Computational Electromagnetics
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Finite Difference Time Domain Methods
Lecture – 12.04
Stability Criteria – Part 1

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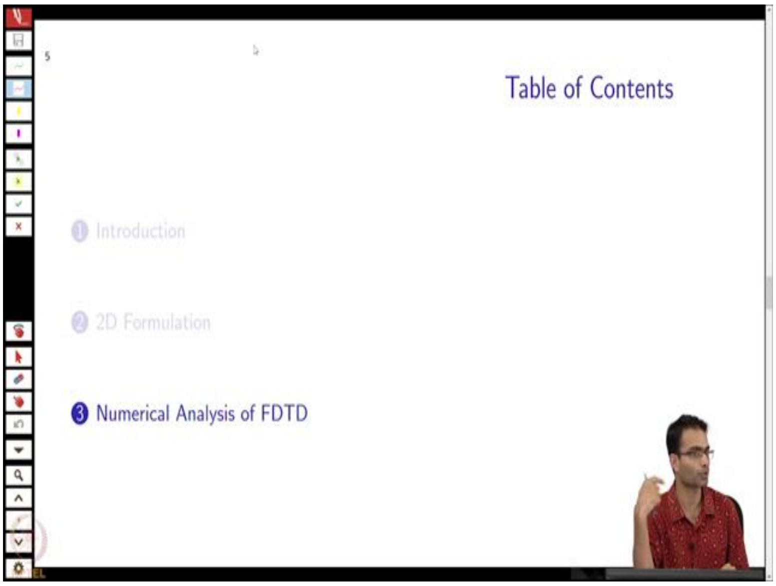

Now, let we have looked at; we have looked at basically how do you deal with the finite difference time domain methods in space and time; space and time updates is what we have seen. Now, we have to actually get it in to the details of the numerical analysis of FDTD. So, this is relatively a new thing which we did not study in the case of integral equation methods of finite element methods, and that is coming because now, we are dealing with time explicitly.

So, for example, a question that can come over here is, does the solution have stability, does it converge or does it explode? In the case of the integral methods we took e to the j omega t, so it always was having a time harmonic behavior. Here, it is not guaranteed, so we want to study under which conditions, are there any conditions?

So, what are the main parameters that we have seen so far in the FDTD? The discretization, $\Delta x,\ \Delta y,\ \Delta z,\ \Delta t$ and so far we have not given any constraints or any design principle that you should choose it to be this much or that much. So, you know how do we know what values of $\Delta x,\ \Delta t$ we should choose in order for us to get a solution which has physical meaning right. I can run it for any I can run the procedure for any $\Delta x,\ \Delta y,\ \Delta t$ does not mean is the correct answer. So, this is the part may be study the numerical analysis of when will I get a meaningful solution.

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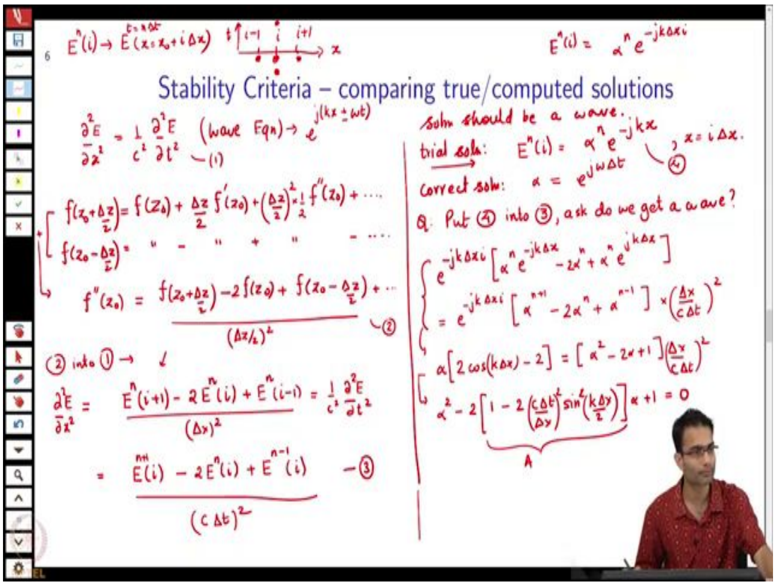

So, the stability criteria that we will look at is something very simple. What we will do is, we will take a trivial example where I know the true solution ok. And I will compare that with the computed solution from FDTD, and we would want both of those to be the same or at least control the error. So, let us see under what conditions does that happen. So, what do you think is the simplest Maxwell as a simply solution to Maxwell's equations, we will pick that and then we will solve that. So, you want what about Maxwell's equations in free space, no current, no sources, nothing right. And we know the solution is plane wave right, so this plane wave comes to our rescue many many times.

So, let us start with the plane wave idea. The plane wave we know again in one dimension we will just take it very simple satisfies the wave equation. And we already know that this has analytical solutions of which form? Plane wave right I mean, so it is what $e^{j(kx-\omega t)}$ right, we know that these are the solutions to this equation. So, it is very good we have a reference solution.

Now, when we begin to discretize these equations, what is what would you do, how would you discretize these equations? So, first of all let us see what order of derivative is there second order derivative right. So, when I write I will use Taylor’s theorem, because that is what we used. So, $f(z_0 + \Delta z/2)$ ok, first term I will also include second order derivative ok.

$$f(z_0 + \Delta z/2) = f(z_0) + \Delta z/2\, f'(z_0) + (\Delta z/2)^2 \times f''(z_0) + \cdots$$

So, now I have got Taylor's approximation up to I have written at up to the second order what is and I want an approximation for what the second order derivative right. So, similarly I can write down for $f(z_0 - \Delta z/2)$ right; same term minus same term plus that is the format.

So, now if I want to approximate the second order derivative what do I do to these two equations, add subtract multiply divide whatever I do. Add these two equations if I add these two equations.

Student: 5.

I want an approximation see the first the wave equation as second order derivatives. So, I want an approximation for second order derivative, so if I add these two equations. So, this says that I can just bring it already on one side f' of over here, so what all terms will have when I add it over here. So, I will have

$$f''(z_0) = f(z_0 + \Delta z/2) - 2f(z_0) + f(z_0 - \Delta z/2)$$

Is that right or are we missing any term? So, in general it is I mean yeah I can just leave it this is the plus higher order terms ok, let us we need not worry about. So, let us call this equation 1 and this is now equation 2. So, if I put these 2 into 1, I can apply this to space, I can apply this to time right, so let us write down the first term right.

So, it is this is what I want to do; now I am looking at electric fields on the grid ok; so space derivatives. So, where do I know electric fields in space how far apart are electric fields?

Δx right; so the so what should I write over here; first in a short hand notations, all of these will happen at the same time, but a different locations in space.

So, E_n what should I write $i, i+1, i+1/2, i-1/2$ what should be the first term be? There is no j let us make it 1D. So, which the grid points are just numbered by integers 1, 2, 3, 4, i right. So, $z_0 + \Delta z/2$, so this $\Delta z/2$ is what, it is the next grid point - $(i+1)$.

Student: Yeah.

If I take z_0 to be i, so it is $i+1$ next term is minus; I am choosing i to denote z_0. So, in what I wrote over here, f is evaluated at three points $z_0 + \Delta z/2,\ z_0,\ z_0 - \Delta z/2$. So, what is the spacing between two adjacent points?

$\Delta z/2$, that is my Δx because that is where I am storing my electric field right. So, this term will be become 2 times right; next final term is easy it is what? $E^n(i-1)$ time index is n space index is $i-1$.

Student: 1.

Because it is $z_0 - \Delta z/2$ it is a previous one. So, think of the time think of space like this x, this is i, this is $i+1$, this is $i-1$ ok. So, graphically what is the second derivative? It is the combination of the function evaluation at these three points ok.

Student: So, we are taking the (Refer Time: 08:39).

No, it is the same and divided by what? Not 1. Come on, what is the denominator of equation 2?

Student: $(\Delta x)^2$

$(\Delta x)^2$ right. So, you are probably forgetting what is meant by this; this means electric field at $t = n\Delta t$ and $x = x_0 + i\Delta x$. Now hopefully it is clear; each of the points in space are apart spread apart by Δx, each of the time instances are spread apart by Δt, that is the discretization in space and time right. And this is, therefore, going to be equal to $1/c^2$ ok; so, now what should I write this as first term. I am trying to approximate the time derivative second time derivative. So, E what do I write, what point in space should we choose? The simplest point in space i.

Student: i.

Very good. Time? n plus 1 that is a right because now I am taking a time derivative so, I have to take different grid points in time minus. Next term $2E^{n}(i+1)-2E^{n}(i)+E^{n-1}(i)$ divided by how far apart are they in time?

Student: 1.

Not 1, Δt. So, what should the denominator here be? Not Δt^{2} something else, forgetting a factor of c that is coming from the wave equation. So, this is what are wave equation which was very nice to look at in continuous coordinates, but in the discretize version these are all the terms that we did right. So, I am discretizing space, I am discretizing time ok. So, in this figure you can think of these other two instances if this is time over here right.

So, this, this these are the five sort of stencils points that are being used, the grid size remains the same I am evaluating my function at grid points spaced apart by Δx, Δt. But now to approximate f'' I need 3 values of f earlier I needed 2 values ok. Is this figure clear or should I draw it again? It is clear right. There is there is x and t is the vertical dimension. So, these are the five points five points stencil which is used to evaluate this function notice the midpoint is common E n i is common alright.

So, now, we want to see what kind of a wave flows on such a grid which has a finite Δx and a finite Δt in the limit of course, its correct right in the limit it will give me the differential equation, but when I choose some finite Δx and finite Δt what happens ok, and I know what the solution should be right solution should be a wave ok.

So, what will do is we will assume this is a common technique in numerical analysis you assume a kind of a trial form of the solution that you expected to be and substitute in. Instead of trying to derive it from the equations we will assume a trial form and push it into the equation and see whether it works ok, and this trail form should be general enough to give wave solution and non-wave solution also and we can see what is the solution that it is giving me ok.

So, the trial solution we are going to take something like this. So, we will keep the wave in space and we will question whether we are getting a wave in time that is the strategy that we will follow. So, I am going to write $E^n(i)$ in this following form. Let us take a movement to observe this what am I saying over here, the trial solution is of this form where there is a wave in space all right and there is some alpha term in time this x is simply the discretize version. So, $i\Delta x$, i is your integer not your complex i . The correct solution has what; for the correct solution what is α? Not 1? Yeah.

Student: $e^{j\omega t}$.

$e^{j\omega t}$ right. So, this α is little bit more general then $e^{j\omega t}$ is a phasor, but supposing α turns out to be purely a pure exponential like a purely decaying exponential or purely increasing exponential that can capture.

Student: Because this.

It can be for example, α could may turn out to be $e^{\beta t}$ where, β is some complex number; then what will happen? Our solution is no longer a wave, it is attenuating or increasing something which is not physically expected in this situation. So, we have chosen this solution because it captures a variety of solutions. You are right α is actually Δt here yeah, so α^n so, $n\Delta t$ becomes my t. What will we do now?

So, what does the question become? Question will become if so, we will take this form over here 4, so the question is as follows; put 4 into which equation, into 3 and then ask do we get a wave. Then life is good and everything is consistent. My system of my way of discretizing is fine. So, I will give you a hint when we try to answer this question we will get some condition on delta x and delta t which will and under those conditions the solution is a wave that is what we will happen ok.

So, let us let us try to substitute this. So, let us take the left hand side of equation 1, what will I get? So E^{n+1} so we will just write this over. So, $\alpha^n e^{jk\Delta xi}$ that is my $E^n(i)$ ok. So $E^n(i+1)$, so what will happen? So, notice that all the terms on the left-hand side and the right-hand side have one factor in common which is $E^n(i)$ there all exponential so there will be multiplied with each other.

So, this $E^n(i)$ term will be basically common so, that is going to be $e^{jk\Delta xi}$ this is going to come out to be common. First term $E^n(i+1)$ so, what will this first term be? So, α is going to remain constant everywhere α will be here because take α^n right, and next terms is going to be e^{jk} what, we are substituting 4 into 3 ok. So, I am asking you what is the first term of 3 left hand side of 3 first term having substituted this.

Student: Delta x (Refer Time: 17:23) delta x.

Δx, very good; next term is going to be a minus 2 that is $n+1$ right. So, look at the term I have taken outside the bracket multiplied that by the term inside, I get $(i+1)\Delta x$, second term is $2\alpha^n$. Anything else? That is all, final term is $\alpha^n e^{jk\Delta x}$ right is equal to; so, the $(\Delta x)^2$ I will take on the other side again over here I can take out $e^{jk\Delta xi}$ common from here. So, the first term will become α^{n+1}. Anything else? No.

Student: $-2\alpha^n + \alpha^{n+1}$

$-2\alpha^n + \alpha^{n+1}$, anything else, no; into these remaining constants right which is $(\Delta x/c\Delta t)^2$.

$$\frac{\partial^2 E}{\partial x^2} = e^{-jk\Delta x\, i}[\alpha^{n+1} - 2\alpha^n + \alpha^{n+1}] \times (\Delta x/c\Delta t)^2$$

So, there is a lot of terms that can that we can see are getting cancelled from here. So, what terms get cancelled, the exponential gets cancelled there is any α gets cancelled, what is the most convenient power of α to cancel from both the left and right side?

Student: Right hand side.

α^{n-1} right, the last term on the right hand side. So, this will further simplify you can cancel it off, so the first term is going to become what will it become? So, α^{n-1} so this alpha will come outside let us say I am cancelling α^{n-1}. So, $e^{jk\Delta x} + e^{-jk\Delta x}$ what is that? $e^{j\theta} + e^{-j\theta}$? $2\cos\theta$.

So that is going to be $\alpha[2\cos(k\Delta x) - 2]$. $e^{j\theta} + e^{-j\theta} = 2\cos\theta$ and that is why I got this and the right hand side I cancelled α^{n-1}. So, the first term will be $[\alpha^2 - 2\alpha + 1] \times (\Delta x/c\Delta t)^2$. So, in terms of your I mean what kind of an equation is this in alpha? α is the guy that I am trying to solve for, so it is a quadratic equation; it is a quadratic equation right.

So, I can write, further simplify this in the quadratic form. So, I am going to get an α^2, this $(\Delta x/c\Delta t)^2$ I will push to the left hand side cos; so, $[2\cos(k\Delta x) - 2]$ can be simplified in terms of half angles. I will write it as $\sin^2(k\Delta x/2)$ right.

Student: Minus 1.

There will be a minus 1 over there. So, I will write down the final form that you get ok.

$$\alpha^2 - 2[1 - 2(c\Delta t/\Delta x)^2 \sin^2(k\Delta x/2)]\alpha + 1 = 0$$

So, I moved this $(\Delta x/c\Delta t)^2$ to the other side that $[2\cos(k\Delta x) - 2]$ became $\sin^2(k\Delta x/2)$ that is your term over here right. Very simple algebra right nothing very complicated about it. And so, this whole term over here I am which is going to call it some A. So, we are almost there we want to now look at what is the kind of solution that I get from here what. So, in other words what is what α do I get?