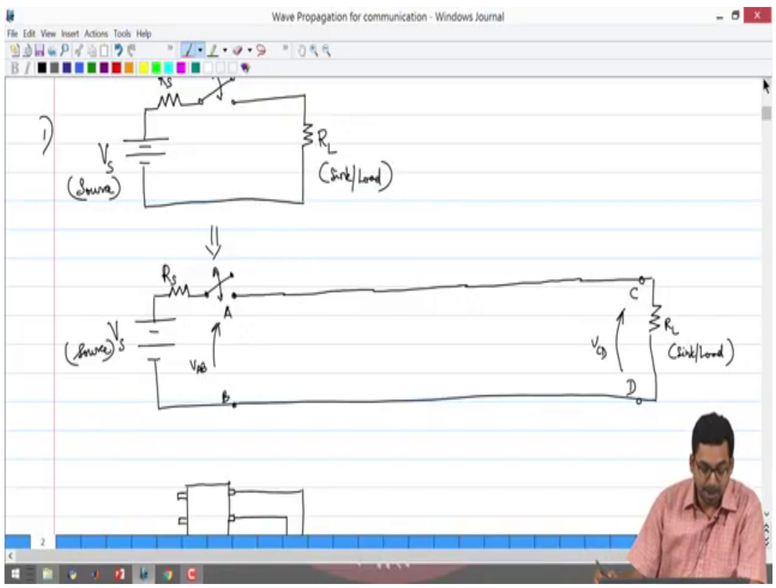

Started with an elementary circuit which had a battery with the series internal resistance and a switch, that can open and close and a load resistance on the other side. A typical low frequency circuit diagram where the lengths of the wire connecting the switch to this resistor is not mentioned. Suppose, we are connecting interconnection between the switch and the load resistance becomes very large. We expect that the signal will not propagate at infinite speeds, but have some finite velocity to travel from.

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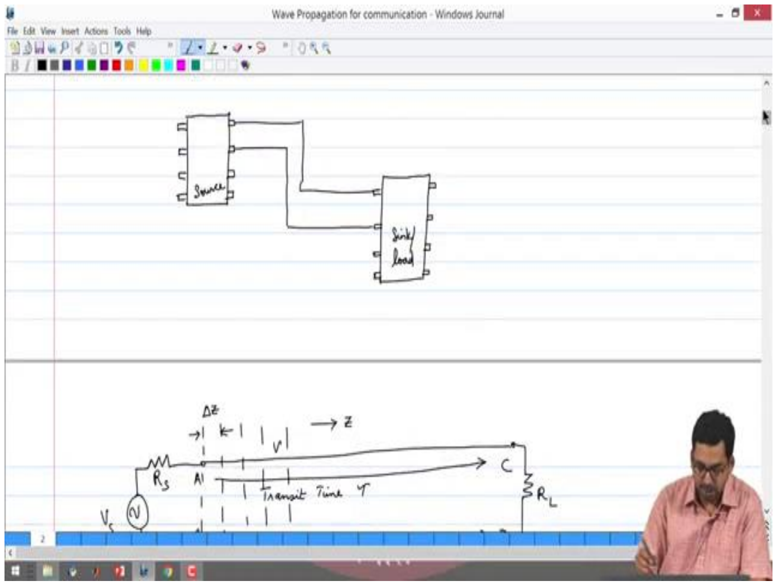

And, this is very important especially if you are connecting different ICs on a board or if you are connecting different components within an IC all of this becomes important, ok.

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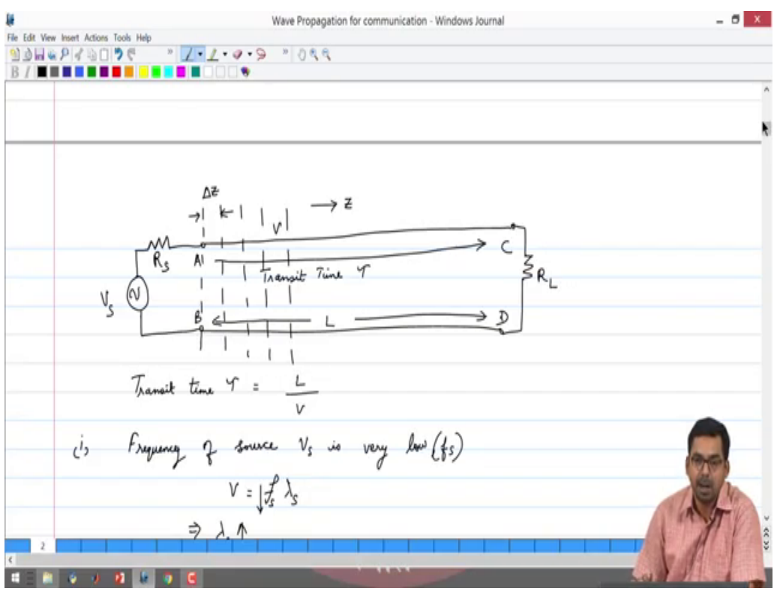

So, we saw that we could not apply KCL or KVL directly when you had a case where the wire length was abnormally large compared to the wavelength of the signal that you are putting in.

So, we had to come up with a technique to apply the low frequency circuit analysis to this problem right, and the simplest way is to divide these wires into a number of small uniform sections and apply KCL and KVL to each of these sections.

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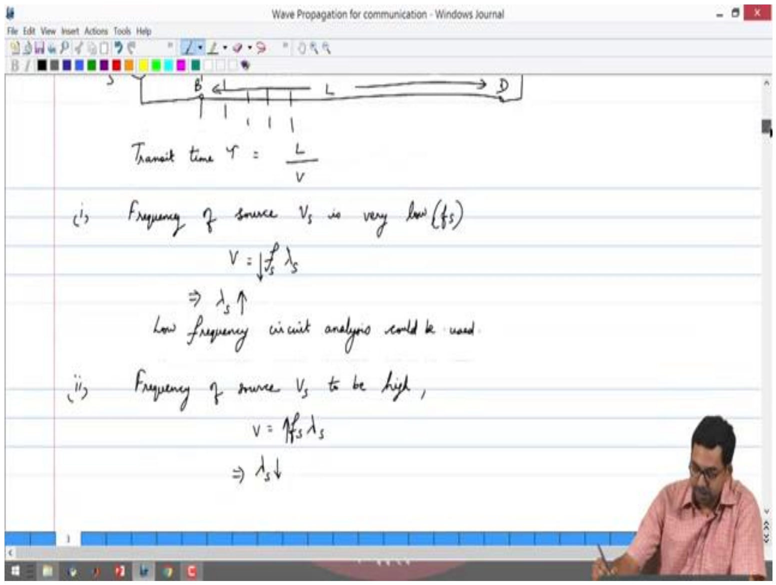

And what we were noticing is that we need to account for a few things that happen in this wire.

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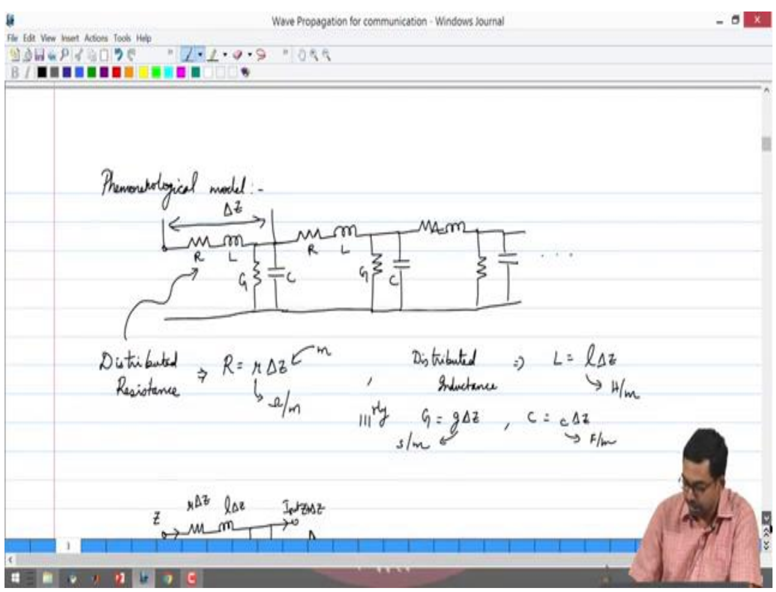

First of all, since the velocity is finite we had to introduce some components that would introduce a delay between one side and the other side. And we already know from our low frequency circuit analysis that you could use R_L or R_C, ok.

This is a phenomenological model which means that we are considering the self inductance of the wire and the capacitance between the forward path and the return path for the current, alright. In the case of a board you will be having traces on top of the board and a ground plan at the bottom of the board and there is a dielectric in between. So, you will end up having some capacitance.

So, the capacitance is connected this way and the capacitance is not going to be the only component connected across because the dielectric is not going to be ideal. You will also have some small amount of current trickling between them. So, that is modeled as this. So, this repeats everywhere throughout the section of the wire and what we are now doing is known as a distributed model of an interconnect or a transmission line.

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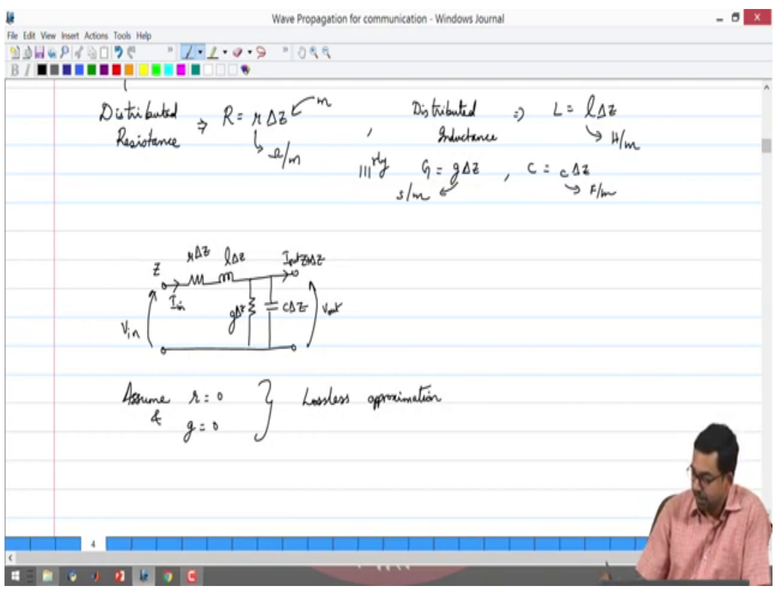

We have made a simple assumption that the series resistance in the parallel conductance is going to be 0 alright, this will avoid any potential drops that you will have in this section of the wire, alright. We are finally modeling a short circuit anyway so, it may be better to start with the lossless approximation. So, we started with resistance and conductance made as 0

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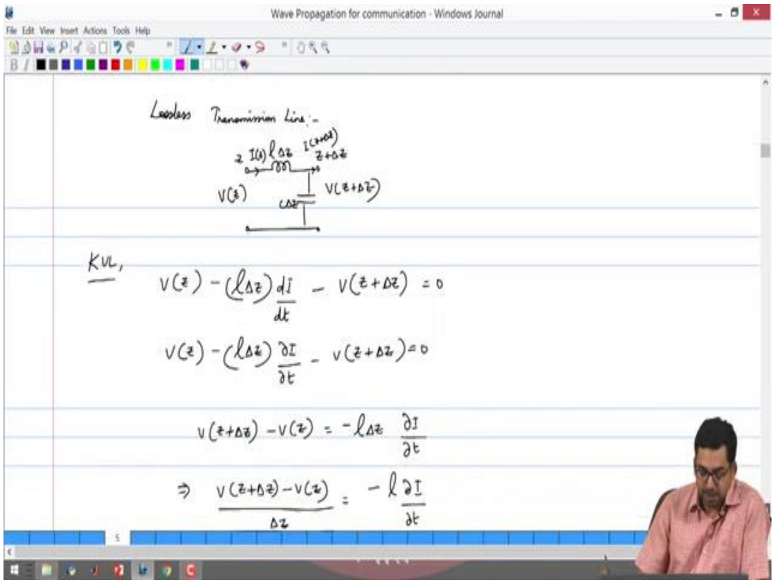

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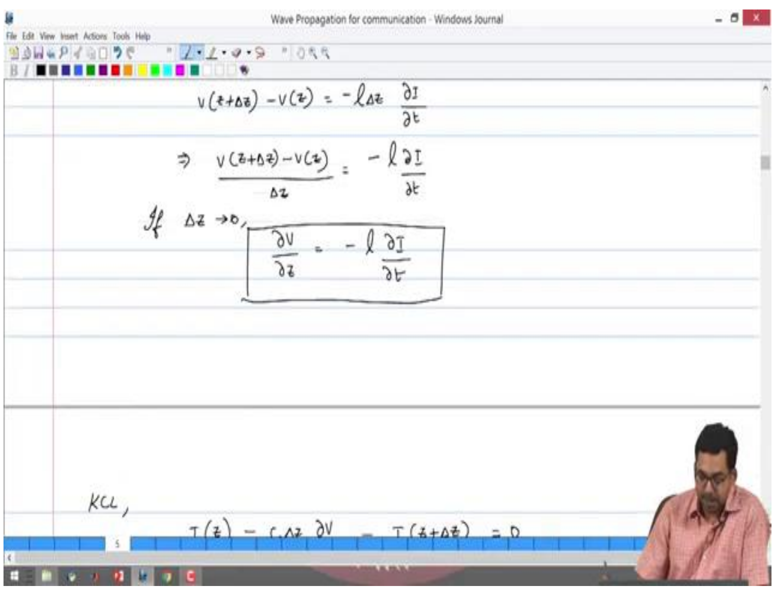

and we proceeded to get some constituent equations by using KCL and KVL.

In one of the sections and we found that we were obtaining space derivatives in circuit equations this is where we had stopped, alright.

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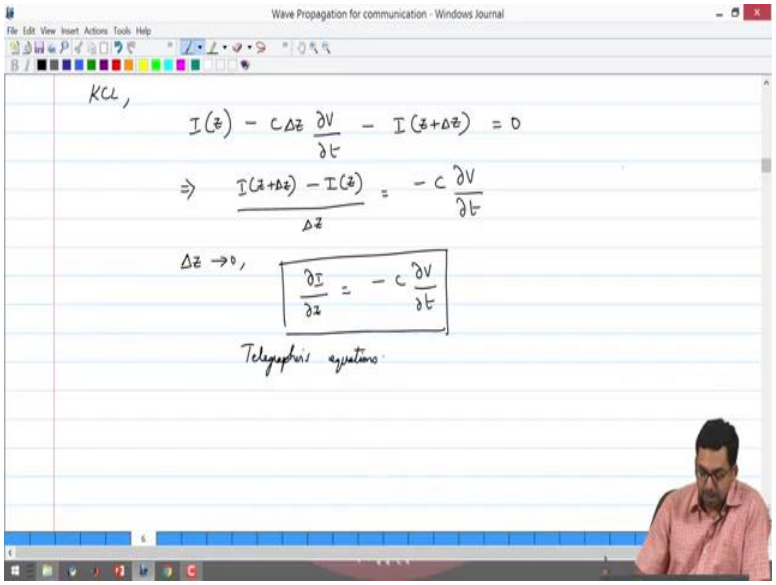

Today, we will proceed a little bit further, we will try to understand what these equations mean and we will try to see whether we can decouple these equations. So, on the left hand side we are having a spatial derivative, on the right hand side you are having a time derivative, but you are having a current component on the left side and a voltage component on the right side.

Decoupling means that you will be having the same quantity on the left and on the right and let us see if that gives us any more information, ok. And, we start today's content. So, let me write down the telegrapher's equations just for the sake of completeness.

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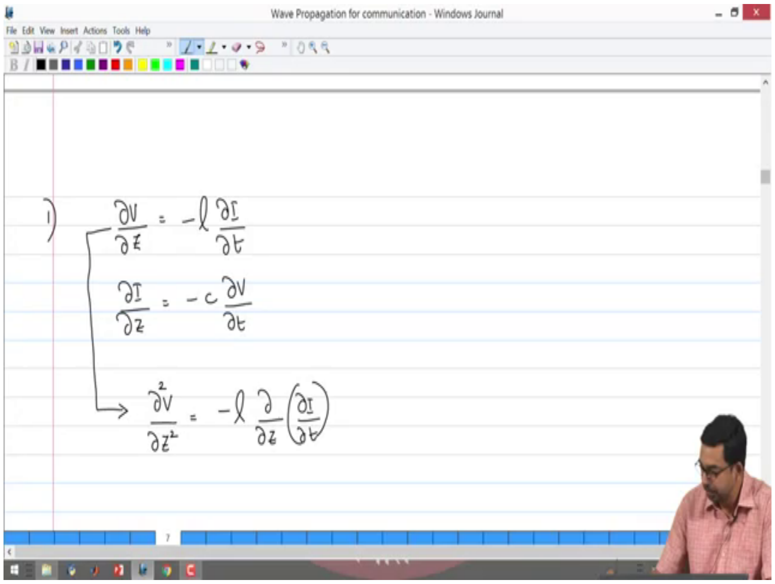

Ok, these were the two equations that we had got at the end of application of KCL and KVL to one section of the transmission line which was lossless, ok. Now, we will take this first equation and try to obtain the second derivative with respect to space, right. So, we will try to write down

$$\frac{\partial^2 V}{\partial z^2} = -l\frac{\partial}{\partial z}(\frac{\partial I}{\partial t})$$

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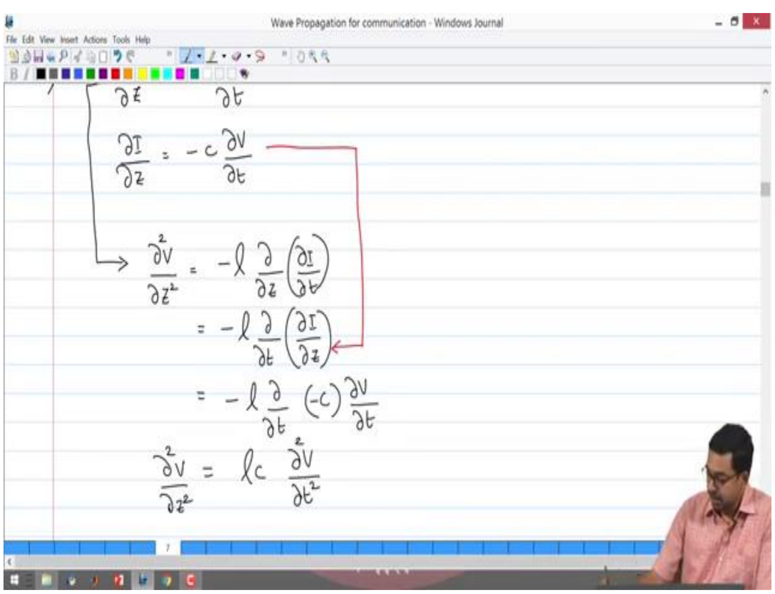

Here since z and t are independent variables we could switch the order in which we are taking the derivatives, right. So, we could go ahead

$$\frac{\partial^2 V}{\partial z^2} = -l\frac{\partial}{\partial z}(\frac{\partial I}{\partial t})$$

$$= -l\frac{\partial}{\partial t}(-c)\frac{\partial V}{\partial t}$$

$$\frac{\partial^2 V}{\partial z^2} = lc\frac{\partial^2 I}{\partial t^2}$$

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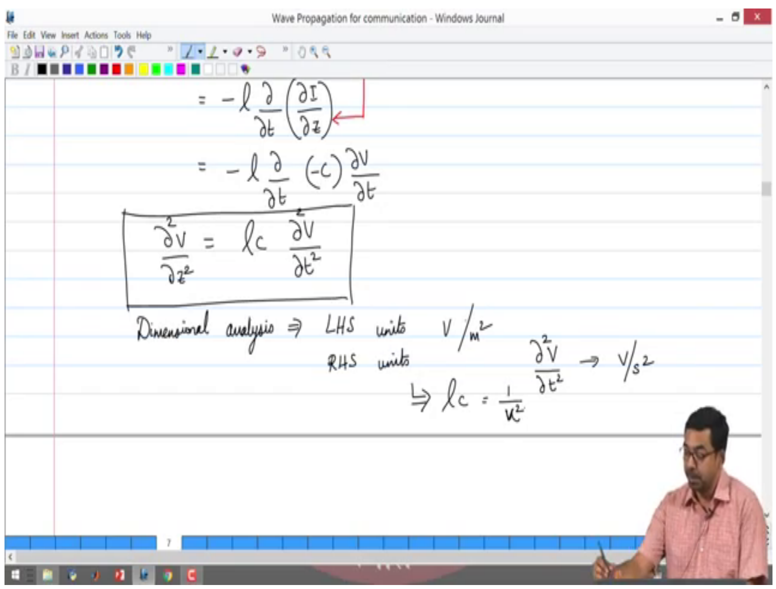

Now, the advantage of this equation here is that on the left hand on the right hand side you have the same quantity, you just have different derivatives being taken on the left and the right hand side. So, we will see what this can mean in the further classes ok.

So, one of the things that we can start with when you have an equation like this ok, start to perform dimensional analysis, that is you start comparing units of quantities on the left and the right hand side to see if there are any new meanings that you can derive from this equation. $\frac{\partial^2 V}{\partial z^2}$ on the left hand side right has the units of V/m^2, ok.

So, let us write down a dimensional analysis, oops is this is V/m^2. And on the right hand side you are supposed to be having the same effective units right, duo square v by duo t square will have the units of V/s^2, right. Now, this gives us that

$$lc = \frac{1}{u^2}$$

Otherwise u and v will become confusing. So, where u is the velocity, I will just go to the next page and has the units of a m/s.

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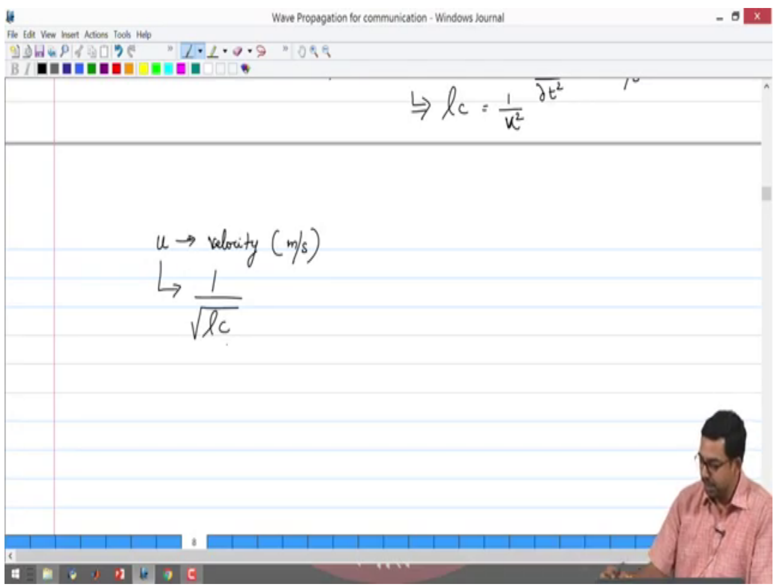

So, this u now corresponds to

$$u = \frac{1}{\sqrt{lc}}$$

seems to be having the units of velocity or m/s, we would not have thought about it in the first class. Because, l would have been described in henries per meter and capacitance would have been in F/m and $\frac{1}{\sqrt{lc}}$ is turning out to be having a unit of velocity which is in m/s.

Now, there are some questions that we can ask using this dimensional analysis, right. In the class yesterday we had plugged in R is equal to 0 and g is equal to 0 and some questions would have come into the mind why do not we plug l equal to 0 and c equal to 0, right. If we were to do that the way we have built this argument it would appear that if you did not have any inductance per unit length any capacitance per unit length you will end up having a velocity that is infinite which is a counter argument for the exact point that we are trying to state that the signals cannot travel at infinite velocity, ok.

So, this components l and c are trying to not only tell you that the velocity is going to be finite ok, they also give you some insight as to how you can model the signal propagation by neglecting the resistance, but you will still not be able to avoid some circuit passives like inductance and capacitance is coming in, ok.

So, though you can neglect your resistance and cap conductance, l and c will play a dominant role in determining how fast your signal is going to be traveling according to this argument. Now, since we did a dimensional analysis for this particular equation alright, a similar thing can also be done for the current which can be decoupled.

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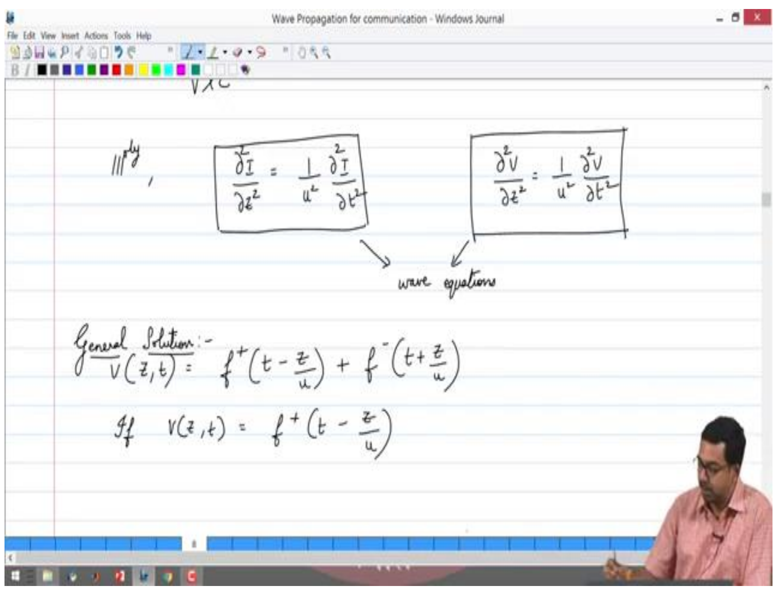

So, you will have an equation for the current that is going to be decoupled using the exact similar steps, right. So, we will be having

$$\frac{\partial^2 I}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 I}{\partial t^2}$$

Now, these two are second order partial differential equations alright and one can start looking at general solutions to these partial differential equations and start building analytical solutions to these equations and trying to derive meaning from it. In this course we will be trying to do two things ok In my experience I have found that the students have difficulty in understanding general solutions to partial differential equations, alright.

At the end they do not have an understanding why it is a general solution, why can they not be some other solution? What is exactly a boundary condition and how do we determine the value of the constants? All these questions start to come to mind. So, we are going to use two approaches.

We are going to go for analytical solutions of these equations, but we are going to back it up with some other approach where we are going to use a computer to do this work for us without plugging in any analytical solution. And, trying to see what the solutions are going to look like and whether it reinforces what you will be trying to learn in the analytical approach ok.

So, these two equations are usually called the wave equations ok and the if you take the voltage equation over here. The voltage wave equation we can write down the general solution to be a function of position and time ok it is going to look like

$$V(z,t) = f^{+}\left(t - \frac{z}{u}\right) + f^{-}(t + \frac{z}{u})$$

So, the general solution looks like

$$f^{+}\left(t - \frac{z}{u}\right)$$

and many students have difficulty in following this general solution and what their implications are, ok.

So, what we will do is we will try to go analytically and in the classes coming in future what we will be trying to do is without plugging in these general solutions into a computer can we make similar meaning of what we had done in the analytical part is the part that we are going to do in you. So, if somebody asks how do you know that this is the general solution, the general answer that is given is take this solution substitute it in this equation, try to find the left hand side, try to see the light right hand side and see if they are equal if they are equal you know that it is the general solution.

So, in this case we can take the first term alone right ok and we can plug this into the voltage equation, and see what left hand side and right hand side are coming to be this is just a refresher of some change of variables techniques that you may have forgotten after your undergraduate, alright. So, we are just getting warmed up. So, I will do this part alright.

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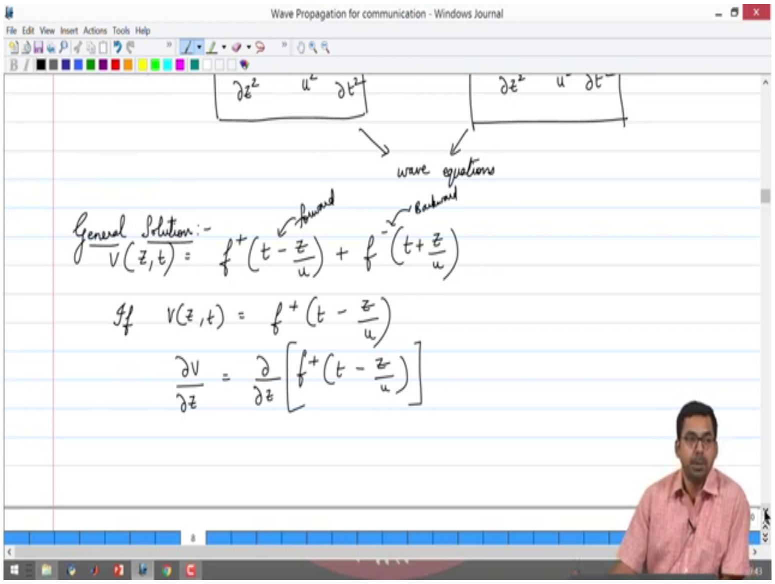

So, we are having v of z comma t is assumed to be some function of t minus z by u, right. So, then we can find out the first derivative with respect to space for the left hand side. So, we can say

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}[f^{+}\left(t-\frac{z}{u}\right)]$$

Alright and we can use a change of variables over here.

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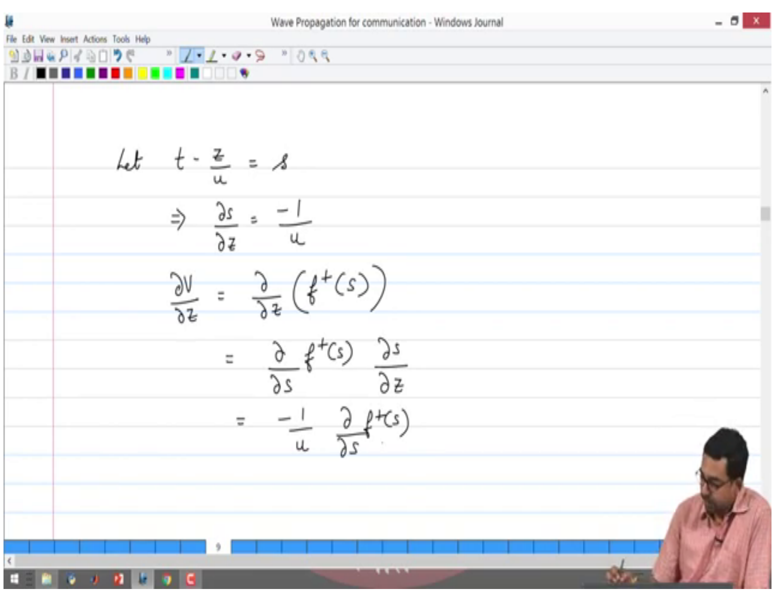

You can say that let

$$t - \frac{z}{u} = s$$

$$\frac{\partial s}{\partial z} = -\frac{1}{u}$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial t}\left(f^{+}(s)\right)$$

$$= \frac{\partial}{\partial s} f^{+}(s) \frac{\partial s}{\partial z}$$

$$= -\frac{1}{u}\frac{\partial}{\partial s} f^{+}(s)$$

The left hand side in our equation had a second order derivative which means that $\frac{\partial^2 V}{\partial z^2}$ is what we need, right.

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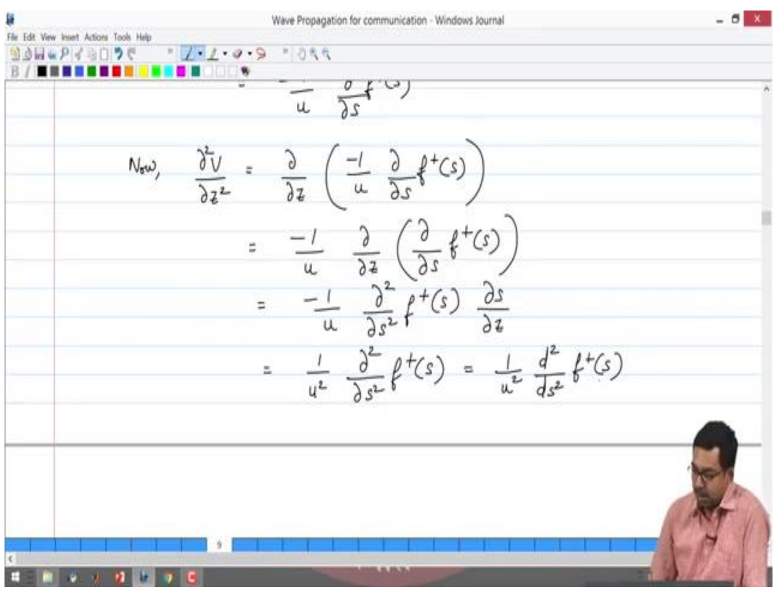

So, that is going to be

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}\left(-\frac{1}{u}\frac{\partial}{\partial s} f^{+}(s)\right)$$

$$= -\frac{1}{u}\frac{\partial}{\partial z}(\frac{\partial}{\partial s}f^+(s))$$

$$= -\frac{1}{u}\frac{\partial^2}{\partial s^2}f^+(s)\frac{\partial s}{\partial z}$$

$$= \frac{1}{u^2}\frac{\partial^2}{\partial s^2}f^+(s)$$

Now, f is simply a function of s alright there is no other independent variable involved after you have made a change of variable. So, this partial derivatives can be converted to an ordinary derivative, you do not need to have a partial derivative over there because there is just one independent variable for your f^+ function.

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{d^2}{ds^2}f^+(s)$$

This is the left hand side of the equation, assuming a part of the general solution right and we can go to the right hand side. The right hand side was right, we can start with the right hand side of the voltage wave equation it was

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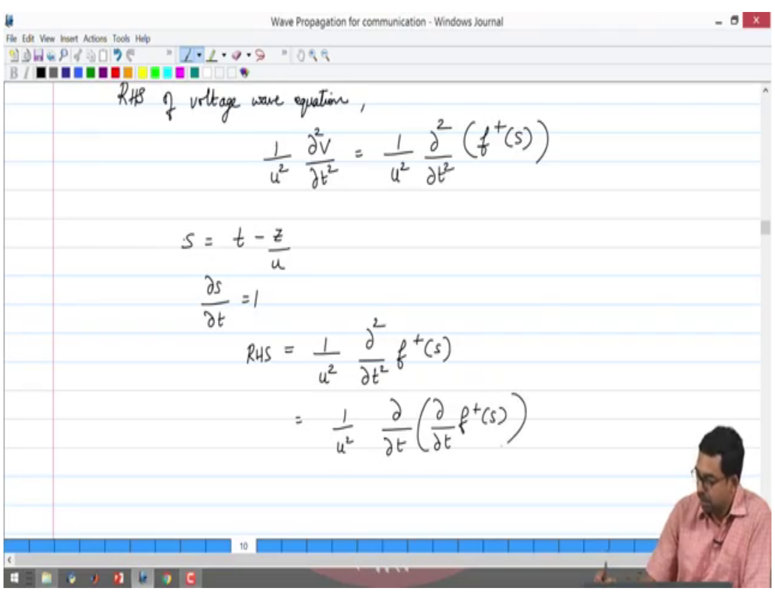

$$\frac{1}{u^2}\frac{\partial^2 V}{\partial t^2} = \frac{1}{u^2}\frac{d^2}{dt^2}f^+(s)$$

So, in this case we can do the exact same change of variables technique, right.

$$s = t - \frac{z}{u}$$

So, here we will be trying to find out 1 by u square duo square by duo t square of, right. So, now, we can go ahead and try to find out what the derivative with respect to time is going to be. So, we have

$$\frac{\partial s}{\partial t} = 1$$

So, we will be having RHS has

$$= \frac{1}{u^2}\frac{d^2}{dt^2}f^+(s)$$

$$= \frac{1}{u^2}\frac{\partial^2}{\partial s^2}f^+(s)$$

$$= \frac{1}{u^2}\frac{\partial}{\partial t}(\frac{\partial}{\partial t}f^+(s))$$

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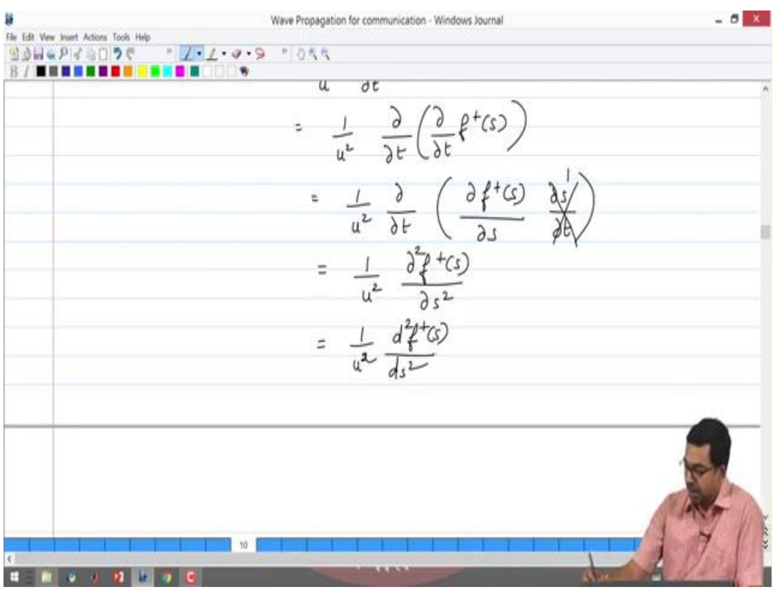

Using the change of variables, you will write in the similar way that we had done for the left hand side duo s by duo t fortunately is equal to 1, right. So, we can cross this term out and mark it as

1 right, now we need to find the second derivative right and we use the same process once again, right.

$$= \frac{1}{u^2}\frac{\partial}{\partial t}(\frac{\partial}{\partial t}f^+(s)\frac{\partial s}{\partial t})$$

The partial derivative can be replaced with an ordinary derivative and then when we look at the LHS and the RHS they are identical. So, it is definitely a solution to the equation.

One can also go ahead and plug the other part and try to see whether it is going to be a solution to this equation, but I will leave that to you as homework you can do out of your own curiosity.

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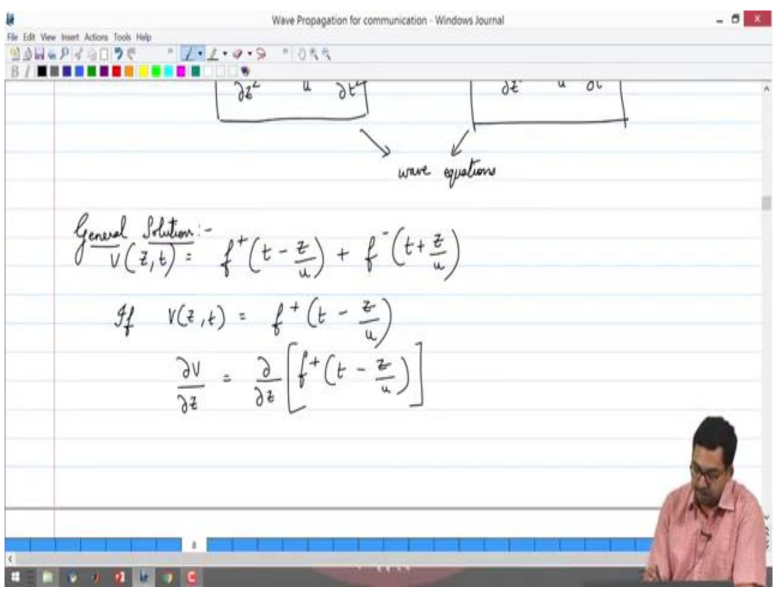

You can always take voltage to be $f^-(t + \frac{z}{u})$. And the general solution also says that a linear combination of these two will also work out as solutions to your partial differential equation ok, having done this, ok. Let us have a look at what the solution actually meant, ok.

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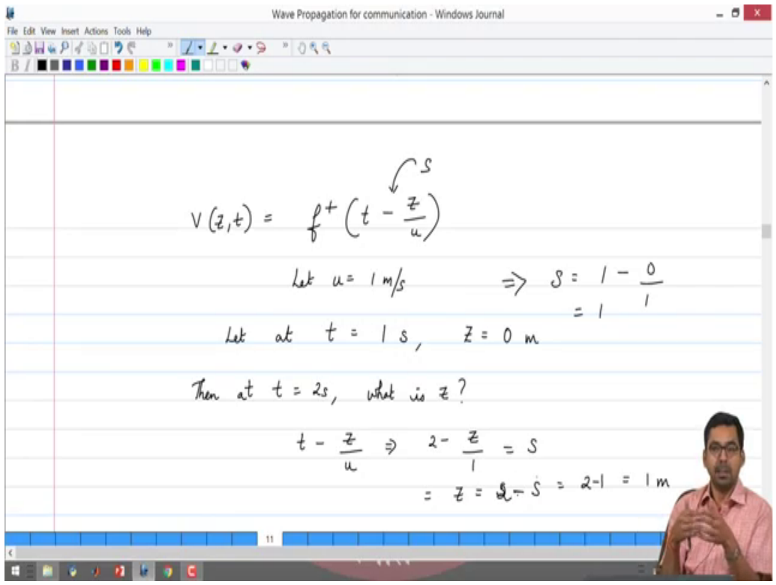

Let us take a scenario where you are representing your voltage which is a function of position and time to be $f^+(t - \frac{z}{u})$ right, and this

$$t - \frac{z}{u} = s$$

Now, let us assume that the velocity of the signal that we are going to be having in the line dictated by l and c of the wires that we are going to be choosing to build your transmission line, let us say it is 1 m/s, ok.

So, wild assumption, but let us make that assumption to just understand this solution, right. Let u be 1 m/s ok and let us say that the moment you have switched on the battery and you have connected it to the transmission line. You are marking some spatial coordinate on your transmission line and along the position is going to be z, ok.

Let us say that at some instant of time ok, let us say that at t equal to 1 second; let us say that at t equal to 1 second the position is going to be let us say 0 meters. This is the place where the peak of your voltage is at the time you are measuring, right.

So, you are taking the transmission line and you have switched it on and after 1 second you are seeing that the voltage has come to z equal to 0 according to you right from there you want to measure what is going to happen. And one of the ways to do this is we can say that at a later time, right at t equal to 2 seconds where would this voltage be would it have traveled along the transmission line to the right or to the left, if it has traveled to the right or to the left how much it has traveled? That is the question, right.

So, we can merely substitute t equal to 2 seconds over here, right. So, you can say that t is equal to 2 seconds; so, you will have $t - \frac{z}{u}$ to be fixed u is fixed over here, right. So, we can say that 2 minus z by 1 right, in the original case you should have calculated what is this $t - \frac{z}{u}$ going to be, right. So, for this choice of u equal to 1 m/s s is going to be equal to 1 minus 0 by 1 just going to be equal to 1 and since this voltage is a function of this quantity assuming s to be constant, right.

So, we have to say that this is also going to be equal to s right, that is the definition of s, s is going to be t minus z by u at all instances of time at all positions for a given velocity 1 m/s. Then what happens is you will be having z is equal to 2 minus s sorry, that is equal to 2 minus 1 that is the position 1 meter, ok. So, when we took this solution part which was

$$V(z,t) = f^{+}\left(t - \frac{z}{u}\right)$$

What we are seeing is that as time increases, ok as time increases from 1 second to 2 second the position at which the voltage was is actually traveling to the right side of your z axis, alright. And if we mark the source to be z equal to 0 and the load to be z equal to some length l, it is actually going towards the source of the load all right. So, this kind of a solution we call it as a forward traveling wave or a forward voltage in this particular case, which means that the other solution part that you had is going to be known as a backward travelling or a backward voltage wave.

And the general solution is telling, you that at any given space at any given time you are going to be actually having a superposition of a voltage that is going forward, and also a voltage that is going backward which is a very very weird solution that we are getting which would not conventionally happen in low frequency circuits and understanding this takes a little bit more time, alright. One can always argue where is this backward voltage coming from if your battery is positioned on one side and your load resistor is on the other side, alright.

Even worse a question can be asked that if your transmission line is going to be infinitely long then it would never reach the end and then actually come back would you have a backward traveling wave or not, ok. So, there are many solutions which are possible for an infinitely long transmission line there is no question of a backward traveling way, because the wave has to travel to the end and then something has to happen and then the backward wave has to be created and you will have to measure a superposition of this forward and a backward wave.

So, when we talk about an infinitely long transmission line, we usually mean that there is no backward traveling wave at all there is only a forward traveling wave ok, but in practice many of the systems that we put together do not have infinite length. Also there is a voltage wave that goes from your source to the load, but we already know that from circuit analysis you have something known as maximum power transfer theorem which says that maximum amount of power can be transferred to the load only if there are impedance matching or resistance matching conditions.

If it is not there then you will not be transferring maximum power, which also means that the remaining power will have to come back through your transmission line. And so, when we talk about a solution having a forward and a backward wave usually it means that it is a finite length transmission line and the impedance may not be matched at the load end. So, you are having some kind of a reflection that is happening at the load end which creates the backward wave. So, for all these classes we are going to be assuming that we will start with the forward wave provided that there is an impedance match or an infinitely long transmission line on the other end, you will have no backward wave.

But, in realistic conditions you are going to have finite length transmission lines with terminations or load resistors which are not going to be matched in some aspects. Then, it triggers a backward wave and at any given point of time you will be measuring the sum of the forward and the backward wave, ok. What that means, in terms of system design is something electrical engineers will need to worry about all right usually when we are designing transmission lines cables or you know tracers to carry some amount of voltage.

We have a safe value of voltage and we say that this is the rated voltage that your transmission line can carry. In low frequency circuit analysis, most of the time we talk about only this forward voltage alright we try we talk about the transfer from source to load, we do not talk much about what happens when it is getting reflected back, but here it is very critical.

So, this means that we have to design in such a way that the entire voltage which is traveling from source to sink, actually has a possibility to get reflected back. And then, travel towards the source backwards while the source is actually supplying some constant voltage. So, it means that at any given point in your wire there is a good possibility that you may end up with double the voltage, ok. This means that you will have to design the insulation between the top and the bottom layers of your PCB to be able to not break down with that voltage, alright and the heat dissipation capacities of your lines have to be determined based upon these things, ok.

It is also quite possible that the voltage that is traveling forward gets reflected at the load end and travels back, but it has the opposite sign. It is quite possible that you are actually sending a voltage pulse from source to sink, but if you are using a multimeter or some other technique to measure what is the voltage across the transmission line at a given position. It is not uncommon for you to read a 0 because the voltage that is traveling forward may have got inverted in sine and actually traveling backward, alright.

So, the conditions that dictate whether the voltage wave is going to travel forward only or it is going to travel backward, alright. If the entire thing is going to travel backward if the backward voltage is going to have the same sign as the forward voltage or inverted sign are all governed by what are known as boundary conditions and we need to understand these boundary conditions very well, ok.

Now, in order to understand the boundary conditions, the best way is to actually have a computer simulation working for this which we will be building, ok and try to use different kinds of conditions on the load end and observe what happens, ok. In order for us to do this there is going to be some fundamentals that we will have to revisit. So, we will start with that now and we will go over it in the next class, and in the following class the aim is to be able to build a simple program that will model propagation in a transmission line, ok.

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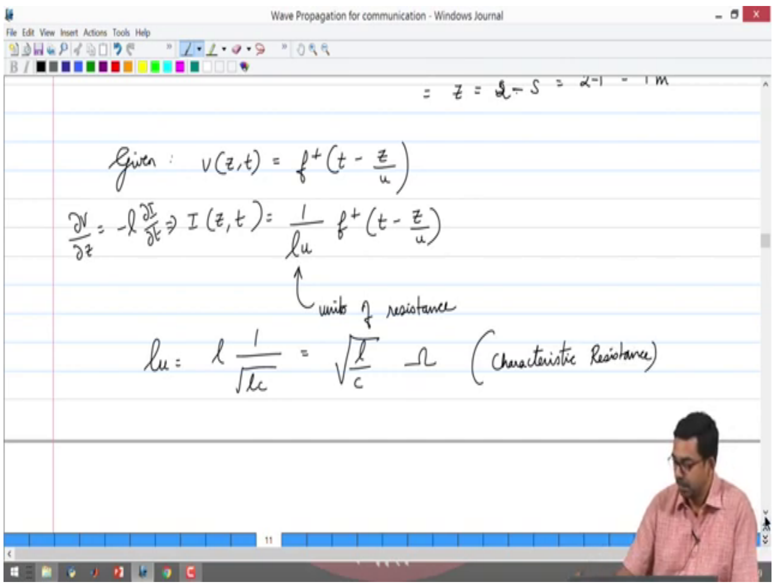

So, just to go ahead a little bit with this.

One can always ask a question that given that your

$$V(z,t) = f^{+}\left(t - \frac{z}{u}\right)$$

One can always try to find out what the current is going to be right. So, can always ask if the voltage is going to be of this form what will be the current in your transmission line ok this can be found out by the equation that we have which is coupling the current to the voltage. So, you have

$$\frac{\partial V}{\partial z} = -l\frac{\partial I}{\partial t}$$

So, you can always use that equation and arrive at the current, and the current looks like

$$I(z,t) = \frac{1}{lu} f^{+}\left(t - \frac{z}{u}\right)$$

So, this is coming from

$$\frac{\partial V}{\partial z} = -l\frac{\partial I}{\partial t}$$

Now, once you have an equation it is very easy to do a dimensional analysis of the left and the right hand side, ok.

Now, if you have a look at this the left hand side I of z comma t it is going to be off the form of the amperes alright which means that on the right hand side you need to expect amperes. We already know that $f^+\left(t - \frac{z}{u}\right)$ is the form for voltage. We already know what ohms law is going to be; so, lu needs to have the units of resistance, ok. So, we will write down, ok.

We already saw that the velocity is going to be

$$u = \frac{1}{\sqrt{lc}}$$

$$lu = l\frac{1}{\sqrt{lc}} = \sqrt{\frac{l}{c}} \quad ohms$$

It is a very very peculiar result that we will not see conventional low frequencies circuit analysis, right. Here, in the transmission line seems like square root of impedance, means inductance per unit length divided by the capacitance per unit length is actually having the unit of ohms which is a resistance, ok.

So, it gets a little confusing why you are having some passive elements which do not actually dissipate any power to look like resistance, ok. So, some results are very very counterintuitive that is why we are going over it step by step. Now, if you also have a look at this lu and say that it is

$$lu = \sqrt{\frac{l}{c}}$$

and the unit is ohms we immediately know that for a given geometry of the transmission line.

Say, if it is a PCB trace with a ground plan at the bottom or if it is some kind of a 2 wire transmission line depending on the geometry, you are going to be having some inductance per unit length, some capacitance per unit length. Suppose you have made a choice of the transmission line which has a specific l and specific c, ok.

Then it is somewhat of a characteristic of that transmission line to exhibit a resistance even if it is lossless, ok. So, this quantity to distinguish it from conventional resistance in low frequency circuit's people call it as characteristic resistance and it is usually represented by R_c to distinguish it from conventional bulk resistors, right.

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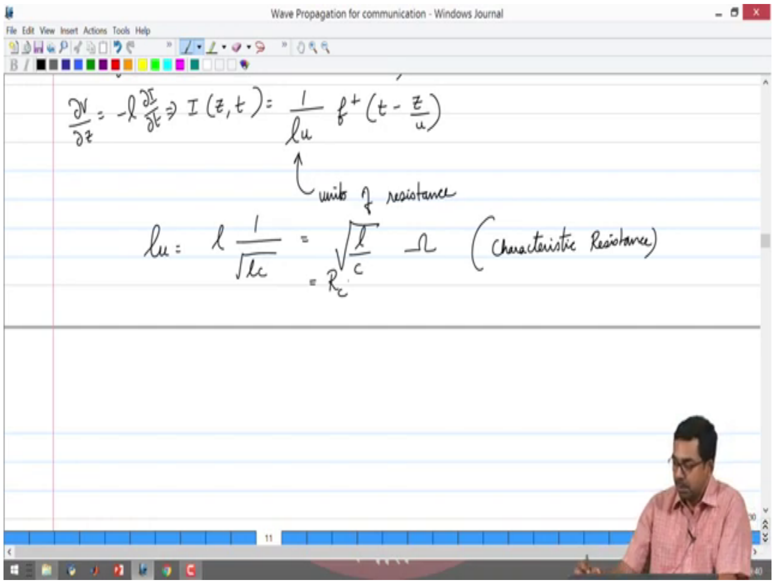

So, when people talk about characteristic resistance it means that they are talking about $\sqrt{\frac{l}{c}}$ for a transmission line. So, now your circuit diagram is going to look a little bit more complicated than before, right. So, if we visit our circuit diagram for the simple case, we will need to start adding quantities to it, alright. In order to say that this is a transmission line you will have to add what is l equal to, alright.

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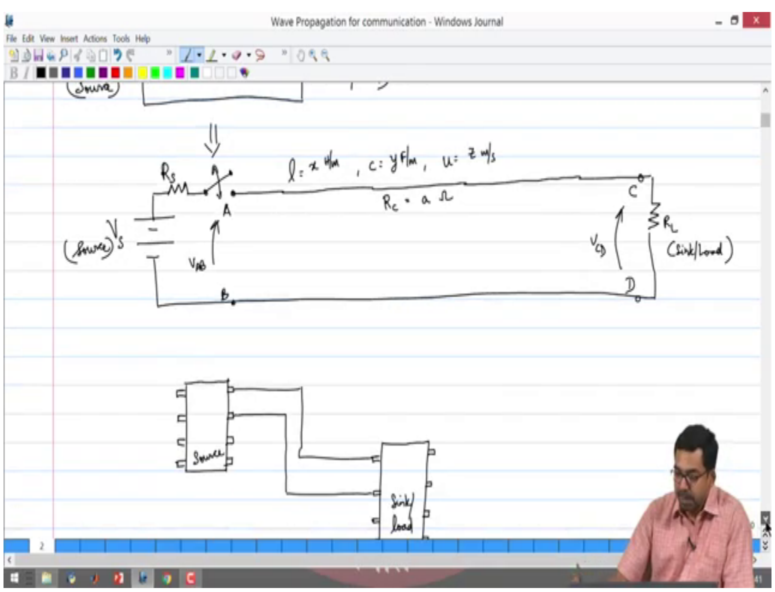

So, l would be equal to some x henrys per meter, c is equal to y F/m you will say that velocity. So, z m/s can also say characteristic resistance some a ohms, alright.

So, previously in your low frequency circuits you did not mark quantities for a short circuit connecting a source and a load, but now you will have to start putting in quantities to represent that it is a really long line that is connecting the source and the sink. And to signify that in your circuit diagram you will start marking some quantities, you could give l, c, u, R. And, in many cases since we will talk about some finite transmission lines they will also mark the length of the transmission line over here, ok.

So, these are some subtle ways of telling that you are not dealing with a simple low frequency circuit, but you are actually dealing with the high frequency transmission line and you will have to take into account all these parameters. ok. To summarize what we are seeing now, from the telegrapher's equations it is possible to decouple the voltage and the current and we can write down wave equations for the voltage and current. At dimensional analysis of the left and the right hand side tells you that the velocity is given by $\frac{1}{\sqrt{lc}}$, right.

And the general solution looks like

$$V(z,t) = f^{+}\left(t - \frac{z}{u}\right) + f^{-}(t + \frac{z}{u})$$

One could substitute some values of t and u and try to find out z as it changes with respect to time and you will find that for $f^{+}\left(t - \frac{z}{u}\right)$. It will appear to be going forward from source to sink, and for $f^{-}(t + \frac{z}{u})$ will appear to go from load back to the source, ok.

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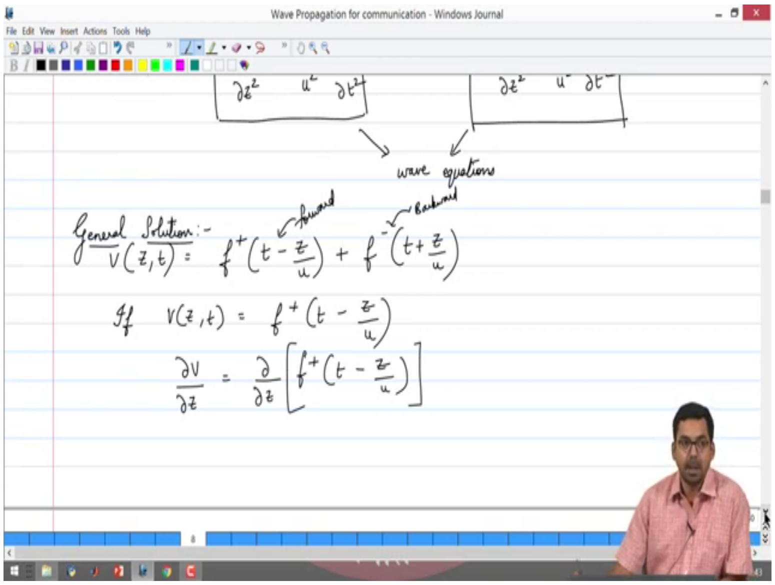

So, correspondingly we would call these as forward and backward voltage waves. Now, in order to prove that these general solutions are valid, one approach is to simply substitute them in the second order partial differential equation that we have obtained using just the wave equation. And by using a change of variables you will be able to show that LHS is equal to RHS and; that means that it is in fact a valid solution.

There is some example over here which we have seen to prove that actually z changes with respect to time and it is changing from source to load for a forward wave the same thing can be done for a backward wave. What is also unusual is for a given voltage of this form $f^+\left(t-\frac{z}{u}\right)$, the current looks like

$$I(z,t) = \frac{1}{lu} f^+\left(t-\frac{z}{u}\right)$$

according to ohm's law the unit has to be that of resistance, and this resistance is also given by $\sqrt{\frac{l}{c}}$. To distinguish this resistance from bulk resistors that we connect in low frequency circuits people call this as characteristic resistance. It depends upon the geometry, the material that you choose for your transmission line and for a given transmission line this can be some fixed quantity, ok.

So, we will stop here for the next class. I want to now go over some absolute basics for solving partial differential equations using computers. Now, you have seen an analytical solution, substitution and LHS is equal to RHS, but in order to convince ourselves and visualize more correctly, it is very important that we take a completely different approach.

So, the next class I will be starting with basic maths that will be allowing us to use a computer to program and find out solutions to partial differential equations, ok. So, I will stop here and try to talk to the TAS if you need help in installing octave. So, I expect that you will be installing octave on your laptops and I will tell you one class ahead whether you have to bring your laptops or not.

And, the way we will be doing these exercises I will be coding in octave in front of you and I expect you to follow, ok. And at home you can go back deliberately tweak the code, make intentional mistakes and try to see what happens, alright and that way I think your intuition will be building up faster than just with analytical solutions, ok.

I will stop here.