Computational Electromagnetics
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Review of Maxwell's Equations
Lecture – 3.2
Boundary Conditions

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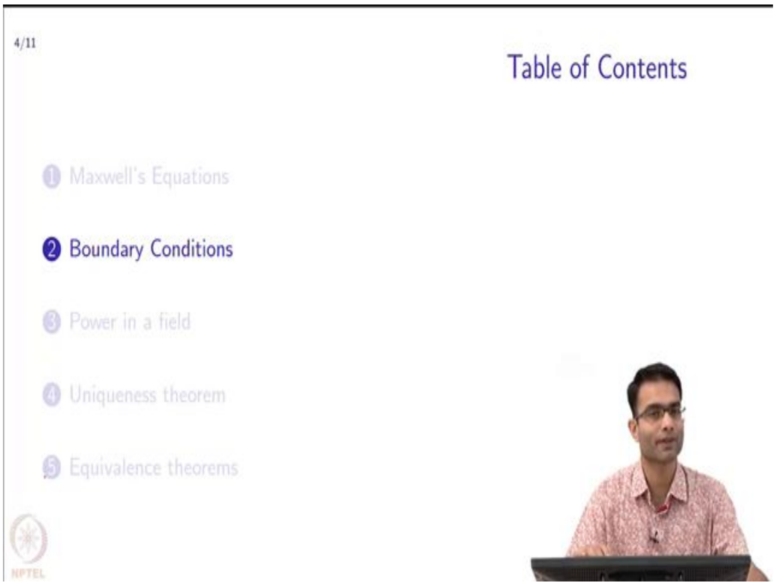

So, now let us move on to Boundary Conditions, again this should be a refresher for most of you.

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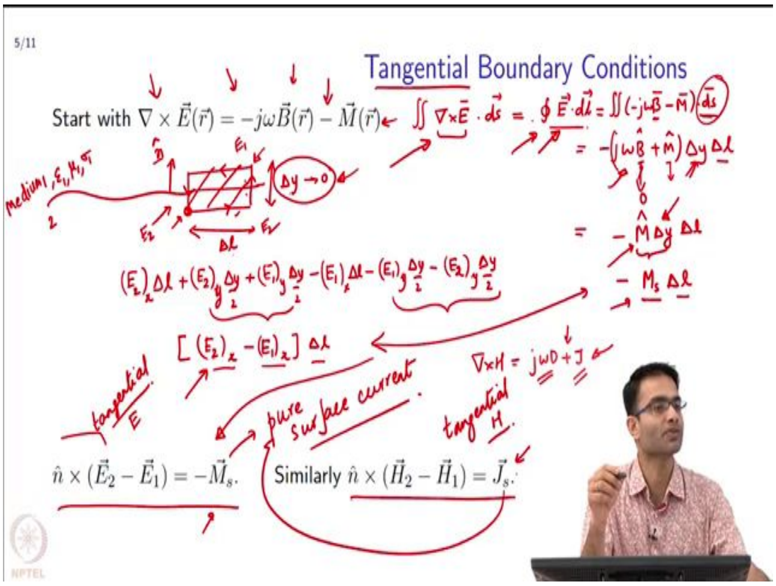

So, we will start with what are called as tangential boundary conditions ok. So, in tangential boundary conditions, I want to get a relation between the tangential electric fields or tangential magnetic fields as the case may be right. So, let us take a medium over here. So, let us call this medium 1 and medium 2 and as I say medium 1 and medium 2 so medium. What is different over here, what differentiates one medium from another medium?

Student: ε, μ

ε, μ and σ.

Student: σ.

Right so that is we can say ε_1, μ_1 and σ_1 and similarly for medium 2 ok. And it is a surface; so surface is going to be characterized by some normal vector over here ok. So, let us say that this is continues over here and I want to get a relation between the tangential fields. So, we have all done this in undergraduate right. What do you do? You take a small little wireframe that straddles the boundary.

So, it is half above the boundary half below the boundary ok, let us take something like this let us call this Δy and let us call this Δl. So, if I want to let us say get a relation between the electric fields above and below, the boundary that is my objective. So, what do you suggest I do?

So, I want the electric field along this curve and what I am going to do is I am going to make Δy tend to 0 that is what will give me the relation just above and just below ok. So, I want to go along the boundary so, which theorem.

Student: Stokes theorem.

Stokes theorem is what is going to help us right. Stokes theorem needs a curl and I see Maxwell's equations over here sitting with a curl like readymade for me. So, what do I need to do to this equation to get it into Stokes form? I need to integrate that is right, but what kind of integration?

Student: (Refer Time: 02:28).

The.

Student: Line integral.

The line integral, but I do not see a line integral if am going to get a line ok, line integral is there on one side of stokes equation what is the other side?

Student: Surface.

Surface integral right so, if I take this surface over here right. It is a open surface bounded by this boundary right.

So, I can say, I am going to integrate over the surface and what way is the surface normal pointing the surface normal is coming out of the board ok. So, I can do this $\nabla \times E.dS$ right and this is my stokes theorem helps us and this is what I get $E.dl$ ok. Now when I look at the right hand side I must I can substitute $\nabla \times E$ from this equation over here right. So, it will, there is no special way to simplified it will remain as it is ok.

So, this so this will become $(-j\omega B - M).dS$ ok. So, so far all I did is I took my Maxwell's equation over here and integrated over this shaded surface over here that is fine right. And

Stokes theorem simplifies this over here as $E.dl$ over the contour. So, when I start from let us say let me start from this point over here and work my way all the way back over here. So, we will assume that these Δl and Δy quantities are small such that the electric field is constant along these edges alright. So, I have E_1 over here and I have E_2 on this side.

Student: (Refer Time: 04:20) end cap.

$\hat{n}$ is just the normal vector for this surface that defines the boundary ok, because whenever I define a interface I how do I define the interface? I give the surface normal the interfaces define right.

So, when I start from this point over here $E.dl$ is going to give me what? So, I will get E_2, but not just E_2 I will get the tangential part of E_2 right. So, E_2 tangential multiplied by Δl then I finished this part then I move to the next segment, what do I get? Plus E_2 tangential.

Student: (Refer Time: 05:05).

So, let us call this so instead of calling a Δt let us call this E_2 along say x direction right, and this E_2 is going to be along y direction. And how much length?

Student: $\Delta y/2$.

$\Delta y/2$, similarly if I go further I am going to get $(E_1)_y \Delta y/2$ next term.

Student: $-E$.

$-E_1$

Student: X delta.

$-(E_1)_x \Delta l$ then minus.

Student: E 1.

$(E_1)_y \Delta y/2 - (E_2)_y \Delta y/2$. So, what I have done? I have calculated this line integral piece by piece. So, I am going to take the limit; this limit Δy tending to 0. So, which so all the Δy terms over here; they are going to become, they are going to tend to 0 ok. Another assumption we are making is that these fields are physical quantities they are not going to blow up to infinity.

So, when a finite term is multiplied by a vanishingly small term, I can get rid off it right. So, what I will be left with is $((E_2)_x - (E_1)_x)\Delta l$ right and on the right hand side what do I have left now right. So, I can say that so this for example, I can write it as $-(j\omega B + M)\Delta l$ that is the surface and we are assuming that we are talking about the component of B and M that is along the ds vector ok.

So, let me so is that is clear right because I have to take the dot product between this vector and this vector over here. So, whatever survives along this that is coming out of the plane of this page is surviving in this term that is the only term that is retained ok. So, if you want you can put a hat over here. What can I do about these term or what can I say about these terms? Which terms will survive which terms may not survive? What about the first term? Magnetic field multiplied by Δy, Δl or $\Delta y \Delta l$; would it be 0 or nonzero as I take the limit Δy tends to 0. It is a magnetic field, it is a physical quantity and we do not expect a physical quantity to blow up. There is no; there is nothing special happening at this, it is just a boundary between two interfaces.

So, what can I reasonable assumption to make is that as I shrink this Δy to 0, I have a finite quantity multiplied by a vanishingly small quantity right. So, this guy is going to actually tend to 0. What about this current over here? It is a magnetic current, forget for a moment whether it is physical not physical its some quantity over here. Will can it be finite or can it be in finite. So, if it is finite; that means, it is this contribution we will tend to 0 right. So, that is one case easy to handle, the other case is supposing it is infinite how can it be infinite?

So, infinitely conductive material right; so, in a like we take a perfect electric conductor and ask you where is the current on the conductor, in how much thickness is it in, what would you say?

Student: σ.

So, no it is a perfect conductor σ is infinity, then what will you say where is how much thickness is the current flowing in?

Student: It is not at the (Refer Time: 09:10).

0 almost right, it is a pure surface current. So, if it is a pure surface current only then in the sense that at the interface itself its value is infinite right, but this M into Δy that will be a finite quantity ok. So, that is what is called a pure surface current ok. So, if I keep this over here the only term that has a possibility of surviving is $-\hat{M}\Delta y\Delta l$ and this term I will call as M_s ok.

So, to sort of state that again this; $\hat{M}$ if it is a pure surface current is going to be very very large, but this very large quantity multiplied by a vanishingly small quantity is what is going to give me a finite surface current. We are not used to thinking about it in such threadbare means, but that is how we arrived at this equation right. So, when I compare both the left hand side and the right hand side what I get is. So, this and this I am going to compare ok. So, this $(E_2)_x$ this is nothing, but the tangential part of electric field above or below the medium and $(E_1)_x$ is the tangential part of the electric field above the medium ok. There is a Δl here and there is a Δl here, these will cancel off.

So, when I combine these two equations I must combine them keeping the vectors in mind right. So, when I combine these two equations you will see this is the equation that I get ok. So, you see this that is why I defined this n over here. So, let us take a electric field over here let us say, like let us say E_2 is pointing like this right. So, $E_2 \times n$ if E_2 is in the plane n is in the plane where is $n \times E_2$ pointing? Outside the plane; will it be tangential to the boundary? n cross any vector will be perpendicular to n and n is perpendicular to the boundary.

So, $n \times E$ will always be along the boundary right. So, this is one very clever way of using vectors to give us the tangential field immediately, the high school way of writing it is like this, $(E_2)_x$ or (E_2) tangential. That is how you would see you know $E_{tan\,1} = E_{tan\,2}$, but we can write it in a more sophisticated way now using the language of vectors. So, just to summarize $n \times E$ is going to be perpendicular to n. And what is perpendicular to n? The interface; so, it is the tan so this what is written over here this is exactly the tangential fields and what is left on the right hand side is simply this thing. So, I will call this the pure surface current all right.

So, I can repeat the same exercise for the second Maxwell's equation right; again there what do I have? I will have $\nabla \times H$ is equal to. So, $j\omega D + J$ right I can mimic the same proof line by line the only thing that will survive. So, this guy is D is a physical quantity or displacement field when I will stick it into a surface integral I can say it is the contribution will vanish I will only be left with a J and this time J is carrying a positive sign in this equation J was carrying a sorry M was carrying a negative sign. So, you can see that this relation is what I will get for tangential H fields.

So, this is tangential E and this is tangential H fields right. So, what is this again over here this is also a pure surface current pure surface currents means it lives in the surface in a vanishingly small thickness, only then can it contribute. So, if I take a non conducting medium unless you take you know glass wall floor whatever right you do not they are there are no its not they are not perfect conductors. So, in that case what will you, what is the most commonly used way of expressing boundary condition? Tangential fields are the same right, that is because these guys J s and M s are 0.

So, you get $E_{tan\,1} = E_{tan\,2}$ and we write that in the language of vectors using $n \times$ have you seen this derivation before very good. So, this is as far as tangential boundary conditions go the other. So, you notice that we use the first two equations of Maxwell right. So, you can ask what happens when I use the remaining 2 equations right.

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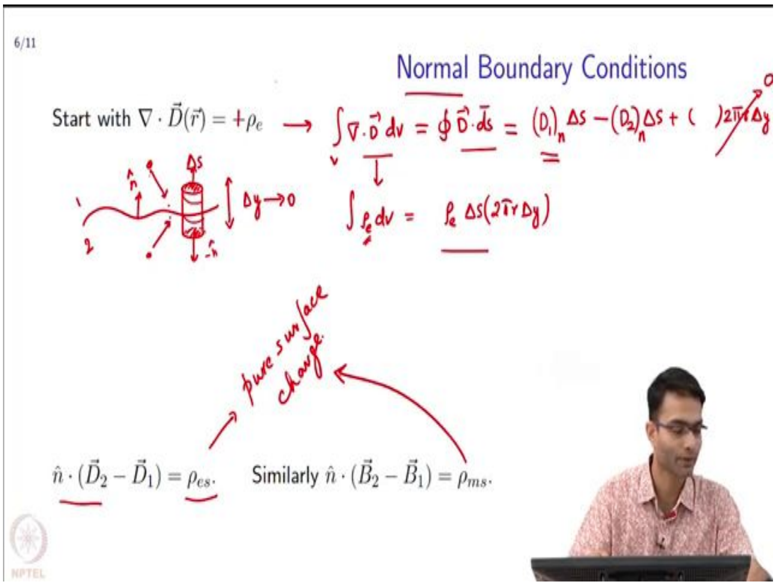

So, that takes us to normal boundary conditions and I will use. So, I think I made a mistake over here it should be plus right normal boundary conditions over here.

So, again I have an interface over here, this is medium 1 and medium 2 what is the way of a finding a boundary condition over here if anyone remembers?

Student: Cylinder.

Take a cylinder right. So, I take a cylinder like this ok. So, this is a medium 1 and 2, the hint for whether you take a open surface or a closed surface how would you get that hint? That hint comes from looking at the.

Student: (Refer Time: 14:59).

Integral form of Maxwell's equations right, over here when I did this equation over here in integral form I got I was taking curl.

Student: Curl.

Curl I know by stokes theorem is going to convert to a line integral. So, I needed a open surface, when I look come to this equation I have a divergence term and I know that the theorem that applies to divergence has a volume integral. So, I think of a volume pill over here. You can try using applying this theorem on a wire frame like the previous example, you will find that you do not go anywhere. So, if I take this over here then as you already mentioned the theorem that we will use is divergence here right.

So, if I take a volume integral on both sides. So, $\nabla .DdV$ and this is going to give me the flux of D right flux over the closed surface right, $D.dS$. So, I have a Δy over here and the ΔS is the area over here right. So, exactly similar to what we did in the previous example I will calculate this integral on by both ways. So, in this case maybe we do not need to go through the entire derivation.

So, what I what we can just sort of intuitively see $D.dS$, how many surfaces should we break this up into to evaluate it. So, it so this for example, this is one surface the top cap then the bottom cap 1 right and then there is going to be some contribution from the curved surface and I know that contribution from the curved surface what will happen to it.

Student: (Refer Time: 16:47).

It is going to go to 0, because I am going to set the take the limit that Δy is going to 0 I want to squeeze this field as much as possible so I get a relation between the fields here just above and just below. I do not want the relation between field here and field here that is all you know you it is not a boundary condition I want to shrink these guys these guys down to as close to the boundary.

So, what will survive will be whatever is contributing from these caps over here. Now what is the surface normal for the cap over here is like this right that is what is content will contribute to the outward flux, what is normal to the surface is flux. So, as I shrink this delta y down to 0 the normals to this to these caps will be along the normal to the surface itself right to the interface, in this case it will be plus n in this case it will be minus n hat right.

So, when I do this accounting over the 3 volumes over here I will be left with for example, I just write it over here $(D_1)_n$ component multiplied by ΔS and $-(D_2)_n$ component ΔS and plus 1 term which is let us say $2\pi r\Delta y$ right that is the curved surface area and this term is going to go to 0 right. Then the remaining tricky part is to look at the right hand side over here. So, the right hand side is going to be $\rho_e dV$ right. So, how much charge is enclosed in this volume integrated over dV and this dV is going to be $\rho_e dV$ I can write simply as ΔS and $2\pi r\Delta y$ right. That is the volume of this cylindrical box right. Now what can you say about this whether will this be 0 or not?

Student: There is no integral.

There is no there is no integral over here we got rid of it, yes. So, will this be 0 or nonzero and what condition will it be 0 or nonzero.

Student: (Refer Time: 19:24).

Student: (Refer Time: 19:27).

Exactly; so, just like in the case of the current discussion if I had a volume charge distribution a charge that is distributed through the volume then what will happen is the finite charge residing in a volume that is shrinking to 0; so its contribution will be negligible. On the other hand if I have a pure surface charge that can again happen for like perfect conductors who take a perfect metallic conductor put charge on it, where does a charge live purely on the surface right.

So, the only possible way for this term to survive is if the charge is a pure surface charge. So, this is a pure surface charge and $\hat{n}.D_2$ and $\hat{n}.D_1$, this captures the normal component of D_1 and D_2 that is the only part that will survive the dot product with in an. And I can repeat the same logic with the second divergence relation and I will get. So, this is also a pure magnetic charge ok, is that fine. So, keep these techniques in mind you will have to apply them depending on the problem at hand and these like you can see we are just using the tools of vector calculus to simplify a surface integral into a line integral or a volume integral into a surface integral that is the basic idea over here ok.

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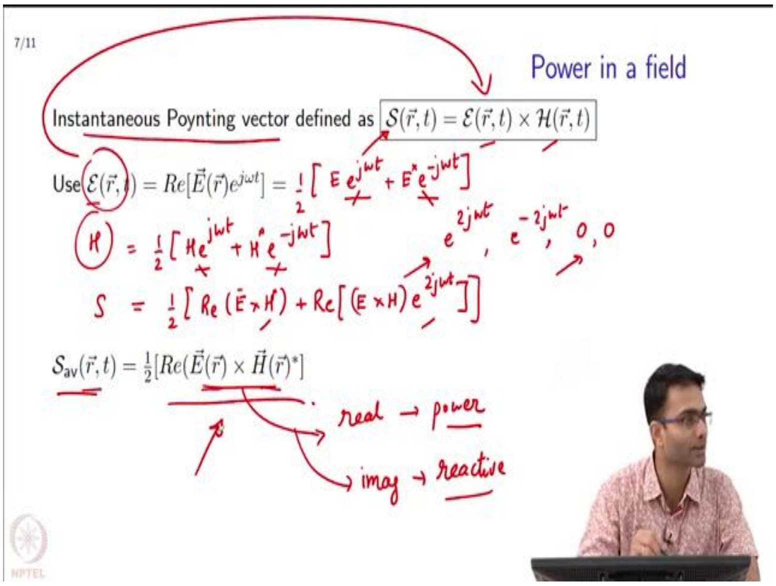

So, let us look at what is the power in a field. So, again this should be familiar to all of you have what is called the instantaneous Poynting vector the definition of instantaneous Poynting vector is simplest when I look at real valued physical quantities ok. So, in terms of that its simply the $E \times H$ ok, it does not get simpler than that. This expression gets a little bit more complicated when I introduce my phasors right. So, I want to now find out, I want to express this Poynting vector in a phasor notation because I am going to do all my calculations in phasor I should have an expression for power in terms of the phasors right.

So, we have already done this the real valued quantity is expressed in terms of the real part of the phasor right. So, if I asked you to simply this what how would you write down real part of a complex number.

Student: (Refer Time: 22:04) cos.

Cos is one way, another way would be supposing I give you a complex number z and I asked you give me a real part of z.

Student: (Refer Time: 22:13).

$(z + z^*)/2$ right. So, I can also write this as half of; so, $(Ee^{j\omega t} + E^* e^{-j\omega t})/2$ ok. So, that is as far as the electric field goes. Similarly I can do the same thing for H I will write it as

$(He^{j\omega t} + H^{*}e^{-j\omega t})/2$ ok. So, what do I do next is I am going to take this expression and this expression and put it into the expression for pointing vector and we will see what we get.

So, when I take the cross product between these two quantities how many $j\omega t$ terms will you observe? So, for example, when this guy combines with this guy what will I get? I will get into the $2j\omega t$; when this guy combines with this guy what will I get?

Student: Minus.

I will get $e^{-2j\omega t}$ similarly when this guy combines with this guy what will I get no.

Student: No.

Dc and what you get 0. And similarly, when this guy combines with this guy what do I get? Frequency terms are the phasor terms are 0 right. So, even without doing all the opening up all these expressions I know that these are the frequency terms that I will get. I will get a 0 I will get a $2j\omega t$ it is not possible for me to get a $e^{j\omega t}$ right you can just intuitively see it immediately. So, when you simplify this further let us write down s what expression you get is first term is this which is at 0 and the second term that you get is ok.

So, all the boundaries are brackets after this is what I get ok. So, you see that the power seems to have 2 components; one is at dc and one is at $2j\omega t$. So, if I take the average of phasors that is oscillating at twice the frequency has how much average? 0 right you integrate $cos(2\omega t)$ over 1 period the way we are undergone 2 fluctuations is average is going to be 0.

So, that is why the average power the only the dc term survives right. So, now, you have a very and usually we are interested in average power ok. So, average power is now given in terms of the phasors the $Re(E \times H^{*})$ ok. So, this is H^{*} and take half of it right. So, you can see that the power it this E cross H term over here E cross H star can be real can be imaginary right and we will have both parts the complex number. So, the real part is going to give me the power. And, what about the imaginary part? What do we call it people from circuits background.

Student: Reactive.

Reactive power so, this concept of real power and reactive power will be very interesting when we look at the radiation from antenna. In the near field of an antenna most of the energy is reactive, as I go into the far field we will find that it becomes more and more real ok. So, it is not just for theoretical fun its actually happening in nature these things ok.