Computational Electromagnetics
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FDTD: Materials and Boundary Conditions
Lecture – 13.03
Debye Model – Part 1

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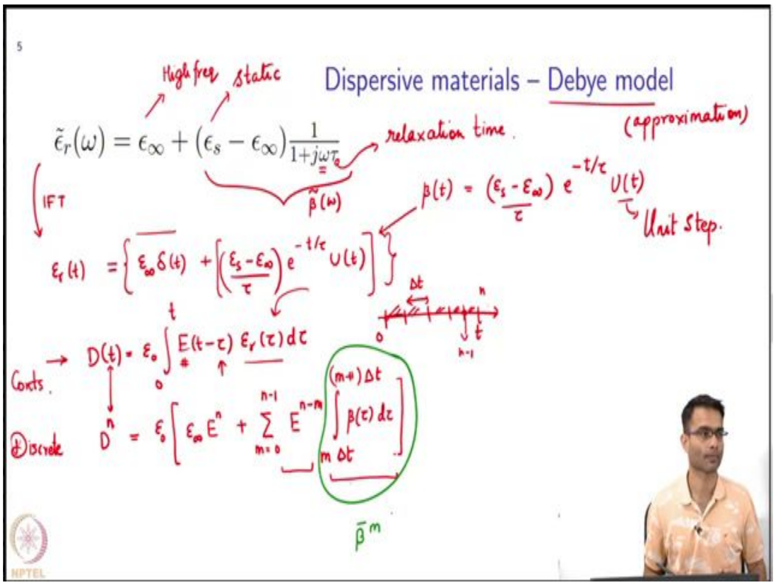

Right so, we will deal with what is called the Debye Model. So, this Debye model is an approximation, but it is something which is reasonable because. So, I mean look at the various features of it what is the frequency term is appearing here in the denominator right. So, at zero frequency what is the term that survives in this expression?

Student: Epsilon s.

Epsilon s right so this is you can call this the static value. Then what happens at extremely large frequencies? Yeah you just get an epsilon infinity. So, that is the high frequency case very high frequency case right and this tau over here is what is called the relaxation time ok. So, this is perhaps one of the simplest models for frequency dependence of permittivity right.

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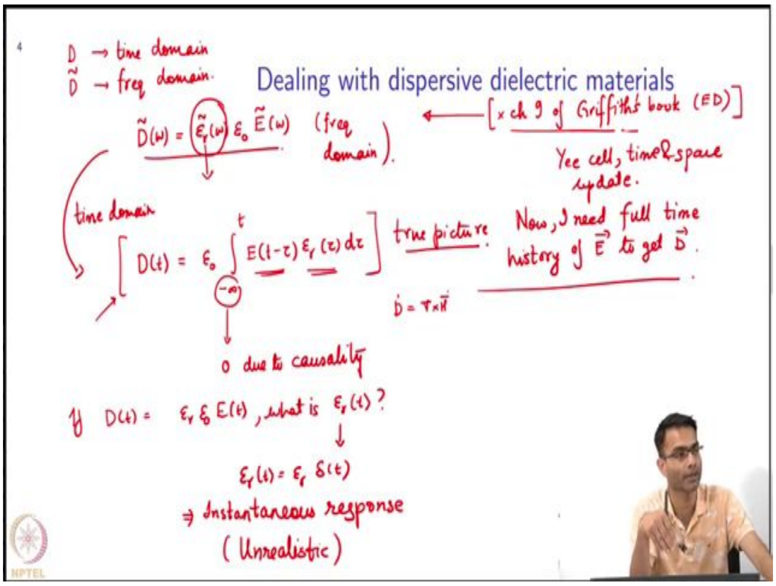

So, in our previous expression over here we did not say anything about what is the form of $\varepsilon_r(\omega)$ right. So, we are going to start with this as one form and see do I still have to store all the past history or is there some clever thing I can do ok. So, that is sort of the objective of what we are trying to do actually. So, this whole term over here it will appear many times. So, I am going to give it a shorthand notation I am going to call it β ok, since it is in the frequency domain I am putting a tilde on its. So, this is $\tilde{\beta}(\omega)$ ok. Again looking back at the time domain picture if I wanted to implement that model what do I need?

So, these guys electric fields they are coming from the calculation from the Yee cells. So, they are not functions they are just numbers what is left is this guy. So, this is in which domain? Frequency or time? It is time domain right. So, I started with some frequency domain version of the permittivity which is the dispersive nature. Now, in order to implement this D because I will need this D to put it into the update equations right; what are the update equations? So, $\dot{D} = \nabla \times H$ supposing I see this.

So, I need D in the time domain then into take a derivative. So, to get D in the time domain I need this ε_r in time domain, but what I have started with is starting point is frequency domain that is the simplest place to start with. So, I need to take now the inverse Fourier transform of this expression in order to get ε_r as a function of time so that I can plug it back in.

Student: (Refer Time: 03:25) starting (Refer Time: 03:26).

Starting it is just put $\omega = 0$. What will happen to ε_r? You will get es, oh sorry ε_s, it is called static because of zero frequency right. So, if I want to take the inverse Fourier transform of this. So, what is for example, let us look at part by part, what is the inverse Fourier transform of this $\tilde{\beta}(\omega)$? What is it? Does it look familiar? $1/(1+j\omega\tau)$ it is exponential that is decaying.

Student: Decaying.

Yeah right. So, there will be some this constant will carry forward epsilon infinity right, I have $e^{-t/\tau}$ one more thing will be there divided by τ because this τ is in the denominator right. So, this is going to be by τ and one more critical thing.

Student: (Refer Time: 04:33).

No in $\beta(t)$. I am missing. I have there is one more term over here.

Student: Unit.

Yeah. Unit step.

Student: Unit step.

$$\beta(t) = (\varepsilon_s - \varepsilon_\infty)/\tau \; e^{-t/\tau} u(t)$$

Everyone knows what are unit steps right. So, this is your very simple right without this unit step if I try to take the Fourier transform of this function is only unbounded at the minus time scale right. So, I need this unit t. So, this is my inverse Fourier transform of $\beta(t)$. Now, I can get the Fourier transform of $\varepsilon_r(t)$. What happens to the first term? It would not be just ε_∞ there has to be something else.

Student: Delta t.

$\delta(t)$ ok. So, now that I have got my ε_r what is my next sort of objective?

Student: Put it.

Put it into the D equation right. So, our D equation:

$$D(t) = \varepsilon_0 \int_0^t E(t-\tau)\varepsilon_r(\tau)d\tau$$

Now what do I have to do, I have to substitute this into this equation right. So, before we do this let us just plan it ok. So, remember this whole term is called $\beta(t)$ right. So, this whole term is remember that we cannot actually do any analytically evaluation of this integral.

So, in FDTD we are discretizing space and time right. So, until the next time instant until a next time update what happens to the electric fields and magnetic fields they remain constant right for a given time interval their constant and then they get updated, that is how FDTD works there is no analytical evolution. So now, this time integral that I this integral in time what should I this how should I go about discretizing it?

Student: Delta t.

Right so I have Δt. So, I will split this 0 to t.

Student: 0 to t.

It will become a summation over each time instant right; so, I have this is my time instant say this is my time 0 to t. I am going to chop it up into intervals of Δt and then do a summation that is what will happen. So, my D^n that is my notation for; so, all of this is happening at a particular point in space.

Student: Typical form of because.

You are asking for this Debye model?

Student: Yeah.

Is it derived analytically?

Student: Yeah.

0 to t. Integral is from 0 to t.

Student: Yeah, what I am asking is does it have an analytic solution.

This equation.

Student: E.

No because I do not know E. If I did not have $E(t)$ in this integral if I just had $\varepsilon_r(\tau)d\tau$ which is known as a function of time then I can perhaps evaluate it. But I do not know the electric field is sitting in there which is itself an object of computation I do not know the electric field. In fact, I am no E and H.

So, this D guy is sort of inconvenient fellow sitting there because Maxwell's equations is $\partial D/\partial t$ not $\partial E/\partial t$. So, I have to go to D and from D, come to E. So, although D is there I have to try to work him out of this equation. So, D^n is the value of displacement field at time instant n ok. So, what will be the first? So, $\varepsilon(0)$ now, this first guy in the integral is going to pop out right. So, this whole expression right this whole expression over here is going to get substituted into here. So, what will be the first term give me?

Student: Epsilon.

ε_∞ multiplied by electric field because electric field is here at what time instant?

Student: t.

t which is n right; so, this t is the same as n, right. On top, I have the continuous time version bottom is the discrete version right. So, this is the discrete and this is continuous all right. Next term, here is where the fun starts right. So, I am going to assume that over each time interval electric field does not change right. So, I am going to have a summation. So, $\sum_{m=0}^{n-1}$. So, for example, this is n right. So, I have over this over this over this over this and finally, over this right. So, this is 0 1 2 3 all the way up to this point which is n minus 1 starting point of the time interval right.

So, I am going to assume the electric field at the start of the Δt remains the same for the Δt right. So, what will I have over here I will have E^{n-m} and this what will be the summation? What will be the limits of integration? I have introduced a variable m. So, what is the starting point?

Student: M delta t.

$m\Delta t$ to $(m-1)\Delta t$ and what is its inside this integral?

Student: Epsilon r.

Not ε_r

Student: Epsilon.

No. It is there staring at you.

Student: Epsilon s by (Refer Time: 11:14).

Yeah, but there is a shorthand notation for I know $\beta(\tau)d\tau$.

$$D^n = \varepsilon_0[\varepsilon_\infty E^n + \sum_{m=0}^{n-1} E^{n-m} \int_{m\Delta t}^{(m+1)\Delta t} \beta(\tau)d\tau]$$

so, let us just check the limits for example, substitute $m = 0$ ok. So, $m = 0$ I will get E^n multiplied by the integral from the first segment right from 0 to Δt. (Refer Time: 11:43) That is a convolution. So supposing I take $\tau = 0$ then its $E(t)$ and $\varepsilon_r(0)$ at zero time instant right. So, its electric field is at the future at the present and ε is at the 0 right. I mean this is a revision of how you calculate how you do a convolution right.

So, $\varepsilon_r(\tau)$ is moving from 0 to t. So, this second part over here is moving from 0 to t as m is progressing and on the other hand as m is progressing this electric field value see τ keeps increasing. So, $E(t-\tau)$ keeps decreasing. So, finally, when $t=\tau$, I get $E(0)$ right. So, if I substitute $m=n-1$ what do I get?

Student: E 1.

$E(1)$ right at the very beginning; so, you can choose whether you want to have the electric field to be constant at the beginning or at the end does not matter it is a piecewise constant right. So, these are the tools.

So, is this clear? So, if I mean this should be really cleared because the rest of it then depends on this part doing very clear what did we do we just. So, we took the Debye model we took its inverse Fourier transform to find out what it looks like in the time domain, substituted the time domain into the constitutive relation between D and E, that is all that we have done so far. So, what are the approximations basically Debye model and second approximation is this piecewise constant, but that is there is no choice we cannot do anything better than that because I know electric field only at discrete time instance I do not know it everywhere.

Student: Suppose 0 to (Refer Time: 13:36) E will be constant.

Why should E be constant from 0 to n. I am calculating E at every time instant of the Yee cell right. So, I have a record of E and this integral needs me to have I mean look at the continuous case electric field continuous case convolution integral it needs $E(t-\tau)$ and τ is varying from 0 to t. So, it needs $E(0)$ to $E(t)$. So, its electric field is not constant.

Student: (Refer Time: 14:08).

Yeah ok. So, let's just call this τ_0. Yeah good point. τ_0 is a relaxation time that has nothing to do with this variable of integration over here yeah good point ok. So, should we proceed? So, our next step is we have got D^n the next step is going to be put it into update equation right that is a logical next step.

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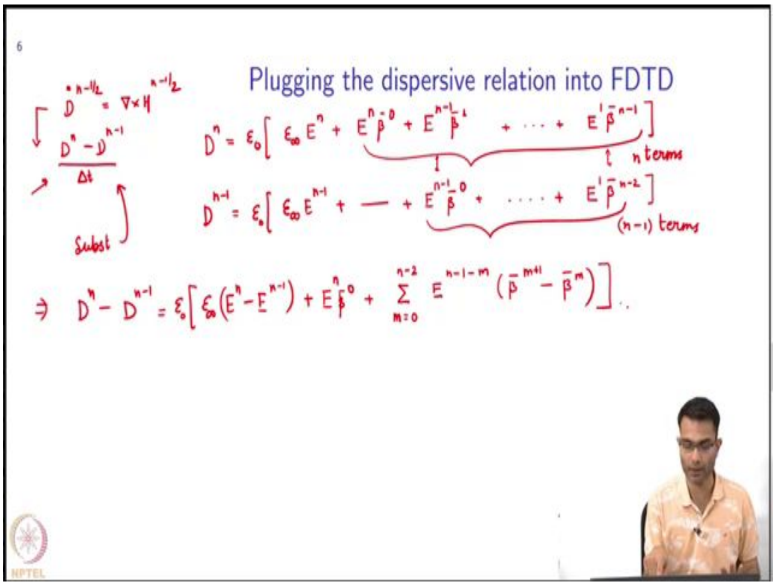

So, what is our Maxwell's equation telling us? So,

$$\dot{D}^{n-1/2} = \nabla \times H^{n-1/2}$$

I am assuming there is no current otherwise there will be one more current term over here. So, now, this will become $\dot{D}^{n-1/2} = (D^n - D^{n-1})/\Delta t$ right. That means, this expression that I derived previously D^n not only do I need to take the D^n I have to write down D^{n-1} then subtract the two ok. So, let us write down let us expanded the D^n term which I had. So, what is D^n equal to ?

So, notice that this integral appears everywhere its. So, we will have a shorthand notation for that also right. So, I am going to call this just this integral right. So, this let us use a different color this guy over here is what I am going to call $\bar{\beta}^m$ otherwise I have to keep writing in integral everywhere right. So, what is the first term is I mean I have ε_0 everywhere then I have $\varepsilon_\infty E^n$. Now I see remember I want to subtract over here from D^{n-1} lot of terms may be common between D^n and D^{n-1}. So, I will just write out the summation and then we will do the subtraction. So, first term over here. So, we will start with $m = 0$. So, it will be?

Student: E n.

E^n right and then $\bar{\beta}^0$ right I am starting with $m = 0$ right. So, this is going to be; so, it is very simple if you just keep track of the indices $E^n \bar{\beta}^0$ right. Next will be $E^n \bar{\beta}^1$.

Student: 1.

And this continuous all the way up to $E^1 \bar{\beta}^{n-1}$?

Student: $E^1 \bar{\beta}^{n-1}$.

How many how many terms are there? These are how many terms?

Student: N term.

N terms ok, now let us write down D^{n-1} this is easy. So, $\varepsilon_\infty E^{n-1}$ first term then I will start over here with. Will I have this guy? Will I have $E^n \bar{\beta}^0$?

Student: (Refer Time: 17:39).

No right. So, this term is not there. Will I have the next, what will be the first term of the summation? $E^{n-1} \bar{\beta}^0$. See if there is an n here there is an n here and there is an $n-1$ here. So, now, if I replace this n by $n-1$ this will become

$$D^{n-1} = \varepsilon_0[\varepsilon_\infty E^{n-1} + \sum_{m=0}^{n-2} E^{n-m-1} \int_{m\Delta t}^{(m+1)\Delta t} \beta(\tau) d\tau]$$

So, the term that will appear over here will be $\bar{\beta}$ what? This is the first term of the summation. So, that is $m = 0$.

Student: 0.

Right. The summation goes on finally, what is the last term of the summation this is still be $E^1 \bar{\beta}^{n-2}$ right. How many terms? $n-1$ terms ok. So, now, what do we have to do we have to subtract these two characters and divide by Δt ok.

So, let us write that over here D^n and D^{n-1} right is equal to. So, we should do this subtraction a little bit intelligently right. So, what would make sense is to keep the. So, let us let us deal with these epsilon infinity terms first and then get to the summation right. So, there is $\varepsilon_\infty(E^n - E^{n-1})$ no contest about that then there is one term who has no one to be with. So, we will just write them as $E^n \bar{\beta}^0$ then I have the electric field will be common and the beta bars are different right I have $E^{n-1} \bar{\beta}^1$ and then $E^n \bar{\beta}^0$.

Now, I am just going to subtract all of these $n-1$ terms from each other right that is why I wrote them one above the other. So, these terms I am going to subtract. So, notice what is nice is the E is the same in both these terms let us $E^1 - E^1$, $E^{n-1} - E^{n-1}$. So, that is right that as a summation over m; so, the E^{n-1-m}

Student: Sir.

Yeah.

Student: In D n minus 1 the first two terms are (Refer Time: 20:46) the n minus 1 term itself right.

In D^{n-1} the first term is.

Student: (Refer Time: 20:50) epsilon into D n minus 1 plus 0 they come in the first term in separately.

There is nothing over here there is no term over here.

Student: That is 0 right.

That is 0.

Student: Yeah that comes in the first term its separate on the (Refer Time: 21:01).

Yeah I mean I am just I left a blank here because I wanted to align it with the first equation that is all. So, there is nothing over here. So, this is a blank over here yeah ok. So, look at the first term E^{n-1}. So, what should m start from?

Student: 0.

0 and what do I have over here $(\bar{\beta}^{m+1} - \bar{\beta}^{m})$

Student: M plus 1.

And the summation m is going to go from 0 to this last term is the entry.

Student: $n-2$.

$$D^n - D^{n-1} = \varepsilon_0[\varepsilon_\infty(E^n \ \ - E^{n-1}) + E^n\bar{\beta}^0 + \sum_{m=0}^{n-2} E^{n-1-m}(\bar{\beta}^{m+1} - \bar{\beta}^{m})]$$

$n-2$ excellent. So, this is not difficult it is just algebra keeping track of which term is where and all right the only difficulty actually. So far there has not been any difficulty. We have, once we assume the Debye model, it is just straight. Fourier transform was simple, the edges are keeping the terms aligned next to each other are what we have to do and then we will substitute this into this term over here. So, we will take a pause over here ok.

So, steps are clear? Start by Debye model, take Fourier take the inverse Fourier transform, simplify the convolution and now I have got a relation purely in terms of E right. The right hand side is purely in terms of E and I will replace equate this to the $\nabla \times \vec{H}$ term.

So, I will get an update relation between E and H which is what I wanted all along then we will see what happened to that running history. So, far I still need electric field at all past time instants. So, we will deal with whether we still need that or not ok.