Computational Electromagnetics
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Review of Maxwell's Equations
Lecture – 3.1
Maxwell's Equations

We will start at today's lecture with a review of Maxwell's equations. All of you have done Electromagnetics at some point in the course. So, this should just be a easy refresher.

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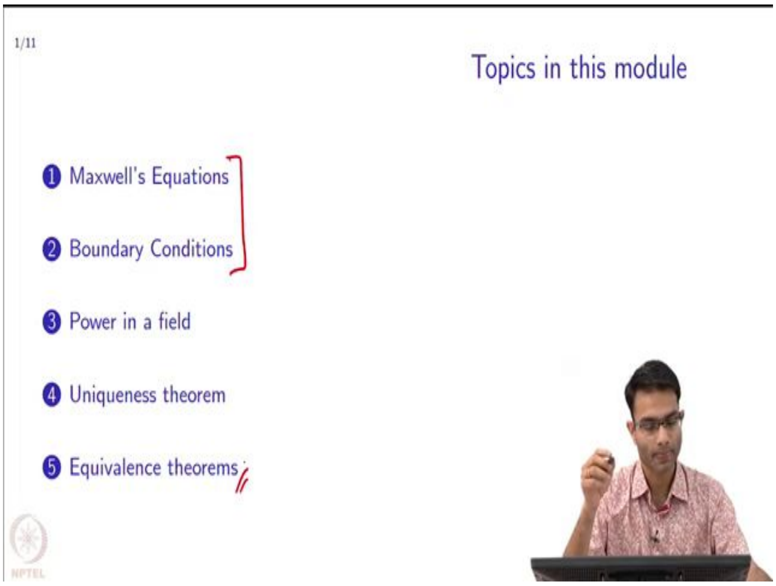

So, the topics that we look at will be starting with Maxwell's equations and then I go to boundary conditions. So, all of you know that in the electromagnetics problem to solve it fully; I need to not just know the equations of Maxwell, but also what are the conditions on the boundary ok. So, these two is sort of essential, then we will look at power, poynting vector and concepts like that and a theorem a couple of theorems which you may or may not have seen in an undergraduate course which is about uniqueness and equivalence theorems ok.

So, one thing I would like you to keep in mind as we go ahead in this lecture and in the course is that there are equations which are written for physical quantities currents and charges, what we will do is we will try to generalize it to make it in a mathematical way which will give us more power. Those equations may not always have a physical meaning, but the mathematical power that we get from it helps us to solve some problems and that will be covered in for example, in the equivalence theorems right. So, let us get started with Maxwell's equations which all of you know.

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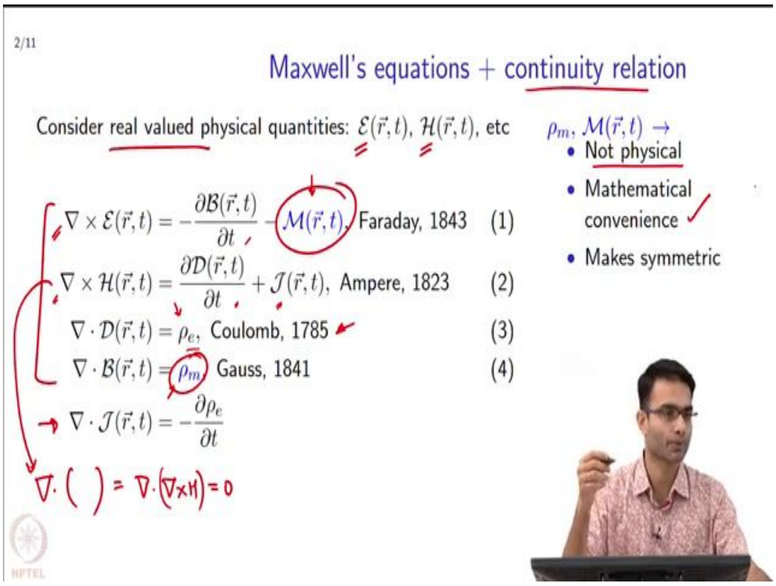

So, we will start with for example, the very early experiments which lets say Maxwell, Hertz, Faraday, Ampere all of these would have done and at that time they would have worked with real valued physical quantities, because that is what you measure in a lab. You measure voltage, you measure current right; you do not measure a complex quantity in the lab right that is a later mathematical abstraction. So, I am denoting these real valued physical quantities by this slightly different symbol like a script E and script H. So, just to tell you that these are real quantities and so as you saw from the previous lecture, Maxwell's equations were these four without the terms in the blue font ok, we all know everything in the black font already.

So, $\nabla \times E$ is rate of change of magnetic field and so on, there is the displacement current all of that is known to us right. So, if I look at these four the first four equations without the blue fonts. There is a sort of asymmetry in these equations right because if I look at rate of I mean the curl of electric field, there is a rate of change of magnetic field and no other term.

Whereas, in the second equation, there is a $\nabla \times H$, there is a rate of change proportional to electric field and the current.

So, there is sort of asymmetry in it and so to fix this asymmetry to give us some mathematical power later on, what we do is we add to quantities to make these relations between electric field and magnetic field symmetric. And what are those quantities? If J was a electric current then, M is a.

Student: Magnetic.

Magnetic current and if ρ_e is a electric charge ρ_m is a.

Student: (Refer Time: 03:20).

Magnetic charge ok; now as of now we say that these are not physical quantities ok, at least magnetic charge is not supposed to be a physical quantity, but there is actually no theoretical reason why magnetic charge does not exist. So, if any of you do find it there is a nobel prize waiting for you. So, that is about; so, these are the questions that we will work with ok. In fact, there is a paper by I think Dirac which says sort of mathematically shows that there should be a magnetic charge, but we have not found it ok.

So, if you are interested you can read more on that. Then there is another equation which is used alongside Maxwell's equations which is the continuity relation for charge. So, do you think; so, that is this equation that I have written at the bottom over here. Do you think that this equation is independent of the first four equations?

Student: (Refer Time: 04:19).

It can be derived right, how would we derive it?

Student: Take the divergence of the second equation.

Take the divergence of the second equation right. So, if I take divergence of the second equation over here, if I do $\nabla.$ this whole equation. So, we all know that so this $\nabla.(\nabla \times H)$ for any vector H this is identity equal to.

Student: 0.

0 right. So, that is it then you can see the left hand side becomes 0 and the right hand side has both the current term over here and $\nabla.D$; $\nabla.D$ we can use our Coulomb's equation and converted to ρ right. So, this is your continuity equation right, it is just written separately just for convenience, but it just comes from Maxwell's equations and vector calculus ok.

So, sort of to summarize what we have done is we have introduced these two new quantities over here for the purpose of mathematical convenience and also to give us a sort of symmetric right. Electric field and magnetic field are interchangeable if I interchange electric and magnetic, I have to interchange magnetic and electric currents magnetic and electric charge right. When I convert this to a real life problem, I will make sure that I am not talking about unphysical quantities. So, one extra step I have to take care of which we will do ok, it will become clear in the rest of the course how we use these properties ok. So, those are your Maxwell's equations right.

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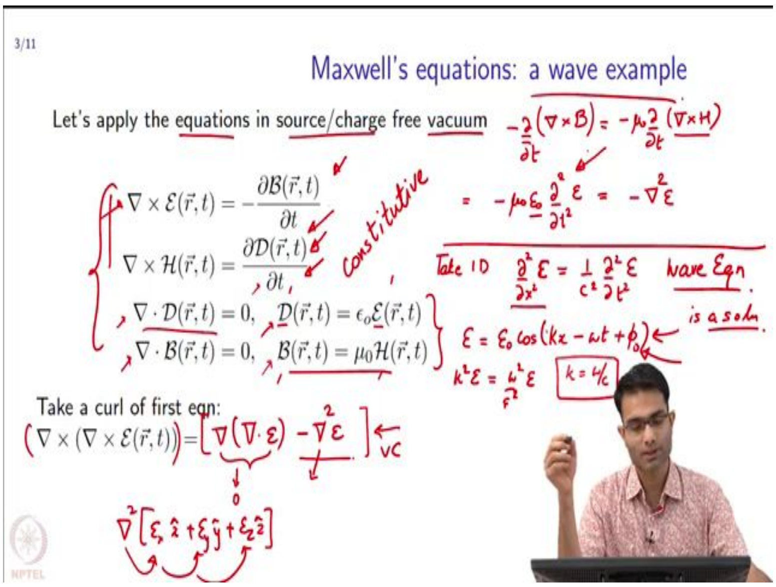

Let us take a simple example of how we use them. We will do much more complicated calculations in this course, but let us just take a simpler wave example. So, I have written down Maxwell's equations in vacuum and vacuum which has no sources or charges, no sources means ρ is 0, no and charges also 0 and current is also 0 right. So, two simple equations, I want to combine them somehow to give me a single equation ok. So, these four equations, we have all seen before. There are two additional what are called constitutive relation which tell us the relation between D and E, B and H ok.

Since its vacuum, it is very simple its related by ε_0 and μ_0 not at all difficult. So, now, if I take I want to, from the first two equations over here let us say I want to eliminate one of the variables right. So, it is like there are two equations and two variables right. The two variables are; if I look at these first two equations what are the two variables in these equations?

Student: (Refer Time: 07:02).

Student: R and t.

Rand t, no; r and t I just telling us the equation that true at some point in space and time. But at a particular point in space and time what are the 2 variables that I do not know? E and H or E and B, electric field magnetic field this is what I do not know and I want to calculate right. So, that will be the general idea of any cem example right, I want to find out the electric field, magnetic field given some charge distribution, current distribution, boundary conditions blah blah right.

So, I have two equations in 2 variables. So, in principle I should be able to solve them right. So, what should we do? We will use some properties from vector calculus. So, we can take let us say the curl of the first equation over here right. So, $\nabla \times (\nabla \times E)$ right; now there is a nice relation that you know from vector calculus for this which is.

Student: $\nabla(\nabla.E) -$

$\nabla(\nabla.E) - \nabla^2 E$ enough ok, that is what we get. Now of these two terms is there any one term that gets simplified?

Student: First one.

The first term that simplified.

Student: (Refer Time: 08:15).

Using Coulomb's law right because $\nabla.D$ over here, $\nabla.D$ and D is related to E just by a constant. So, $\nabla.E$ is going to be?

Student: 0.

0 right; so, this whole term is 0 ok. So, we have reduced this first term. So, the left hand side is become $-\nabla^2 E$ ok. Just a quick check is this a scalar or a vector is the left hand side, is this expression over here is this a scalar or a vector.

Student: Scalar.

Is this expression over here a scalar or a vector?

Student: (Refer Time: 08:57).

So, how can a vector be equal to scalar? So, left hand side is clearly a vector, right hand side therefore, also a vector why because how does this guy operate this is actually going to be $\nabla^2(E_x\hat{x} + E_y\hat{y} + E_z\hat{z})$ right. So, ∇^2 will act on each of these guys one by one and give me a vector ok, ∇^2 can act on the scalar it can act on a vector, but right now its acting on the vector so output will be a vector.

So, that is one simple check that you should do, like in school we should do dimensional analysis right. So, this is similar over here alright. So, I have a vector on the left hand side also, I need to simplify the right hand side right. So, I have so I am going to take a curl on the right hand side also alright. So, I will have minus del by del t, the del cross operator acts only in space so I can take it across the time derivative alright.

So, I am going to have a $\nabla \times B$. And then what do I do? How do I simplify further, do I have a relation that tells me what is $\nabla \times B$, do I have a relation that tells me $\nabla \times H$? Yes; is there a relation between B and H? μ_0 right. So, this will therefore, become so $\nabla \times B$, I can write as be as I can use this over here $B = \mu_0 H$. So, this becomes minus $\mu_0 \frac{\partial H}{\partial t}$.

Now $\nabla \times H$ I know right, $\nabla \times H$ is related by this equation over here. So, this expression will become minus $\mu_0 \varepsilon_0$; what else? So, one derivative will come from the second equation. So, this becomes $\frac{\partial^2}{\partial t^2}$. What am I left with? I am substituting $\nabla \times H$ as $\frac{\partial D}{\partial t}$ and $D = \varepsilon E$ so that ε_0 came out over here. What am I left with over here?

Student: E.

E exactly ok; so, what I have got this is the right hand side so, therefore, this is equal to the left hand side over here then squared E ok. So, we will write this in a familiar simpler form, let us this side say take let us take a one dimensional example; that means, the physics varies only in one dimension let us take the dimension to be x. So, in that case electric field etcetera is just a function of x. So, this ∇^2 operator will only become $\frac{\partial^2}{\partial x^2}$ right, partial derivative with respect to x.

So, if I simplify the what I have got so far I will get $\frac{\partial^2}{\partial x^2}E = (\frac{1}{c})^2 \frac{\partial^2}{\partial t^2}E$ and this many of you will know is called the wave equation ok. Now, why is it called a wave equation?

Student: It describes a plane wave.

It describes a plane wave; we can see what is the solution to this equation. So, for example, does if I take something like this cos of; cos of what do I write?

Student: Kx minus (Refer Time: 12:29).

$kx - \omega_0 + a$ that is a phi naught right. So, when I take a derivative, two derivatives in x, what will I get? I will get back cos right and some constants will come so I will get a k^2 on the left hand side. On the right hand side, what will I get? I will get E back over here, right hand side I will take two derivatives with respect to time; cos will become sin will become cos I will get back cos right. So, this will become omega squared by c squared back to E right. So, if k is equal to omega c, then this guy is a solution right and this is exactly the equation of a wave that is traveling.

So, is this derivation clear? So, we used very simple things, we use Maxwell's equations and all four of them were used right. We did not leave even one of them, we use the fact that

$\nabla.D = 0$ and the relation between D and E, we use the fact that $\nabla.B = 0$ and the relation between B and mu. So, all of these relations were used along with some simple vector calculus identities right. So, you will find that this is going to be the story of CEM, using Maxwell's equations, boundary conditions, vector calculus you can calculate everything ok.

And so this, when this kind of a derivation was first understood was a very big deal because if you look at these equations Maxwell's equations like I mentioned in the previous lecture these were derived in a lab dealing with resistors, capacitors, inductance currents. That is how these equations were derived and somehow you are able to get out of it a wave equation which suddenly then explains to you how light from the sun reaches as its coming as a electromagnetic wave.

So, this connection between the electromagnetics and optics really was made possible by Maxwell. And if you notice if this term over here if this displacement term were not here this thing would not have worked right, I would not have been able to get the second derivative with respect to time, this guy would not have happened right. So, it is all comes together very nicely alright. Any questions before we go ahead? Fairly simple many of you have probably seen this wave equation derivation earlier ok. So, this is all fine, this is how Maxwell and company had written it in terms of real valued physical quantities ok.

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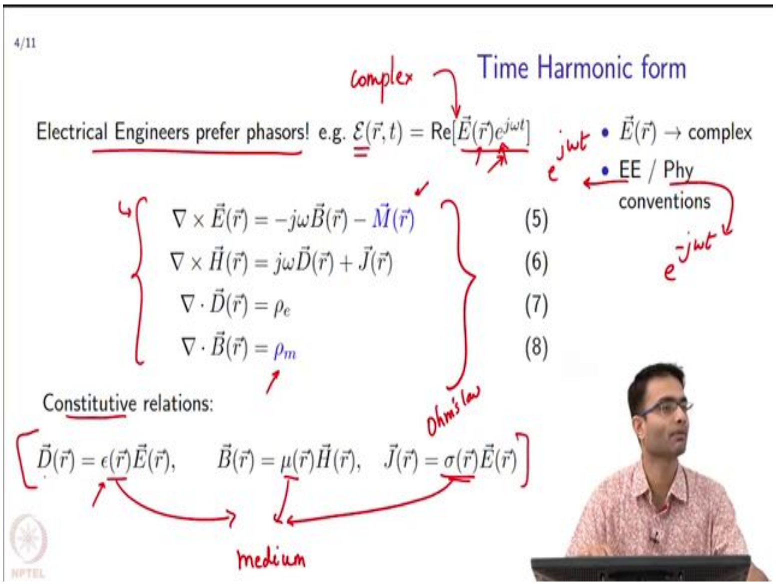

Now, as it turns out we are all electrical engineers and we like phasors and why do we like phasors? Again this is it is a simple property of electrical engineers we do not like pain so we want to make math easier right. So, this using phasors helps us to reduce the mathematical complication right. So, what I do is I keep I take my physical quantity over here that is E and I write it in terms of phasor.

So, what is a phasor now? It is a complex vector; so this is complex. And I have put the e to the j omega t that is the phasor and I this is; obviously, going to be a complex quantity and since I want to only deal with real valued quantities in the lab world so I related by taking the real part so, that is what we will do. By introducing this I am also separating the role of time and space, I made it into two separate variables and I can that is convenient for me with Maxwell's equations because Maxwell's equations are nicely linear right.

So, what happens to all the time derivatives, wherever you see a $\frac{d}{dt}$ there is a $j\omega$ that will come out and the time derivative is gone right. So, that is what happens to my four equations over here, I am not writing the continuity equation anymore because its understood. I have retained the magnetic current and the magnetic charge keeping in mind that they are not physical quantities ok, they will they are introduced for mathematical convenience ok; so, if someone asks you what are the equations that are valid in lab you will drop the blue terms ok.

One other thing I want to mention which will happen to you when you read books from different disciplines. If you are reading a book from electrical engineering the convention that electrical engineers is $e^{j\omega t}$ usually. If you read a book from the let us say the physics of the optics community they prefer to use $e^{-j\omega t}$ ok. So, you should whatever book you pick up make sure you go right to the beginning and see what is the time convention they have used. Both are correct, but if you get confused you will all your answers will be wrong by minus sign everywhere ok, because of the simple difference ok.

Of course, another differences that are physicists will not use a j they will use i, that is another thing for the imaginary constant ah. So, those are the phasor relations ok. So, these are the equations that we will work within the course ok, once I solve this. If I if someone says ok, now give me the lab quantity all you have to do is supposing you solve these equations you got some E and H out of it. What do you have to do?

Student: (Refer Time: 18:02).

Put.

Student: (Refer Time: 18:05).

No you have solved this ok. Now you have got your E and H has come out as a result of a calculation and someone says now give me the actual quantity in lab, the physical electric field in lab, what would you do?

Student: Rho and then m.

That is ok, rho and m the use is gone I have already used them to solve for E and H. Now what do I do? She says take the real part ok, common mistake; now that is not enough. What else do you need to do?

Student: You have got a mu r (Refer Time: 18:31).

I have to multiplied by $e^{j\omega t}$ right, if you just take the real part of E r you will get a time invariant quantity, we do not want that. So, remember to stick in the $e^{j\omega t}$ and that gives you the full time dependent ok. So, those were the phasor forms of Maxwell's equations; the next thing that is used is how do I relate D and E, B and H and the current. So, I am writing down these so called constitutive relations over here. So, the relation between D and E is in terms of epsilon permittivity right, relation between B and H is in terms of mu permeability ok. And here is anyone who knows what this law is called?

Student: Ohm's law.

Ohm's law right this is your famous Ohm's law ok. Now fields we have spoken about where does the material actually come into picture, whether its glass or wood or free space?

Student: (Refer Time: 19:38).

Right; so these guys, these guys and these guys this is where the medium properties come into place ok. I have what I have written is a very simplified version of material properties. In fact, in the strictest sense these equations that I have written a wrong and we will correct them as we go further in the course this. For example, this seems to be a simple constant; actually it could be a tensor for an isotropic media.

And in lenses in optics you all we have that way of travels differently in one direction then another direction and isotropic things are there that is captured by making this epsilon into a tensor or mu into a tensor ok. There are further complications with these equations these constitutive equations and I will point them out when we come across ok. So, these are the main sets of equations that we will work with alright.