Computational Electromagnetics
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Finite Difference Time Domain Methods
Lecture – 12.06
Stability Criteria - Higher Dimensions

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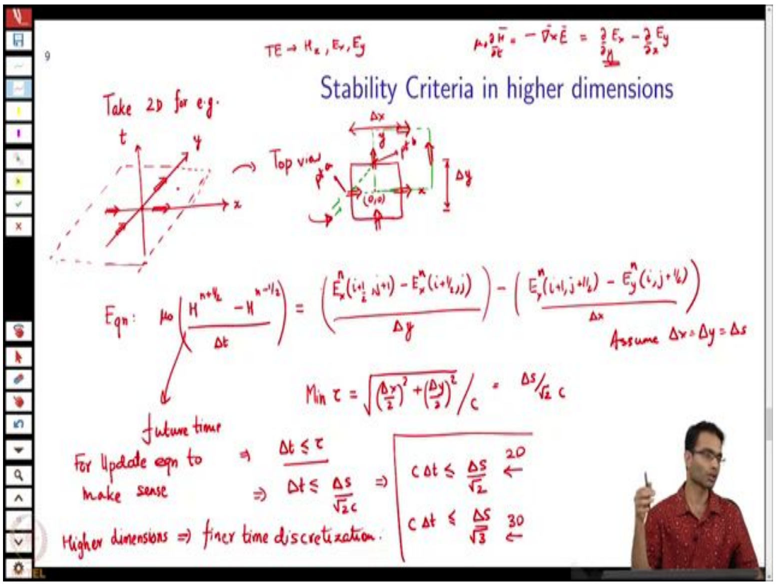

So let us take 2 D for example, ok. So, I have to try to draw something in 3D. So, this is my x y what should I what should this axis be? Not z its a 2 D there is no z and within this I have you know my variables are like this. These are the coordinates that are on my Yee cell. So, how many Yee cells have I drawn 1 or 4?

Student: 4.

4 Yee cells which I have drawn ok; so, yeah there is only so much we can go from here. So, let us look at the top view ok. So, a top view of the same thing what does it look like? So, I will draw this here. So, there is an x. So, this is spaced apart by how much? Delta y. What equation do we want to use? We can use the equation $\partial\vec{H}/\partial t$ is related to $\nabla \times \vec{E}$ ok. So, we are looking at again which polarization? We are looking at TE polarization. So, TE

polarization will have (H_z, E_x, E_y). So, this arrows refer to values of what? These unknowns are unknowns corresponding to what?

Student: E x.

E_x, E_y and where is my H_z? It can it will at the middle over here or I can also choose it over here that is not the matter ok.

So, let us write down this equation so, μ_0. So, $\partial B/\partial t$ right that is the first thing. So, let us $\mu_0 H$. So, $\mu_0(H^{n+1/2} - H^{n-1/2})/\Delta t$ that is my $\partial B/\partial t$ and that is going to be equal to $-\nabla \times \vec{E}$. So, that is again right this down $E_x{}^n(i + 1/2, j) - E_x{}^n(i - 1/2, j)$. What should I divide this by? Δx this is the first term and the second term is. Actually something is not right.

Student: (Refer Time: 04:006 10 times.

Yeah. So, this is E_y. So, E_y is so, this origin over here. So, E_y is being evaluated at i plus no. So, this should be E_x only and this should be Δy.

Student: That j should be (Refer Time: 04:34).

Yeah.

Student: (Refer Time: 04:36).

So, yeah, so we are changing only x coordinates over here. So, this should be Δx over here and then this should be E_y and E_y is actually not being evaluated at. So, if this is the origin over here (0,0). E_y is being evaluated at not half integer is next, but at integer instances right. So, this should be I will rewrite this whole thing.

Student: (Refer Time: 05:10).

Yeah. So, this should be $(i, j + 1/2)$ right. Is that right?

Student: (Refer Time: 05:26)

And, $(i, j-1/2)$. Does that look right? So, we are taking the difference between this guy and this guy. So, the i location is the same. $i-1$?

Student: (Refer Time: 05:48).

Then this should be and this becomes Δy, this is right.

Student: (Refer Time: 06:08).

So, let us write actually what we should we do is write down the equation. So, we have

$$\partial \vec{H}/\partial t = -\nabla \times \vec{E}$$

and we are only looking at the z component. So, this will become what does this become? The first term will be $\partial E_x/\partial y$ there is a minus sign right. So, $\partial E_x/\partial y - \partial E_y/\partial x$ this is the expression that we want.

Student: (Refer Time: 06:40).

Yeah.

Student: (Refer Time: 06:50).

Yeah. So, this is actually ok. So, the grid is let me use a different color the grade is this, this is still the grid I am considering. Now our expression is correct. So, the first one is E_x partial derivative with respect to y. So, then this expression that I have written is incorrect right this is x ok. Let us start from scratch over here once again. This time we should get it right. This is the first time we are looking at E_x right the partial derivative with respect to y ok. So, that is E_x. What is the location? So, I am subtracting this and this right. So, $E_x(i+1/2, j+1)$ right and.

Student: j.

$E_x(i+1/2, j)$ Now we are getting that.

Next term is $\partial E_y/\partial x$ right. So, E_y so it is this minus right. So, this is going to be

$$\partial E_y/\partial x = (E_y(i+1, j+1/2) - E_y(i, j+1/2))/\Delta x$$

$$\mu_0(H^{n+1/2} - H^{n-1/2})/\Delta t = (E_x(i+1/2, j+1) - E_x(i+1/2, j))/\Delta y$$
$$- (E_y(i+1, j+1/2) - E_y(i, j+1/2))/\Delta x$$

We did not actually need to write it, but it's good practice for us to write it ok. Now look at let us get back to what we are trying to do stability criteria right. So, we have to find out the shortest distance that wave could take. So, these in this right top view diagram over here these are sort of we can imagine four nearest I mean in one Yee cell all the four values are involved in calculating H at a future time instance right. So, this is my H at a future time that I want right, everything else is past $H^{n-1/2}$ is past these are all happening at what n right correct.

Of all of these things which of the distances is the shortest? It is going to be this distance between these four points that have four arrows that I have drawn over here, 1 2 3 and 4 which is the shortest distance is it the dashed green line right. So, so we are talking about the wave that is physically travelling ok. Now we want to do time, the philosophy is, we want to do a time update before the wave travels to next due to the nearest point ok.

So, the minimum time taken to travel right the best case can happen when the wave is travelling for example, in this direction like this. Then this is the shortest, because this is longer than for example, this distance right because it is on the diagonal. So, how much is this distance equal to? So, Δx, Δy is the grid spacing.

$$\tau = 1/c \times \sqrt{(\Delta x/2)^2 + (\Delta y/2)^2}$$

where c is the speed of light.

Student: (Refer Time: 10:44).

No before I do a time update.

Student: (Refer Time: 10:47).

Yeah, it's going to travel. The wave is travelling from let us call this point ‘a’ and let us call this point ‘b’ before the wave travels from point a to point b. I should do my update because my update equation has all of these points in it. So, before I before the wave travels from point a to point b is when I do my update right. So, in other words we can say that $\Delta t \le \tau$.

Student: (Refer Time: 11:30).

Need not be who said that they are the same I mean I can make the simplifying assumption that $\Delta x = \Delta y$ in that case. So, assume, that is the simplest case $\Delta x = \Delta y = \Delta s$ ok. So, what does this τ turn out to be?

$$\tau = \Delta s/(\sqrt{2}\, c)$$

And for update equation to make sense this delta t should be less than tau. So, it implies that

$$\Delta t \le \Delta s/(\sqrt{2}\, c)$$

$$\Rightarrow c\Delta t \le \Delta s/\sqrt{2}$$

So, an additional factor of $\sqrt{2}$ has come right, just because these points are arrayed on a square and you are looking at the diagonal between the midpoints. So, can you guess in 3D what will happen? Exactly I will get a root 3.

Student: (Refer Time: 13:05) that is what is.

I am just looking at the shortest distance I want to do my update before the signal gets there, in computation ok. So, which is harder - 1D 2 D 3D? You want to classify them in terms of computational resources. 3D is the hardest because Δt has to be even finer right another the bounds on Δt are getting tighter and tighter.

Now, Δt in 1D could be $\Delta s/c$. Now $\Delta s/c$ is not good enough so $\Delta s/c\sqrt{3}$ in 3D right. So, higher dimensions imply finer time discretization right. So, what we have done if you summarize what we have done so far, we have looked at the, I mean, we did the space update, we did the time update, then we looked at accuracy under what condition, not accuracy but stability, under what conditions there is a wave, appear like a wave and we find out what is called this courant stability criterion ok.

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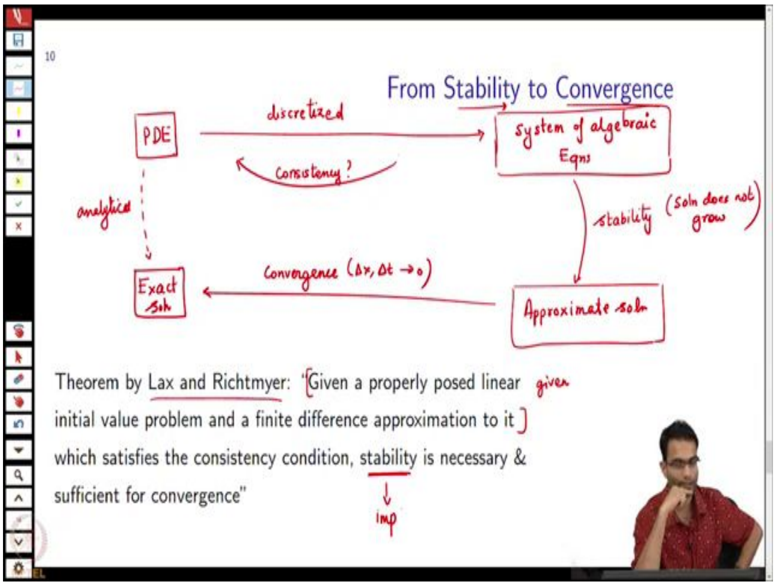

Now, the next question we will ask that is it good enough ok. You can ignore the text written for now let us just get to it step by step. So, stability I have got, but is my answer correct that is the next question that is that is meant by convergence ok.

So, we will go through this very systematically. What do we start with what was absolute starting point for FDTD before Yee cell? Maxwell's equation, partial differential equations; so, we start with PDE or a set of PDEs right what did we do to this PDE? Next we went to Yee cell right in other words we discretized right. So, we went, let's draw long arrow here, we discretized, once I discretized what kind of equations should I get? Are they linear?

We use Taylor's theorem, we substituted a derivative by function value itself. So, I got a system of equations right. So, I can write this, I can write this as a system of algebraic equations ok. So, what does ah? So, I have got this I got this discretization which is given me system of algebraic equation, another question that you can ask is this discretization depends on delta x and delta t right; obviously, that is the discrete.

So, you can in fact, go in the reverse direction and ask consistency. Means this system of equations, which is I have discretized, is it consistent with the actual PDEs. That one question you can ask, a legitimate question. Then let us leave that for now. Once I got the system of algebraic equations what did I do? I solved them and solving them gave me an approximate solution and under what conditions did I get this approximate solution? Stability right. Under stability, the approximate solution that I got I made sense. So there is a notion of stability in going from here to here ok.

So, from these three sort of square boxes is there something that is missing still? I have the PDE s, I have a system of equations and I have an approximate solution what other things can you think of? I mean the hint is that I would have drawn three boxes.

Student: (Refer Time: 17:56).

The actual solution right exact solution; so, we can call this is the, in the olden days where there was no computation you try to go from here to here by doing what analytical right, but we know that is not possible many times right. That is why you have numerical methods.

So, what is this, what should I write for this arrow? Convergence right. That is what is convergence. Mathematically how will I write convergence under what conditions I mean how will I have an approximate solution and I wanted to convergence. So, what is convergence? You say $\Delta x, \Delta t \to 0$ that is when I will get the true solution. In real life do I actually have a true solution to compare the convergence with? No. So, what is the notion of convergence then?

Student: Numerical.

Numerical convergence. Subsequent discretizations do not differ very much in their answer right. Beyond a certain point, discretizing it even finer is giving basically the same answer to machine epsilon. So, that is our numerical convergence right. So, stability we have seen solution does not ok. So, now, because convergence is hard to do, here is the theorem by Lax and Richtmyer which is very crucial in these things.

So, what is it saying? So, it is given a properly posed initial linear initial value problem and a final difference approximation. So, this is what is given to me ok. So, this is given what is it saying? It also should satisfy the consistency equations, that is fine then its saying that if you can give me stability, stability is necessary and sufficient for convergence ok.

So, this is a one of those theorems which mathematicians make which in practice as engineers we do not tend to use very much, but you should know, if someone asks you: what is the guarantee that your solution will converge to the true solution. If you do not know the true solution how will you answer that person? You cannot, but this theorem can be quoted in reference saying that my solution is stable for example, in FTDT what will you say?

Look my choice of Δt respect the courant stability criteria therefore, it is stable, by this theorem it is good enough for me to have a convergence in my solution and what kind of convergence will you be able to show numerically I mean you will be able to show numerical convergence. So, then you have solve the PDE, otherwise someone will say you have solved it you have got some solution, but what is the proof that this is the true solution right. So, stability what I mean the bottom line is stability is the important thing ok.

So, when you look at commercial FDTD software, one of the most important knobs is this courant stability factor. You have to make sure that it is obeying it such that your stability is satisfied therefore, convergence is also satisfied ok; any further questions on this? As I mentioned in practice this is apart from just making sure that the courant factor is satisfied there is not much practical use in the engineering that we will have for this.