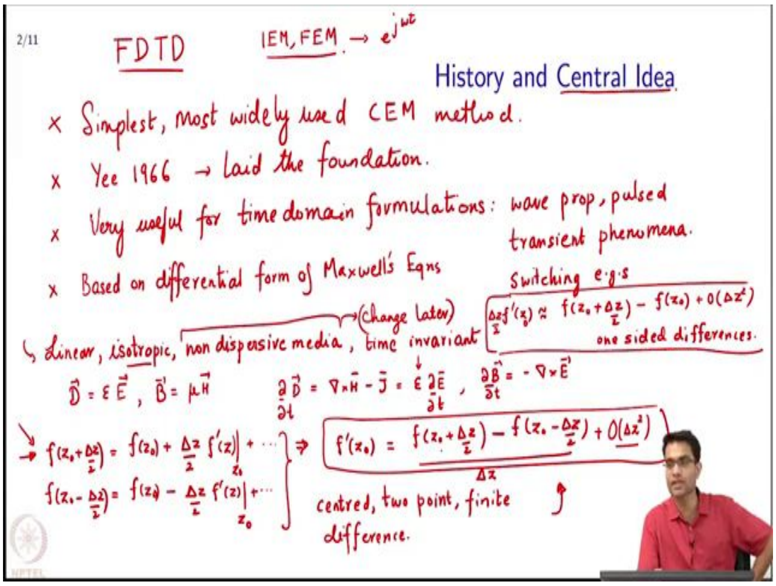
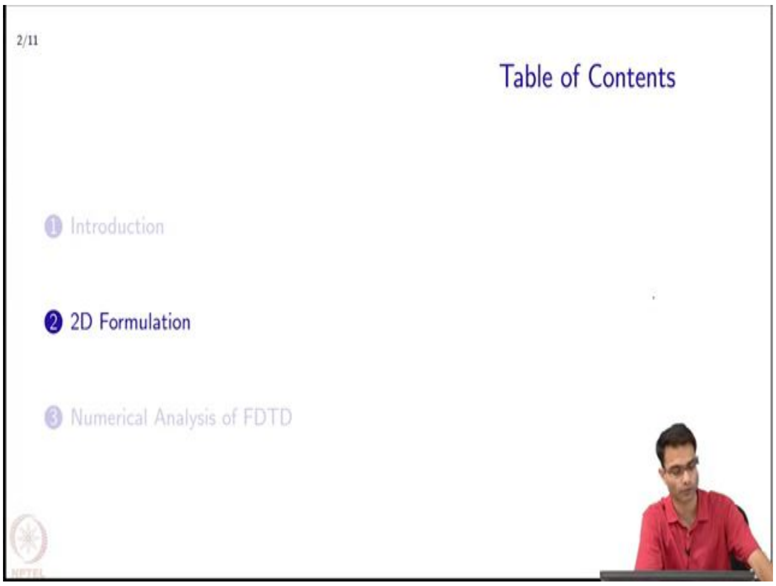
Computational Electromagnetics
Prof. Uday Khankhoje
Department of Electrical Engineering
Indian Institute of Technology, Madras

Finite Difference Time Domain Methods
Lecture – 12.01
2D FDTD Formulation : Stencil

(Refer Slide Time: 00:14)

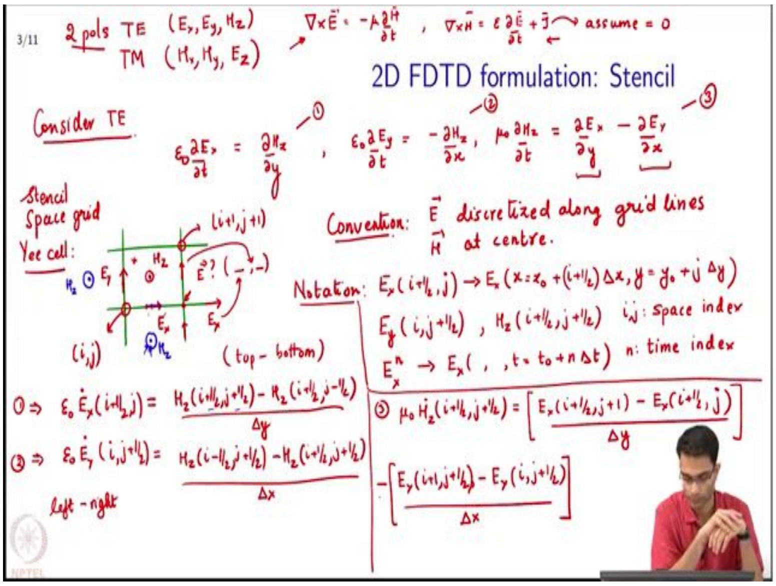


(Refer Slide Time: 00:15)



So, now, let us instead of doing a 1D formulation, we will jump to a 2D formulation. It conveys the idea a little bit better ok.

(Refer Slide Time: 00:24)



So, when we look at 2D, again this is common to what we did in the integral equation methods and finite element methods also, how many polarizations are there independent that we can talk about? 2 polarizations right. So, there are 2 polarizations TE and TM Transverse Electric Transverse Magnetic and any problem any arbitrary problem can be written as a linear superposition of both of these. So, I can deal with each of these two separately and then worry about superposition later right. So, TE as we have said it consists of which variables?

Student: E_x, E_y, H_z (Refer Time: 01:07).

E_x, E_y, H_z. So, far and TM was H_x, H_y, E_z.

Student: H_x, H_y, E_z.

So, let us take the TE polarization ok. So, we will just write down our Maxwell's equations once over here $\nabla \times \vec{E} = -\mu\, d\vec{H}/dt$. So, given our previous experience with Maxwell's equations is tempting to always replace the d/dt by a $j\omega$, but we should remember not to do that. That is the whole point of FDTD. $\nabla \times \vec{H} = \varepsilon d\vec{E}/dt + \vec{J}$.

To keep matters simple for now we will just we will assume that there is no current, but we will add it in later in the module ok. Now in TE polarization I can use these two Maxwell's equations to give me few scalar equations right because the unknowns are only E_x, E_y *and* H_z. So, the first equation will become and let us further just take vacuum ok.

So, ε_0 right. So, I am taking the second Maxwell's equation and looking at the x component and assuming current 0 ok. So, this is $\varepsilon_0 dE_x/dt$ and on the right hand side I will get spatial derivative of the H-field and what will that be? You can work it out right. It will be $\partial H_x/\partial y$. Then I get the y part of this equation again time derivative, I will get a $-\partial H_z/\partial x$. ok. So, I have used the first Maxwell's equation and finally, what is left is the first Maxwell's equation; In that, how many components are there which are non-zero in this equation?

Student: One.

Only one because $\vec{H}$ vector is only H_z so, I only have to take the z component. So, I will get the spatial derivatives of $\vec{E}$ now right. So, in terms of time derivative, I will have $\mu_0 \partial H_z/\partial t = \partial E_x/\partial y - \partial E_y/\partial x$.

Student: This is an extra minus (Refer Time: 04:28).

This is an extra minus sign where?

Student: Sir whatever you wrote at the right hand side.

Yeah.

It is correct right ok. This is just a simple working out the curl. Here you should not make minus sign mistake otherwise all your computation is gone. So, now, we want to before I sort of late out for you let us try to design this. My variables are (E_x, E_y, H_z) that and they have to be discretized in space and time ok. I have seen that the discretization scheme needs me to know E at different points in space different points in time.

So, what you I mean the way to do this is, you have to make a grid; obviously, it is a 2 dimensional problem, the unknowns are on a grid right. So, I make a grid. So, it is a repeating grid I am just showing one unit cell ok. And what I can do is I can call this E_y. So, at the centre of this and this over here at the centre of this I can call E_x, and here is where I will do H_z.

So, it will as we work more with this it will become clear why I did not define all of these unknowns all at the centre of the cell or at the corner of the cell; I have staggered them out in space, later on we will also stagger them in time, but will come to time first right now we are just dealing with the space grid. So, this grid was what was proposed by Yee in 1966.

So, it is also called a Yee cell. Now these it is like a lattice right. So, these points are given some shorthand numbers. So, I will call this point (i,j) and therefore, this point over here is $(i+1,j+1)$. So, people also called this a stencil, various names for the same thing, which I repeated everywhere in space as well as it is called stencil.

Student: The grid will end by.

The grid ends at some point I mean it let us say it ends on a straight line.

Student: Yeah.

Right. So, then.

Student: E_y will be there.

Only E_y will be there. So, I mean it ends where it ends where. So, the convention is the electric field is being discretized along the along the grid boundaries and $\vec{H}$ at the center. Can you think of a practical reason for doing this? I could have done at the other way also.

Student: (Refer Time: 08:01).

Student: Continuity equation.

Continuity equation. So, most materials that we work with are electric I mean they are not magnetic materials the electric material. So, we have to many times worry about tangential continuity of electric fields. So, by choosing a variable to be at the grid line, it is easier to work with boundary conditions. So, it is not a golden rule you can have it the other way, you can define you electric fields to be at the centre of the domain and magnetic fields on the boundary. So, this is just one choice ok. So, this is the convention. Now we also need a notation. So, let us write that out.

I will show you what will do with TM polarization.

Student: You can have H_z .(Refer Time: 08:47).

At the corners, we can have H_z at the corners also. In fact, that is what we do in the case of the TM polarization. So, let us just write down some notation over here. So, E_x for example, at i plus half if I write something like this $E_x(i+1/2, j)$ you know; that means, right. It means that I am actually evaluating $E_x(x = x_0 + i/2\Delta x,\ y_0 + \Delta y)$.

So, instead of writing this very long right hand side, I just use this shorthand for the left hand side I just mention which are the points. So, E x for example, is not evaluated at integer x is, but it is evaluated at integer y. So, that is why is $i + 1/2$; similarly E_y what will where are we discretizing E_y ?

Student: $E_y(i+1/2, j)$.

$E_y(i+1/2, j)$ and it's clear what I mean by this. Similarly, when I talk about H_z ?

Student: $i + 1/2$.

$H_z(i+1/2, j+1/2)$

Student: (Refer Time: 10:27).

What about E_x ?

Student: I did not get how do you get $i + 1/2, j$.

$i + 1/2, j$.

Student: $x = x_0$

Yeah x_0 is some origin Δx is the discretization right.

Student: Discretization.

So, $i\Delta x$ will bring me to the corner plus $1/2\Delta x$ will bring me to where this electric field is being discretized. It is being discretized in the middle right; middle of the line so that is why the plus half is necessary. Similarly in the case of E_y it is in the y part its $j+1/2$ not j.

Student: But where in going the expression their why will be constant (Refer Time: 11:07).

When we go in the.

Student: Expression, but the phi coordinates will not change.

Yeah they will not change. So, that is why it is just j yeah. I am just trying to discretise a 2D grid in terms of node locations as well ok. One thing we are missing so far is time. Time has not been indicated over here and we should discretize time also right. So, again the notation that will have for time is, we will put it in the superscript right. If you see this it means E_x at some point in space and $t = t_0 + n\Delta t$ ok. So, we will use these very compact notation is to describe where we are talking about space and time.

So now, that we have done this let us try to convert equations 1 2 3 using our central difference idea. So, the first equation what we write for our first equation? So, it is spatial, sorry time derivative of E_x. So, for now we will just write. So, equation 1 it becomes. So, we will just write this as $\dot{E}_x$ we will put a dot on top to indicate time derivative, what coordinates in terms of i and j?

Student: (Refer Time: 13:00) i plus half.

$(i+1/2, j)$ is equal to. Now we come to the $\vec{H}$ part. So, partial derivative of H_z with respect to which coordinate? y coordinate ok. So, what should I write on the right hand side? So, this is H_z let us. So, what should I write this as?

Student: Epsilon (Refer Time: 13:25) delta x by 2 minus.

Yeah.

Student: y plus delta.

Yeah y. So, H_z evaluated at in terms of (i,j) let us try to write it down.

$$\varepsilon_0 \dot{E}_x(i + 1/2, j) = [H_z(i + 1/2, j + 1/2) - H_z(i + 1/2, j - 1/2)]/\Delta y$$

Right the difference between these 2 points. Do not thing that these I mean only the notation is (i,j) does not mean that everything is separated by you not distance right the separation is their you can see Δx and Δy is the discretization.

Student: So, we are not (Refer Time: 14:25).

I have mentioned it this has a dot on top we will work it open it up later. First I want to get the space parts done correctly second equation again simple. So, it is the time derivative of E_y, but evaluated where coordinates are $(i, j + 1/2)$.

Now I have partial derivative of H_z with respect to x with the minus sign never mind the minus sign H_z first term. Actually, let us just absorb the minus sign. So,

$$\varepsilon_0 \dot{E}_y(i, j + 1/2) = [H_z(i - 1/2, j + 1/2) - H_z(i + 1/2, j + 1/2)]/\Delta x$$

So, there is a sort of trick short hand trick you can use to remember this E_x derivative can be thought of as top minus bottom, right the top value minus the bottom value. So, I write this as top minus bottom right and this over here can be thought of as.

Student: Bottom.

Second.

Student: Left minus.

Left minus right ok. So, just to show you what we have done, we have considered which points so far? We have considered this point and these are the points that have been involved in these finite differences right H_z at these points. So, this blue and this red are involved together in giving me E_y, this red and this blue over here they are involved in giving me E x information right look at the first one. Time derivative of E_x, I am talking about this guy over here and what are the values being subtracted?

$H_z(i + 1/2, j + 1/2)$, that is this guy over here and $H_z(i + 1/2, j - 1/2)$ that is this guy over here right. So, it is very neat that when I am taking the difference and approximating the value at a grid point in between. That is why I choose my E_x to be discretized over there and not somewhere else. Similarly second equation you see $\dot{E}_y = \partial E_y \partial t$ it started on the left and right and their differences giving me this.

Student: Sir, you are taking both of these medium.

Yeah, I am taking vacuum.

Student: Vacuum.

ε_0 is here. So, it is a vacuum then what is left?

Student: Equation.

Equation 3. What becomes of equation 3; So, equation 3 alright. So, left hand side is simple $\mu_0 \dot{H}_z$. What are the coordinates?

Student: (Refer Time: 18:24) $(i + 1/2, j + 1/2)$

$(i + 1/2, j + 1/2)$. No problem now I have to write this in terms of.

Student: Space.

Space derivatives of E_x and E_y; so, the first term, this is the first term and this is the second term. First term is $\partial E_x/\partial y$.

Student: E x (Refer Time: 18:46).

So, look over here I am trying to find out approximate H_z derivative over here. So, first term is E_x derivative in space. So, what will the first term be?

Student: E x.

$$\mu_0 \dot{H}_z(i + 1/2, j + 1/2) = [E_x(i + 1/2, j + 1) - E_x(i + 1/2, j)]/\Delta x$$

$$-[E_y(i+1, j+1/2) - E_y(i, j+1/2)]/\Delta x$$

So, once you keep this grid in mind it is very geometric you can just see what minus what and so on right. So, now you can appreciate why we have staggered (E_x, E_y, H_z) in such a way because all the three equations are giving me an approximation at the correct point; when I take for example, if you look back here look at this expression over here. When I take and look at the right hand side, the right hand side is $f(z_0 + \Delta z/2) - f(z - \Delta z/2)$ This is an accurate estimate of f' at which point?

Student: z naught.

$z = z_0$ that means, the average of the 2 points right. So, that is the reason why this staggered grid is chosen. So, when I take the spatial the spatial derivative of E_y right; when I take the special derivative of E_y with respect to x, the approximation is valid at the midpoint of those 2 places which is where I have chosen H_z to be. So, everything is at the midpoint of the other when I start taking finite differences right. So, this is the use of this Yee cell alright.

Student: (Refer Time: 21:46).

E_x *and* E_y are not at the midpoint of the grid, but at the midpoint of the grid line not in the middle of the cell.

Student: (Refer Time: 21:51).

Yeah.

Student: What about the.

Correct. So, E_x so, remember we have Maxwell's equations which are continuously varying everywhere, we want to solve it numerically. So, we are forced to discretized it. So, you are saying that I know E_x only.

Student: (Refer Time: 22:04).

E_x is only known at $(i+1/2,j)$. What about E_x? add (i,j) for example, you can ask what would you do?

Student: Discretized average.

You could discretize it.

Student: Average (Refer Time: 22:17).

There are two ways. So, if I want , I can discretize it at half the grid size. So, I will get at that point also or the smart away would be average, I know E_x on the left hand side I know E_x on the other side right. So, if I take the average of those two I will get an approximate value of the electric field at (i,j). So, and remember these $\Delta x \; and \; \Delta y$ they are in your hands you chose how find you want to discretize it.

Student: What would be their (Refer Time: 22:52).

Direction of.

Student: The E x and E y (Refer Time: 22:54).

These are.

Student: (Refer Time: 22:56) direction.

There is no direction I mean I am assuming E I mean it is along a line. So, E_x whatever supposing (Refer Time: 23:03) value of 5, then I say along the plus x axis a value of 5; if you get minus then pointing in the opposite direction. So, it is a scalar, it is not a vector because the already because E_x is a scalar right. So, x component of $\vec{E}$ it is a scalar can either point in the plus x or the minus x there actually.

Student: (Refer Time: 23:22) something was. So, if I want to tell (Refer Time: 23:24) here the point.

Yes.

Student: So, you should look it E_x *and* E_y and that (Refer Time: 23:29).

Yes. So, supposing we want to find out. So, good question I want to find out what is the value of the electric field at (i,j) vector now what will you do?

Student: Average.

Average the x components average that is the and that is the x part of your electric field, average the y components that is the y part (Refer Time: 23:50) net you have got a vector.

Student: Average (Refer Time: 23:51) because the points (Refer Time: 23:53) right.

Yeah. So, for example, let us say that here, I want what I am asking what is electric field over here that is the question.

Student: Somewhere in the grid (Refer Time: 24:05).

Somewhere.

Student: In the grid.

You want somewhere inside the grid yeah. So, if I want to find the electric field at this vector over here. So, I will take the x component to be the average of this E_x and this E_x right. So, that will give me the x part.

Student: Sir then there will be points where we cannot do anything.

Yeah, they will be point. So, let us come to it step by step. So, at this grid point what do I do? I take the average of this in this there is also E_y here and E_y here I take the average of this and that pops in over here in to the second component right. So, this goes into the first component now what are the vector E? I can repeat this at, I can repeat this at every grid point easiest thing. So, basically what I have done is on a square grid on rectangular grid I have got at every node location I have got the electric field; now you want even finer than that then there are two things you could do.

Student: Sir, but not (Refer Time: 25:03).

Yeah you want somewhere inside the cell right.

Student: In the middle also we can (Refer Time: 25:08).

Yeah, you want to know now what is electric field let us at the location exactly you want to.

Student: That also we can average.

That also we can average.

Student: But any other points (Refer Time: 25:15).

Add any other point. So, what would you do? So, what is that mathematically procedure called? You all know it.

Student: (Refer Time: 25:25).

All of you know what. So, you want to find out the question is to find out the electric field at some random point inside here like this.

Student: (Refer Time: 25:34).

What is that word, technical word? You all know it interpolation right. Once you know values on the grid, if you want finer then that interpolate ok. Now, typically this turns out to not be a real issue, I mean because you are seeing it for the first time this question comes up the discretizations $\Delta x \text{ and } \Delta y$ typically they are so small, there on the order of lambda by 10 lambda by 20, where the wavelength is lambda right. So, for example, if I have a 1 gigahertz your mobile phone works at 1 gigahertz 30 centimeters. So, your discretization is 30 centimeters say by 10 3 centimeters or even smaller than that let us say a 1 centimeter.

So, every 1 centimeter you are anyway calculating the electric field. It does not vary so much that you want to know at some very minute location. This is also it will turn out to be quite a disadvantage of these finite difference time domain methods because you have to discretized everything so finely. So, that is why these are called brute force method; so, you just brute force in space and time calculate everything even though you know do not need it finally, you do not need it right.

The radar cross section example finally, what do I want to know? I want to know the tangential currents. So, that I can apply Huygen’s principle to find out the field everywhere, but when you do FDTD I have to find out the fields everywhere in space, from that only take out what is along the edges. So, there is a lot of work that needs to be done to just get one small thing.

Student: (Refer Time: 27:05) time.

What about the time derivative yeah; so, that is what we will talk about in the next slide how do you deal with the time derivative alright. So, hopefully everyone is got this stencil very clearly in their mind set right. So, the main point is staggered by half a grid.

Student: (Refer Time: 27:24) delta x (Refer Time: 27:25).

Yeah (Refer Time: 27: 25) right. So, top minus bottom left minus right these are the keywords alright.