Computational Electromagnetics
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Finite Difference Time Domain Methods
Lecture – 12.07
Accuracy Considerations – 1D

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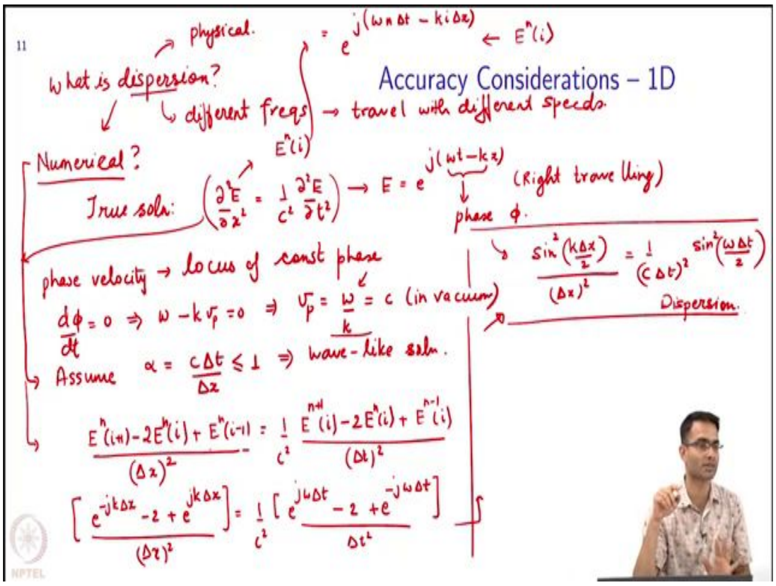

Alright so having looked at having looked at stability as a criterion ability the next thing that we want to look at this accuracy of electromagnetic field propagating in the Yee grid. So, under Accuracy Considerations we will pose a very general question; so for example, if I ask you this what is first you what do you understand by dispersion, what is the first thing that comes your mind?

Student: Frequency dependent.

Frequency dependent right; so, in a very practical sense what happens? Different frequencies.

Student: (Refer Time: 00:57) velocity.

Travel with different velocities right. So, different frequencies travel with different speeds ok. So, this everyone knows is a physical phenomena right. this is very easy and why for example, you have a rainbow because the optical I mean light is travelling at different speeds and the water bubble it disperses ok. Can there be dispersion numerically? It could happen right. So, one is physical dispersion which is ok, the other is numerical.

So, what would; so, numerical dispersion supposing I am trying to model wave propagation through uh lets say free space, physically should there be any dispersion? Obviously, no; but numerically dispersion may happen depending on how I have implemented FDTD so, would it be a good thing or a bad thing; obviously, a bad thing right.

So, a what we will look so, that in general that tends to destroy the accuracy of your calculation, if your waves are travelling at different speeds whereas physically they should travel at the same speed then that is what is called numerical dispersion. So, we will explore now under what conditions this numerical dispersion happens and when does it not happen ok; so, that is what we going to look at ok.

So, let us first look at the true solution, will notice in all of these calculations we use the wave equation a lot right because it is very easy to handle. So, our wave equation gives us the second derivatives of space and time for the electric field and this gives us the solution of the form we all know this $\pm kx$ right. So, $E = e^{j(wt-kx)}$ is a right hand, right travelling wave let us call this term over here what do what would be call this term over here inside the phasor?

Student: Phase.

Phase right. So, this is the phase ϕ.

Student: Ok.

So, if I ask you if I want to find out at what speed is this wave travelling then what is that physically known as is the locus of.

Student: What was.

What is phase velocity did you find there.

Student: (Refer Time: 03:50).

Right; so, phase velocity is the locus of constant phase right, how do I obtain this?

Student: (Refer Time: 04:08).

Right; so, constant phase means phase should not change with time right. So, I will do $\frac{d\Phi}{dt} = 0$ right. So, that gives me the first term gives me $\frac{d\Phi}{dt} = \omega - k\frac{dx}{dt} = \omega - kv_p = 0$ ok, right is equal to 0 that gives me $v_p = \omega/k$ ok. And I know that this is going to be equal to c in vacuum we know this already.

Now the question is numerically what is it? So, what approach should we take, what we did in the theoretical calculation? Obviously, the phase velocities ω/k which is c in vacuum; now, I want to answer the same question, but now on a in a discrete world where I have discretized phase and time. So, what should my approach be?

Student: (Refer Time: 05:10).

Student: Sit at a point.

Sit at a point and.

Student: Points that use.

But how do I bring in the discrete world?

Student: Make some (Refer Time: 05:19).

Correct. So, I mean this method that we are studying is FDTD right. So, we have finite differences in time and space and we have a second derivative. So, we should take what how should be approximate these things? We should approximate the second derivatives by finite differences, and see what kind of relation results from it ok. Then we can for example, in that condition look at the locus of constant phase and see is it speed of light or is it something else ok. Now uh under what conditions, so we have looked at this before under what conditions will the electric field be a wave like solution in FDTD?

As long as the courant stability criteria is satisfied; so, when courant stability is satisfied my E will be a wave; we derived that ok. So, now, let us assume that your say alpha, $\alpha = c\frac{\Delta t}{\Delta x} \leq 1$

ok. So, that says wave like solution is what is obtained right now, what do I do; so, this equation over here, I can add it over here and replaced by what kind of a second derivative can be replaced by a finite differences right.

So, let us do that; so, first is the spatial derivative. So, remember our notation for electric field, it is going to be $E^n(i)$; n is the time index, i is the space index ok.

$$\frac{E^n(i+1)-2E^n(i)+E^n(i-1)}{(\Delta x)^2} = \frac{1}{c^2}\frac{E^{n+1}(i)-2E^n(i)+E^{n-1}(i)}{(\Delta t)^2}$$

And this $E^n(i)$ itself is short hand notation for remember it is a wave like solution with space and time being discretized right. So, this simplified expression over here is going to be; so,

$$\frac{e^{-jk\Delta x}-2+e^{jk\Delta x}}{(\Delta x)^2} = \frac{1}{c^2}\frac{e^{j\omega\Delta t}-2+e^{-j\omega\Delta t}}{(\Delta t)^2}$$

$$\frac{\sin^2(\frac{k\Delta x}{2})}{(\Delta x)^2} = \frac{1}{(c\Delta t)^2}\sin^2(\frac{\omega\Delta t}{2})$$

Right;So, it is a nice looking expression right we did not expect that it will simplify so nicely, but that is what we got ok.

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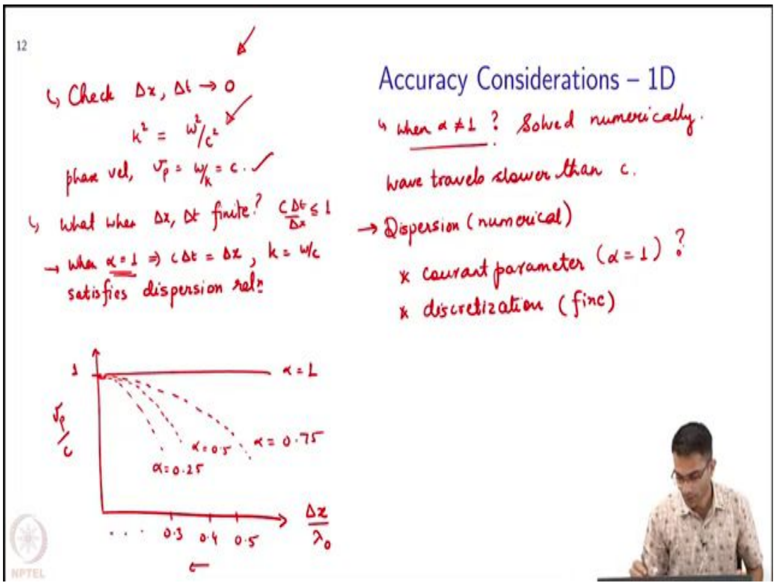

Now, let us see first of all let us first check what happen when I set $\Delta x \to 0$, $\Delta t \to 0$ because when they tend to 0, I should I should get the correct solution right. As I make the dis the differences more and I mean smaller and smaller I am going to approximate the actual derivative right.

So, as $\Delta x \to 0,\ \Delta t \to 0$ what happens to this expression over here. So, $\sin\theta \to \theta\ as\ \theta \to 0$ right; so, what will I what will I get over here ?

Student: Derivative.

So, I am going to; so, let us look at over here. So, this is going to become $\frac{(k\Delta x/2)^2}{(\Delta x)^2} = \frac{(\omega\Delta t/2)^2}{(c\Delta t)^2}$ So, all are cancelled off right. So, k will remains; so, I will get $k^2 = \omega^2/c^2$ ok, and our phase velocity which was $v_p = \omega/k = c$ right. So in the limit everything is well behaved because this is giving me the phase velocity equal to c right, so life is good this is what we expect.

Now the question is what happens when Δx and Δt are finite ok? So, we are still in the courant condition right; so, we are still saying that $c(\Delta t/\Delta x) \le 1$, but they are finite numbers; I mean if the courant criteria is not satisfied I do not get the wave like solution, but now I am within the courant condition, but these are finite. So, will this condition still be satisfied; so, first of all looking at this expression over here I can no longer replace sin theta by theta. So, it is going to be what it is right; now.

Student: (Refer Time: 13:58).

No, that is to enforce the I mean let us assume courant stability satisfied ok. So, can you think of a situation where Δx and Δt is finite they are not tending to 0 they are finite still I get $\omega/k = c$, this relation has to be satisfied you will agree because you have derived it by discretizing the wave equation. So, this has to be satisfied.

Student: (Refer Time: 14:26).

Can you think of a relation or a condition where you will get the ratio that like we got in this case, we got the ratio that $\omega/k = c$ over here when I took $\Delta x \to 0, \Delta t \to 0$. Can I get the same relation for finite Δx and Δt. So, I will yeah, we can we can look at this expression over here.

Remember you can play with α where $\alpha = c(\Delta t/\Delta x)$ it is in your hand ok. So, as a as a hint if I have alpha in my hand; that means, if supposing I fix alpha then I only need to fix one of the two either Δt or Δx; because they are related. So, supposing I fix $\alpha = \alpha_0$ then I just have to specify Δx let us say; I mean Δt gets fix automatically, does that give you a hint of what under what for what value of alpha is this relation going to simplify to; what we want? The questions clear? Question is clear right when I look at this expression, it is not at all clear that $\omega = ck$, it is not necessary that $\omega = ck$ is the solution to this equation.

But under one special case it is equal to; yeah, so just look at what happens when $\alpha = 1$ it is allowed; right, so, then what will happens, what happens over here? If I now substitute $\omega = ck$ is it consistent in both sides; so, that value is the solution to this equation it is satisfying it right. So, what is the dispersion dispersion relation is this one right, and interestingly does not depend on the value of Δx or Δt right.

Student: (Refer Time: 17:20).

That is I mean you would not have expected that to happen. So, what is its saying therefore, $\alpha = 1$ whatever discretization you choose my wave is travelling at the speed that I expect it to travel right.

Student: (Refer Time: 17:33).

So, if I plot something like this; so, what will I plot over here, let me plot the ratio of the phase velocity to speed of light ok. So, $v_p = c$ ideally, I am just stimulating vacuum only; so, ideally this ratio should be 1 right so, this is 1, and here what I will plot is Δx to make it sort of normalized to wave length because as a frequency changes, everything Δx will have to change I will just do this the wave length that this electromagnetic wave is going.

So, regardless of; so, what should this for when $\alpha = 1$ what kind of a curve I will expect here, straight line, curved line, angle, vertical to x axis, y axis, 45 degrees what kind of curve do I expect; straight flat line right answer is equal to 1 right. So, I can have different values over here let us say 0.5, 0.4, 0.3 and so on; right, the discretization is getting finer and finer as I approach the origin is everyone convinced about this.

Student: (Refer Time: 18:53).

By λ_0 that is the free the wave length of the electromagnetic wave, whatever ω convert that wave length ok. But what happens when $\alpha = 1$; right, so maybe I should write this expression once again over here. So, what is it we can just have a look here $\frac{\sin^2(\frac{k\Delta x}{2})}{(\Delta x)^2} = \frac{1}{(c\Delta t)^2}\sin^2(\frac{\omega\Delta t}{2})$. So, that sin term will not cancel off on both sides then what will you do?

Student: (Refer Time: 19:35).

Then you have to solve this transcendental equation to find out what the relation between I want the phase velocity; so, I want the relation between ω and k that is what I want to get out. So, this can be done it has to be solved numerically because it is a transcendental equation right it can be done, but so, I mean I am not showing that the calculation here, it is not very hard to do what happens is that as I have so, I mean one thing is that we know that as $\Delta x \to 0, \Delta t \to 0$, I get the perfect wave behaviour ok. So, on this graph that I am showing you as I approach the origin as I go to finer and finer discretizations what should happen; the what will happen is the ratio of v_p to c it should approach.

Student: 1.

1 right asymptotically as I make my discretization finer and finer whatever be alpha right, that sin theta can get replaced by theta and everything will get cancelled off. So, I know that everything should somehow land up over here, but as I go to finite values of Δx as I go to larger and larger values of Δx what happens is the performance begins to degrade. So, this v_p/c begins to look something like this this is fall something like say alpha is equal to 0.75, then as you begin to reduce alpha even more then you will get say alpha is equal to 0.5 and.

Student: Is it because the (Refer Time: 21:19).

Is it because the.

Student: Computation (Refer Time: 21:21).

Computation that has nothing to do with whether it takes less time or more time, it has to do with the fact that the solution to this equation is giving you a certain relation between ω and k. In the ideal case it is giving you the ratio between ω and k to be c, but when Δx and Δx are finite and not related with alpha equal to 1 then there will be some other value of ω/k with satisfies this equation other than c, and that other than c is coming from here.

So, this is this is what is happening? So, this is your numerical dispersion right; so, so the wave travels slower than c which we do not physically expect. Physically its a wave, I have not even put a medium or a material here, free space wave should travel at speed of light, but; because I am simulating it on a finite Δx Δt and I chose $\alpha < 1$ I have this problem ok.

So, if you want to summarize; so, the dispersion numerically if I wanted to have a knob to control how bad this I mean this is clearly an undesirable effect. So, if I wanted a knob or a several knobs where, which I can keep a control on this what are those knobs.

Student: (Refer Time: 23:02).

Say that.

Student: (Refer Time: 23:06).

Yeah.

Student: Cannot (Refer Time: 23:09) value you get for consecutive work.

Student: Find it the over all.

Student: (Refer Time: 23:18).

No, that is numerical convergence that is once I have fix my Δx Δt. α I will check whether the solution is converging on that is not what I am talking about. I am saying, I know based on this discussion I know that my wave will have numerical dispersion I want to mitigate that effect. I do not want my wave to be numerically travelling at a speed different from what it should physically travelled, what knobs do I have in my hand to reduce this effect.

Student: So, alpha is.

$\alpha = 1$ thing right; so, I can check it by courant parameter.

Student: (Refer Time: 23:53).

I will come to it, courant parameter is 1 knob any other knob discretization. So, discretization fine and courant parameter is equal to 1 right, but I am going to put a question mark over here we will come why ok. But; I mean there sort of summary of what I wanted to sort of illustrate over here, is that by changing the courant parameter and by changing the discretization. I can put a check on how much numerical dispersion will be happening in my simulation ok.

Whatever else happens, you know that if your courant parameter is less than 1 you will phase numerical dispersion. In that case your only way to beat it is to make your discretization finer and there is a big prize you pay for it, you cannot just make Δx small you are also make Δt small and. So, your RAM; memory requirements increases, your computation time increases everything right.