Computational Electromagnetics

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FDTD: Materials and Boundary Conditions
Lecture - 13.10
Implementing PML using Coordinate Stretching

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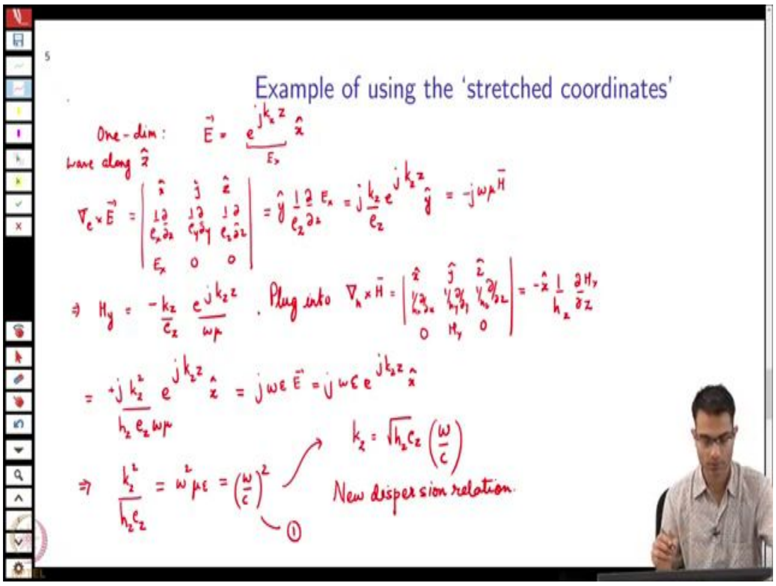

So, let us take a 1D example so, one-dimensional ok. So, the solution will be of the form $\vec{E} = e^{jk_z z}\hat{x}$. So, let us say something like this ok. So, one-dimensional wave going which direction is this wave traveling?

Student: (Refer Time: 00:40).

Right so, along $\pm z$ whatever along the z - direction is going. So, now, what do I have to do? I have to use those operators which I have defined and use the two Maxwell's equations ok.

So, the first equation will be for example, curl of E, this is the first equation that I use. Now, I write down this $\nabla_e \times \vec{E}$ operator right so, $\hat{x}\,\hat{y}\,\hat{z}$ so I will just write it ones for clarity. So, $1/e_x$ $1/e_y$ $1/e_z$ and then here, what do I write?

Student: (Refer Time: 01:33).

This is my E_x component right. Which component survives from here?

Student: (Refer Time: 01:46).

Right so, only the $\hat{y}$ direction, $\hat{y}$ component is going to survive and its value is going to be?

Student: (Refer Time: 01:57).

$$\nabla_e \times \vec{E} = \hat{y}\ (1/e_z)\ \partial E_x/\partial z$$

Student: Del by del z of E x.

Del by del z of ex right and that can further be simplified as.

Student: There is a.

There is z component, no because, d/dy of E_x is 0 right. So, this is just going to give me a $jk_z/e_z\ e^{jk_z z}\hat{y}$ and I know that this is going to be equal to my Maxwell's equations;

$$\nabla_e \times \vec{E} = \hat{y}\ (1/e_z)\ \partial E_x/\partial z = jk_z/e_z\ e^{jk_z z}\hat{y} = -j\omega\mu\vec{H}$$

Student: (Refer Time: 02:44).

So, my as expected electric field is along $\hat{x}$ magnetic field is along $\hat{y}$ ok.

So, the y component can be written as

$$H_y = -(k_z/e_z)\ e^{jk_z z}/\omega\mu$$

That is the y component of magnetic field. Then what should I do? If I want one more relation, I need to use the second Maxwell's equation right. So, I need to plug this into, this is where I will get now a relation between the e's and the h's small e's and the small h's, when I use this relation ok; so, again $\hat{x}\ \hat{y}\ \hat{z}$ right, $(1/h_x)\partial/\partial x$, $(1/h_y)\partial/\partial y$, $(1/h_z)\partial/\partial z$.

And what do what components do I have? $0\ H_y\ 0$ ok, what component survives from here?

Student: x hat.

Right so, I am going to get a $-\hat{x}\ 1/h_z\ \partial H_y/\partial z$. So, let us just we already have an expression for H_y . So, we will just substitute it over here. So, I am going to get a

$$\nabla_h \times \vec{H} = +jk_z^2/(h_z e_z \omega\mu)\ e^{jk_z z}\ \hat{x}$$

So, I just substituted the value of H_y that are already had, that gave me a $+jk_z$ plus j k z right and H_y already had a k_z . So, it begins $k_z^2\omega\mu$ was in the denominator $h_z e_z$ right. So, the algebra is fine. So, this is $\nabla_h \times \vec{H}$, what do I relate this to? Right, this whole thing should be by Maxwell's equations, equal to $j\omega\varepsilon\vec{E}$, which is $j\omega\varepsilon\ e^{jk_z z}\hat{x}$. So, all this simple hard work there is not much to it right, I just redefine del that is all that I did. So, if I compare these two things over here, what everything I mean a lot of terms cancel off.

For example, the exponential cancels off, the j cancels off, anything else cancels off? No right, I get a k_z^2 , denominators going to be my $h_z e_z$ let me take the rest of the terms on the other side, I am going to get an $\omega^2\mu\varepsilon$ and $\mu\varepsilon$ can be related to the speed of light, $1/c^2$ right. So, this is $(\omega/c)^2$ right. Earlier, what was our is this is like a dispersion relation, a relation between wave vector and frequency is dispersion relation. In plane free space, what was the relation? $k = \omega/c$. Now, what is the dispersion relation? Right so, this is now giving me a dispersion relation of the form that

$$k_z = \sqrt{h_z e_z}(\omega/c)$$

So, what is effectively happened? I have got a new dispersion relation, let us just say that right the relation between wave vector and frequency is now altered by some factor right. So, this is, you can say that this has no relevance to physical reality because, in physical reality $e_x\ e_y\ e_z$ are all 1 right, at least what we can say is that what we have derived is consistent right. Now, let us let us stick to this equation over here and $(\omega/c)^2$ right that is going to remain over here.

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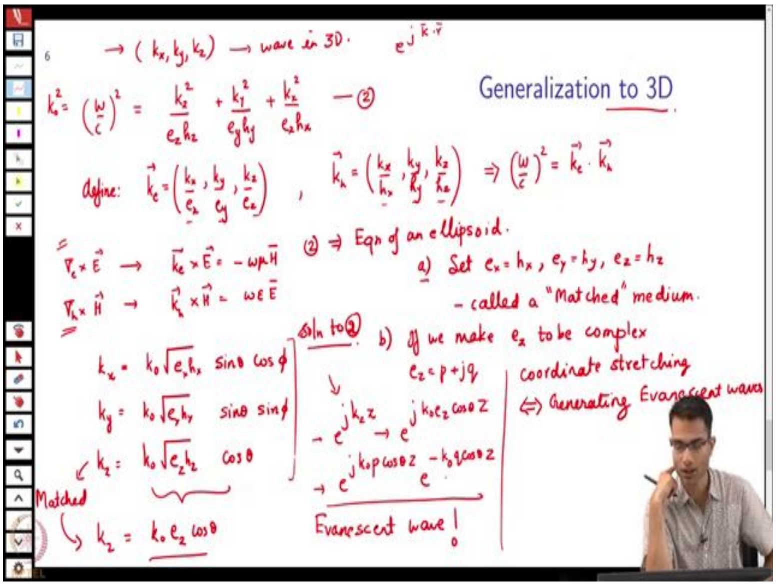

Now, let us go to 3D ok, what we did was in the 1D case, where there was only one h_z, but in general there will be sorry one k_z right that is the wave in one direction, but in general there can be a $(k_x\ k_y\ k_z)$ right this is a wave in 3D ok. So, then what happens? So, without doing the math, what do you intuitively expect? Equation 1; so, we will have $(\omega/c)^2$ will appear every always right. So, I will have an $(\omega/c)^2$ right. In the case of 1D, this was $k_z^2/e_z h_z$, what do you expect will happen? What happens when?

Student: (Refer Time: 08:55).

Under root yeah. So, when there was no coordinate stretching, what happened?

This became $k_x^2 + k_y^2 + k_z^2$ now, what will happen now?

$$(\omega/c)^2 = k_z^2/(e_z h_z) + k_y^2/(e_x h_x) + k_x^2/(e_x h_x)$$

So, I mean you can go and derive it, but you can see this is the expression that you will get right so, this we will call this our equation 2 right and this $(\omega/c)^2$ is what we called the sort of free space k naught square right, this was some physics $k_0 = (\omega/c)$, but now we are saying that oh this is actually, this is the new expression that we are getting. So, the other two

Maxwell’s equations that I mean the first two Maxwell’s equations that I had right this one $\nabla \times \vec{E}$ let's go back further right.

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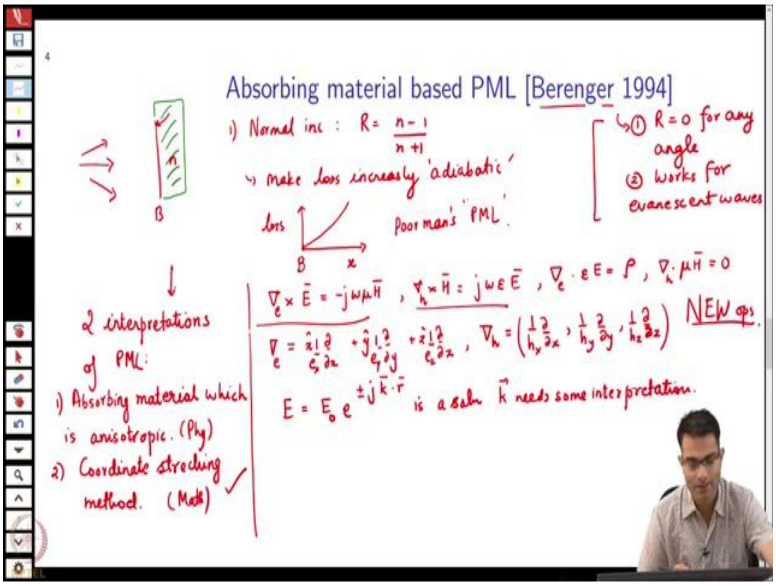

So, this expression and this expression, supposing I substitute the E in terms of this new waves that we have got, what do we get? So, you will get $\vec{k}_e \times \vec{E}$ right before that, let me do the following, let me define you notice that this k_x is always coming with a e_x and h_x in the denominator right. So, let me define two new wave vectors so, $\vec{k}_e$ corresponding to the $\vec{E}$ part and a $\vec{k}_h$ corresponding to the $\vec{H}$ part. So, what I do is, I define this as k_x/e_x , k_y/e_y , k_z/e_z .

And $\vec{k}_h$, I will define as same way k_x/h_x , k_y/h_y , k_z/h_z z, I am only introducing these two to make the math a little bit more. So, when I put these two together, what is, can I rewrite equation 2? Does it look like the dot product of something and something.

Student: (Refer Time: 11:32).

The dot product of these two wave vectors right so, that was the motivation behind this so, $(\omega/c)^2$ now becomes $\vec{k}_e \cdot \vec{k}_h$, just the simplification right and your first equation that I had the $\nabla \times \vec{E}$ this gets translated into remember when I have a plane wave the curl gets replaced by $j\vec{k}$ right we have seen this a simplification.

So, this simply will become $\vec{k}_e \times \vec{E} = -j\omega\mu\vec{H}$ the j gets canceled, the second equation curl h therefore, gets translated as $\vec{k}_h \times \vec{H} = j\omega\varepsilon\vec{H}$ ok. We notice that k h has exactly the $h_x,\ h_y,\ h_z$ terms. So, it is consisting with the definition of ∇_h and these $e_x,\ e_y,\ e_z$ they are captured in this ∇_e term ok. So, I am just in this make belief new world that I have gone into, I am just rewriting all the equations for simplicity ok. Now, looking at equation 2, what form of $k_x,\ k_y,\ k_z$ do you think I can say is a solution to this?

So, let us spend some time interpreting equation 2 so, equation 2 it is a three dimensional equation in the variables $k_x,\ k_y,\ k_z$ right and this $e_x,\ e_y,\ e_z$ all of these are numbers some numbers that I choose. So, what does equation 2 remind you of?

Student: Ellipsoid.

It is an equation of an ellipsoid. This ellipsoid gives me a whole family of Maxwell's equations for different values of e's and h's and I get back to reality, so called physical reality when that ellipsoid becomes a circle right or a sphere over here. So, there is a very simple parametric way of writing down the solution to an ellipse or ellipsoid, if you just use polar coordinates ok. So, you may have seen this earlier. So, I can write it as

$$k_x = k_0\sqrt{e_x h_x}\ \sin\theta\ \cos\phi$$

So, the radius I mean the right hand side is my omega by c whole square, which I am calling ${k_0}^2$ right and $\sqrt{e_x h_x}$ is like the length of the semi major axis for that and $\sin\theta\ \cos\phi$ phi. Similarly

$$k_y = k_0\sqrt{e_y h_y}\ \sin\theta\ \sin\phi$$

Now let you got the hang of it this should be

$$k_z = k_0\sqrt{e_z h_z}\ \cos\theta$$

So, now, this solution actually is what is going to lie at the heart of PML ok. So, now, let us because it does not seem that way, just seem that we have done some make belief world where we have introduced some e's and h's and solved this Maxwell's new Maxwell's equations, but let us look at the conclusion right. So, first a bit of notation so, if I set

$$e_x = h_x,\ e_y = h_y,\ e_z = h_z$$

This is called a matched medium it just terminology, having pose the problem in a very general way now, I begin to see where do I want to get to ok.

So, when I make the electrical parameters equal to the magnetic parameters, I call it matched medium that is just terminology, but the more important conclusion comes over here ok. Now, all the e's are equal to all the h's. So, take any one of these three solutions ok. So, for example, this solution over here, what will it become? $k_0 e_z \cos\theta$ that is what it becomes, under this matched medium condition right and usually we expect these wave vectors. So, I mean these are the wave vectors and they correspond to $e^{j\vec{k}\cdot\vec{r}}$. We are used to thinking of $\vec{k}$ as a combination of three real numbers and that gives me a wave traveling in a particular direction right. Now the question is this, if we make e_z to be complex.

What happens? What happen to my k_z right? So, this is my k_z for a matched medium. The wave becomes.

Student: (Refer Time: 17:27).

Right so, it will become what? So, supposing I mean so, let us say $e_z = p + jq$ right. Then a wave which was of the form $e^{jk_z z}$ become $e^{jk_0 e_z \cos\theta\, z}$. Now, k_z, I replace from here so, this becomes $k_0 e_z \cos\theta\, z$ and that becomes $e^{jk_0 p \cos\theta\, z} e^{-k_0 q \cos\theta\, z}$. What does that mean?

Student: Decaying.

It is a decaying wave right so this is, in other words, an evanescent wave. So, the conclusion I mean now if I when I try to interpret from here, my interpretation is, coordinate stretching is equivalent to generating a or representing an evanescent wave. How? By choosing my e's appropriately, if I chose e's to be some numbers.

Then I will always get a traveling wave, but if I chose e’s to be complex, I am able to get an evanescent wave right. So, this is the connection between the two interpretations that coordinate stretching is the same as generating evanescent waves ok. So, this is why I said that this a solution to equation 2, this is at the heart of PML because, by doing a so called coordinate stretching, effectively I am generating a medium where the waves are evanescent ok.

Further you can see that in each direction, I have independent knobs, in x I have e_x, in y I have e_y, in z I have e_z right. So, that corresponds to what? Anisotropic propagation because propagation constant is different in every direction and; I have control over each one of those. So, this is actually anisotropic lossy medium right. So, in the very beginning I had mentioned that, these are the two interpretations of PML over here you can see straight away how it is coming right and I did not have to deal with tensors or any such things, I got is just by using our usual Maxwell’s equations and stretching the coordinates right. So, this is the principle of PML the rest are all details. You know since its z since it is in k_z then it is z.

Student: (Refer Time: 20:54).

Where?

Student: (Refer Time: 21:00).

This equation over here?

Student: Yeah.

The second line I wrote x.

Student: (Refer Time: 21:05).

This is z still; it looks like x is it? You mean these two guys.

Student: Yeah.

We should write it bigger. The modulus of e_x should be less than 1? Well, it is up to me, how to design it.

Student: Because when you (Refer Time: 21:31).

Ok so good so, the question is how do we, what we have done so far is we have made the e's equal to h's and we have said that e should be complex, interpretation is of lossy medium, my next question is now ok, I want most specifics, what value should I set? So, I should look at the reflection coefficient and see how to make it zero and that itself will tell me what values of e to set, should the modulus less than one equal to one greater than one. So, we will come to that in subsequent.