Computational Electromagnetics
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Finite Difference Time Domain Methods
Lecture – 12.03
2D FDTD Formulation Divergence Conditions

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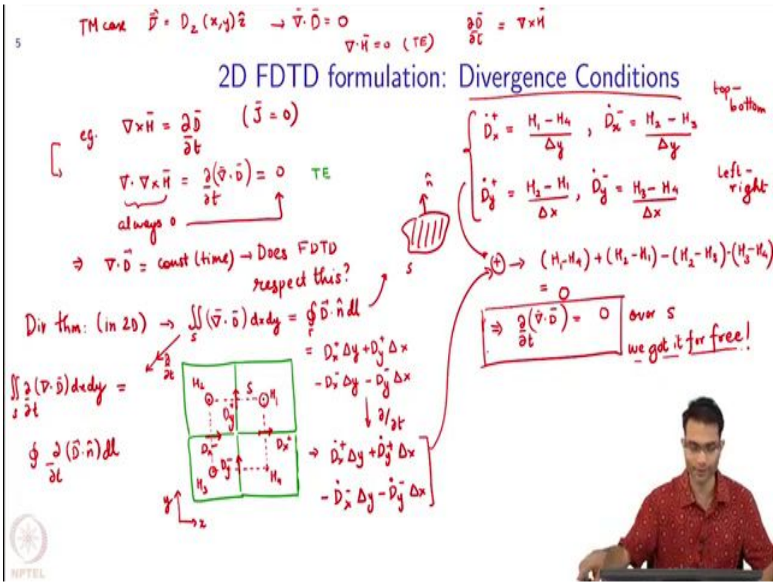

So, let us move to the next discussion which is what Divergence Conditions, now notice that when we use our Maxwell's equations we actually use only the first two Maxwell's equation, we do not touch the other two equations. So, the actual question comes is when I am doing this discretization in space and time, what happens to the other conditions are they beings violated or are they being respected right. So, in particular taken example of your one of these equations, so $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$ we are assuming J=0 for now (Refer Time: 00:56).

Now, if I take and let us also assume a charge free situation. So, if I take a dot product on both sides ok, $\nabla . \nabla \times \vec{H} = \frac{\partial \nabla \vec{D}}{\partial t} = 0$ .Left hand side is a identity in vector calculus right, this is always 0 right implies, so I am will just write this equal to 0 over here. So, what is that in terms of time what does that tell us about divergence of D?

Student: Constant.

Constant right; so, this implies that divergence of D is constant in time ok. So, if it for example, if it is 0 to begin with it will continue to be 0, this has nothing to do with FDTD, this what Maxwell's equation that telling me right. But now let us see that the update equation that I have written, so does question is this, does FDTD respect this that is the question we are asking.

Yes, the other equation will tell me what is the value of this $\nabla.\vec{D}$ for example, ρ; right, but ρ may or may not be a function I mean may or may not be a function of time in the depend. But under these conditions this is what Maxwell's equation is telling us. So, let us reflect on this divergence condition a little bit, and we want to find out what is FDTD tell us about this. So, what; how will we get an answer to this?

So, first of all this is a condition that is in continuous space $\nabla.\vec{D}$ right. So, if I want to look at it in discretize space then I have to think of these stencils and discretized versions of which variable D right. So, which of the two polarizations will be more interesting to answer this question TM or TE? TE, because I will have a access to E_x and E_y and I can evaluate this divergence in a way that is easy to interpret, otherwise in the case of a your TM polarization I have to be looking at the z axis right.

So, let us see what should be do about this, so $\nabla.\vec{D}$; when you see $\nabla.\vec{D}$ what is it remind you of in which theorem of vector calculus comes to your mind, divergence of a vector field; Gauss's theorem or divergence theorem right. So, divergence theorem let us say in 2 D, because we are in 2 D now what does it tell us $\int\int_S (\nabla.\vec{D})dxdy = \oint_\Gamma \vec{D}.\hat{n}dl$ right. So, this is our divergence theorem in; So, that is going to be one nice way of calculating $\nabla.\vec{D}$ over some space ok. And the right hand side now has become the outward flux of D ok. So, now let us draw a grid, let us go back to our TE polarization which had E_x E_y or D_x D_y right. We can; now since we are working with divergences let's use D instead of E; I mean since your working with the displacement field. So, what is the variable that I have a here H_z is here at the centre right and the other two variables are.

Student: E_x E_y.

E_x E_y or D_x D_y, one and the same. So, this is my D_x this is my D_y, this is the stencil that I have. Now I want to evaluate this divergence theorem in 2 D over here. So, for that I need a surface and a line enclosing it, so what might be good way to do this. And remember I want to in that be able to evaluate $\vec{D}.\hat{n}$ ok, so just to give you the physical interpretation over here, this is some region over here S and this is my n hat right. So, something like that is what I want.

So, what kind of a grid or what kind of a region S will be suitable now? So, the left hand side is not going to help us very much, the right hand side is a line integral. So, what kind what kind of integration contour do you think of.

Student: Square.

So, it should be well a square will be the most natural thing and since I want to evaluate $\vec{D}.\hat{n}$. So, if I take my contour to be along the grid lines will it help me. Supposing I take the blue the blue box itself to be S; sorry not blue green box itself to be S will it help me. So, what will happen to $\vec{D}.\hat{n}$? Well, not exactly 0 because let us say at this point where D_x is there is also value of D_y; D_y is not 0 over there, but I am not storing that variable over there I will have to calculated by interpolation.

So, that is the inaccuracy over there, if I wanted to directly calculate this at the point; that means, at this place where I have D_x, the normal vector should be which way along D_x, then when I go to D_y which where show the normal vector be preferably along D_y right. So, what if I take now think of extending this there are going to be four adjacent grids, four Yee cells right what should I do? So, let us call this value H_1 ok, now there is going to be D_y over here also this is going to be a D_x over here also ok. Now do you have a hint has to what a square right, so if I take it like this ok. At each centre there is going to be some value of magnetic field H_1, H_2, H_3, H_4 ok.

Similarly I should give some names to this D_x is to distinguish them. So, this D_x which is going outward over here I will call it D_x^+. It is leaving this red box, this D_x on the left hand side which is coming in I will call it D_x^-, this D_y^+ over here plus because it is going out and D_y^- because it is coming in. So, we have we have minus to get this all right.

So, now, what happens to the right hand side? Let us evaluate $\oint_{\Gamma} \vec{D}.\hat{n}dl$ ok. We can assume that the value of D_x is approximately constant over this right.

$$\oint_{\Gamma} \vec{D}.\hat{n}dl = D_x^{+}\Delta y + D_y^{+}\Delta x - D_x^{-}\Delta y - D_y^{-}\Delta x$$

Now that is done, we have not yet actually used much of the FDTD, FDTD was one of the main points was linking time derivatives to space derivatives. So, for we have not done that so now, let us see how do we evaluate let us say, so which is the equation that we are trying to do? $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$, this is what is going to help us right.

So, this equation I can look at the x-part and the y-part, we have done this when we look at the TE stencil right, we had those update equations for E_x E_y basically that is what is needed. So, what is D_x^{+} if you look back at your notes dot right that is the time derivative. So, what was that equal to just look back at your nodes they should be proportional to?

Student: $\frac{\partial \vec{H}}{\partial y}$

So you remember the mnemonic we are made top minus bottom right. So, top minus bottom this will gives us what, in terms of the variables;

$$\dot{D}_x^{+} = \frac{H_1 - H_4}{\Delta y} \qquad \dot{D}_x^{-} = \frac{H_2 - H_3}{\Delta y}$$

$$\dot{D}_y^{+} = \frac{H_2 - H_1}{\Delta x} \qquad \dot{D}_y^{-} = \frac{H_3 - H_4}{\Delta x}$$

If you do not remember this no problem we can just go back to your notes and derive the difference equation. So, finally, what I am wanting to look at, not this equation, but the time derivative of this equation right.

So, what I want is;

$$\int\int_{S} \frac{\partial}{\partial t}(\nabla.\vec{D})dxdy = \oint_{\Gamma} \frac{\partial}{\partial t}(\vec{D}.\hat{n})dl$$

I have put a time derivative on both sides of this equation and that will therefore, be equal to the time derivative of right. So, what I will do over here, I just did a $\frac{\partial}{\partial t}$ in both sides. So, if I take a $\frac{\partial}{\partial t}$ over here on both sides, what will happen?

$$\oint_{\Gamma} \frac{\partial}{\partial t}(\vec{D}.\hat{n})dl = \dot{D}_x^{\ +}\Delta y + \dot{D}_y^{\ +}\Delta x - \dot{D}_x^{\ -}\Delta y - \dot{D}_y^{\ -}\Delta x$$

The time derivative will come on top of each of this.

Now, do I have these expressions with me? Yes, I have evaluated all of this right. So, now, if I take this expression and combine it with this over here, what do I get?

$$\oint_{\Gamma} \frac{\partial}{\partial t}(\vec{D}.\hat{n})dl = (H_1 - H_4) + (H_2 - H_1) - (H_2 - H_3) - (H_3 - H_4) = 0$$

Everything has very neatly cancelled off right. So, implies that over this tiny cell $\frac{\partial}{\partial t}(\nabla.\vec{D}) = 0$, we cannot say that this is true at every point because we are not evaluating these fields at every point, but we are evaluating over a finite grid. So, over this grid S which I have shown over here, it is clear that the divergence of D does not change in time. So, we can say we got this for free, we did not do anything special; what was the key reason we got this for free?

Student: The design of where this.

The design of this e cell by staggering everything by half a grid those finite and those taking finite difference as two sided center right.

Student: Centre.

Centre differences.

That is how these all is H_1 got cancelled from another H_1 from somewhere else and so on, all the Hs got cancelled over there. So, this divergence condition being satisfied that the divergence remains constant with time we are just getting this for free right. So, this is one of the sort of elegant aspects of the FDTD Yee cell ok. For the TM case how will be prove it?

Student: Yeah, because there is only E_z it also there.

Right so, there we will have to take our S not in the xy plane, but will have to take it in let say the your zx plane or zy plane or something like that right. Then I will have, but that again want help me very much because my E will be along the $\hat{n}$ I mean sorry perpendicular to $\hat{n}$. So, the dot product will give me 0.

Yeah.

Student: There by less than.

The other divergence condition will be easier in TM.

Student: The (Refer Time: 18:30) had some.

Yeah, so in this is when the case of TE polarization you could show that divergence of D remains constant, in the TM polarization we will be able to show that divergence of magnetic field remains conserved.

Student: Right.

Yeah that is also true.

Student: So, will have a constant.

Actually yeah in the in that TE case sorry in the TM case, E_z is what? Pointing so, let us say take the TM case what is electric field, only $E_z(x, y)\hat{z}$ and what is it a function of?

If you want to be very precise let us make this D. So, what is divergence of D?

Student: 0.

0 automatically nothing has to be done, because is $\partial/\partial z$ of something which is not a function of z right. So, this is the easier thing to do, but in TE you have to take this loop and direction whatever. Here curl of which quantity?

Student: H.

Which is what we did no. $\nabla \times H$ will be use curl.

Student: (Refer Time: 19:49) the other Maxwell's the other divergence Maxwell's.

Yes the other divergence equation which is what, $\nabla.\vec{H}$ right, that will automatically we satisfied here by the same logic because H_z is the function of x, y. So, $\nabla.\vec{H}$ is going to be 0 right ok. So, in other words I mean what sort of final line is that if divergence starts out as 0, my FDTD scheme guarantees that at remain 0 throughout the simulation ok, there is no accumulation of error on that account ok. So, we yeah as I said we just got at for free ok.

So, we will close this discussion now and next we look at numerical analysis of FDTD.