Computational Electromagnetics
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2D Finite Element Method
Lecture - 11.10
Numerical Aspects of 2D FEM

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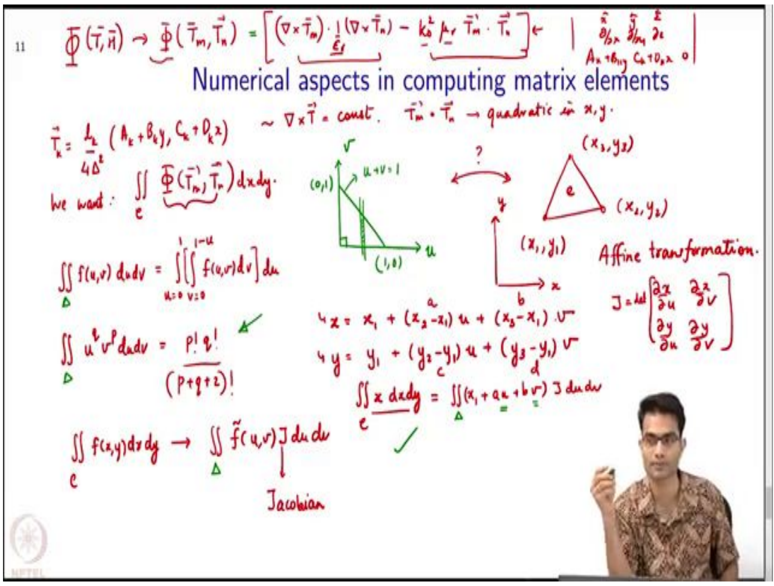

So, today so now we will look at the final aspect of the 2D FEM edge based FEM, where we will look at some of the numerical aspects that go into calculating the matrix elements. So, far we did not talk at all about how we will calculate matrix elements right. We had if you remember a function $\Phi(\vec{T},\vec{H})$ that is what we had said, and then later we said that when we choose a particular testing function this would become say $\vec{T}_m$. H itself would be expanded in the basis of these T’s. So, you would have to think of something like $\Phi(\vec{T}_m,\vec{T}_n)$ that is what we have to worry about.

So, what we did we just left it as some expression $\Phi(\vec{T}_m,\vec{T}_n)$, but how do we actually calculate this ok. So, there are there are there is a little bit of subtlety here which we will discuss.

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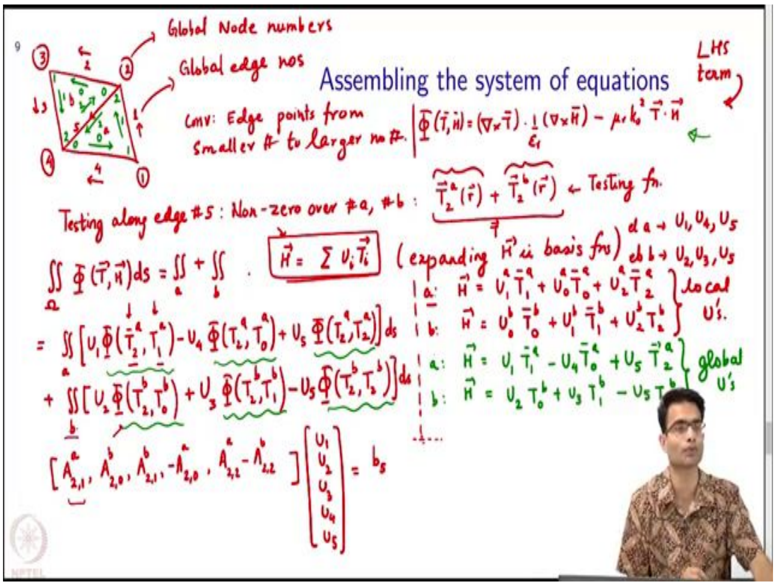

So, here is where we discuss the assembly of equations you are all familiar with this. So, you notice that your matrix elements over here these capital A's are consisting of these kinds of terms right. The terms underlined in green, so they are all the coupling we call them the coupling coefficients. So, to speak between the different T's that is what we are interested in.

Now, yeah so here is the other this is the main expression that we have to keep in mind what is the form of T itself. So, there is a

$$\Phi(\vec{T}_m, \vec{T}_n) = (\nabla \times \vec{T}_m).\frac{1}{\varepsilon_r}(\nabla \times \vec{T}_n) - {k_0}^2\mu_r\vec{T}_m.\vec{T}_n$$

These are the kinds of terms that we have to calculate ok. Now, T itself if you remember what was the form that we had for T?

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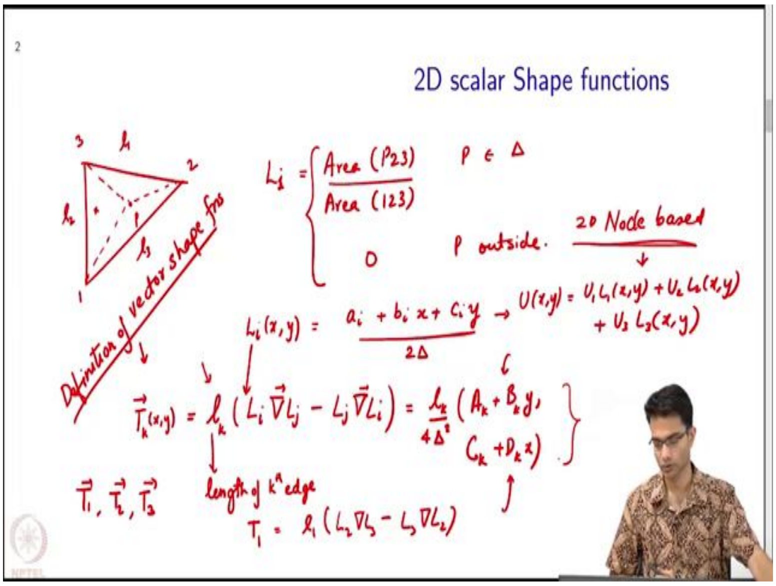

Let us go back to let us go back to the basis functions quite a way quite a while back here you go right. So, this is the form that we have right. So, you can see that they are linear in x and y right. So, let us go all the way back and put them down ok. So, I had $\vec{T}_k(x,y) = \frac{l_k}{4\Delta^2}(A_k + B_k y, C_k + D_k x)$ correct right.

Where these capital A B C D they come from the coordinates of the element right. So, it is all given to me. So, one thing you have 2 terms over here first term, second term. For the first term I need to calculate $\nabla \times \vec{T}$ ok. So, when I calculate $\nabla \times \vec{T}$; what do you think will happen over here? So, this is a vector to calculate curl what do I do.

Student: (Refer Time: 03:43).

so $\frac{d}{dx}(C_k + D_k x) - \frac{d}{dy}(A_k + B_k y) = D_k - B_k$

But basically $\nabla \times \vec{T}$ constant right, it is not a function of x and y good it is easy to integrate it ok. What about terms like $\vec{T}_m.\vec{T}_n$? what kind of expression will I have. So, you have a $\vec{T}_m$ and $\vec{T}_m$ which is of this form it has some A B C D, and I am going to take the dot product ok. So, when I when I take dot product means I multiply the x components

Student: you will have y^2 (Refer Time: 04:46).

Right so I have y^2.

x^2 and.

Student: xy.

Will I have a xy term? Only the x components get dotted with each other. So, I will not get a xy term but I will get a constant term into y and a a constant term into x.

Student: (Refer Time: 05:05).

We will get linear exactly. So, basically this is going to give me a quadratic in x, y. Some terms may not be there like xy term may not be there and so on ok. But we need not get into those specifics ok, we will try to give a general approach. Now if I look back at the assembly term over here, it is not just capital once I get this capital phi over here I have to integrate it over an element. That is the that is the integral I have to work out right.

So, what we want is $\int_e \int \Phi(\vec{T}_m, \vec{T}_n) dxdy$ ok. Where this term has constant, I mean constant and quadratic up to quadratic is what we have to do ok. So, now, this element e is that is something that is going to be very general right, this is going to be your very general triangle. Your CADD software has given you these coordinates (x_1, y_1) you typically have zero control over where these elements will be ok.

Now, what remains is to calculate this integral and the integrand is a polynomial in general. How would you calculate this integral? once I know how to calculate this integral I am done.

So we have to do what how many dimensional integration.

Student: 2.

2 dimensional integration does it look very easy from the looks of it no it does not seem very easy. Do we know how to integrate over a special kind of triangle, supposing I give you a unit triangle ok? So, let us let us consider a triangle of this form ok. So, this is a 90 degrees let us call this let us say u and v. Let us call this (1,0) and call this (0,1). Now on this triangle

can I can you do an integral of let us say $\iint f(u,v)dudv$, where f is some polynomial over this triangle.

Do we know how to do this integral? we had all we have all studied this some time in early undergrad 2 dimensional integration how do we do; we break it up like this into a strip here right. So, what are the limits of integration over here for v? So, what is the equation of this line u+v=1 right so if I can write this expression as integral from. So, first I can integrate over let us say. So, I can this is integral $\int_{u=0}^{1}\int_{v=0}^{1-u} f(u,v)dudv$ right so, this is how I would do it and this is if f is polynomial you can see this will be very simple to calculate right can I get it can I get a closed form expression for it if f is polynomial yes.

Student: yeah.

You will get a closed form equation because finally, there will just be polynomial I can integrate them substitute the limits and I will get it so. In fact, turns out that if I have the most general expression like this $\int_{u=0}^{1}\int_{v=0}^{1-u} u^q v^p dudv = \frac{p!q!}{(p+q+2)!}$ turns out to be very a very nice expression. So, surprising that it turns out to be so simple; so, you can check for example, if I put p=0=q. So, that is $\int_{u=0}^{1}\int_{v=0}^{1-u} dudv$ what should I get? when both p and q are 0 what is it physically.

Student: Half.

Half because the area of the triangle is half right which is what this expression also gives you ok. Simple check we know how to integrate this triangle, but what we want is this red triangle can we make use of this information somehow.

Student: perpendicular bisector of (Refer Time: 10:38).

Perpendicular bisector of what.

Student: (Refer Time: 10:42).

The green tri the red triangle is not a unit triangle. So, now, you want to put a perpendicular bisector from.

Student: Any node.

From any node to the opposite edge and then.

Student: Bisect the edge.

Bisect it ok, but then it is still not unit you are saying break up the triangles into 2 right angled triangles and do the integration of each other,, but remember the your you then you also have to rotate your triangles because right now your co ordinate axis are what your triangles are not aligned with the co ordinate axis like the green triangle right this is your x and y. So, that is not enough and anyway that sounds like a brute force approach is there something cleverer.

If I want so I mean the hint I am trying to lead you towards is change of variables, is there a change of variables that can map the 2 triangles to each other this what I am asking. So, can you map any triangle to the unit triangle physically think about supposing I give you know some clay and I ask you to map it into a unit triangle.

Student: (Refer Time: 11:53).

Student: (Refer Time: 10:55) they have a same area.

Need not have the same area, you can stretch your clay or compress your clay. So, if I want you to map this red triangle to green triangle what is the first thing you would do.

Move (x_1, y_1) to the origin then.

Student: (Refer Time: 12:10).

Well not projection you would some kind of expansion such that (x_2, y_2) goes on the x axis, (x_3, y_3) goes on the y axis right. So, it is in general possible there is I mean at least intuitively it should be I mean it sounds reasonable that there is a transformation that goes from this to this ok. So, let us lets try to work that out. So, x y let us try to write this out this is the transformation. So, well before we come to that. So, let us see. So, I will give you the answer turns out there is a transformation from here to here and it is called an Affine transformation.

How does it work? So, supposing I write $x = x_1 + (x_2 - x_1)u + (x_3 - x_1)v$, $y = y_1 + (y_2 - y_1)u + (y_3 - y_1)v$. Now supposing I substitute u=0, v=0 what happens?

Student: x_1.

I get x_1. put u=1, v=0.

Student: (x_2, y_2).

Right so; I will get?

Student: (x_2, y_2).

(x_2, y_2) and the other point (0,1), I get (x_3, y_3). So, I have got a transformation that is very neatly mapping my red triangle to the green triangle right very clever. So; that means, I can in general convert an integral of the form $\int_e \int f(x,y)dxdy = \int_\Delta \int \hat{f}(u,v)\, Jdudv$

When you do change of variables what is the main thing that you should not forget the Jacobian of the transformation right. Which is so that Jacobian term is what will come in over here the simplest so called (Refer Time: 15:26) check you can do is put f(x,y)=1. These 2 triangles have different areas. So, the Jacobian is nothing but the area of.

Student: (Refer Time: 15:45).

Area of the red triangle, because $\int_\Delta \int dudv = 0.5$. So, I mean this is nothing new we have all seen this in undergrad maybe even before undergrad when you how do how to do 2 dimensional integration? How to do change of variables right? So, with this so for example, you know I have supposing I wanted to evaluate integral $\int_e \int xdxdy$. So, what would you do?

So, let us call this term say a b c d ok; just short hand notation and this is the integral that I want to calculate ok. We will get a term like x, x^2, y, y^2 these are the terms I will get. So, supposing I got x integrated over this triangle. So, now, what is my recipe? So, x I have to write as now substitute in terms of the new variables right. So, this will be so $\int_e\int x dx dy = \int_\Delta\int (x_1 + au + bv) du dv$ right and integrate it now over the triangle.

Correct yeah so what will I mean, so we can is this easy to evaluate this is very easy to evaluate because I know the analytical expressions as given from this formula. Any power of u and v I can evaluate. So, I can just substitute it over here and I get it ok. So, in my code when I go to evaluate the $A_{m,n}$ terms these are the tricks that I have to do. So, it is important that you let the Jacobian.

So, this is very very important when you get down to implementation you calculate the Jacobian exactly do not do not try to take mod of something in between ok, because the way the points are oriented you may get a 1 or a -1. In the definition of Jacobian you should not try to put an absolute value over there let it be what it is that is the source of lots of coding mistakes fine. Any other questions I mean any questions regarding this.

Student: Sir, what is the formula for Jacobian?

So, formula for Jacobian so, the Jacobian has terms like d u sorry. So, you will have $J = det([(dx/du)\ (dx/dv);\ (dy/du)\ (dy/dv)])$ right. The convention for this Jacobian differs from people to people some people take the transpose of this as the Jacobian whatever, but transpose does not alter determinant.

Student: so the new integration limits will remain change with respect to (Refer Time: 18:58).

The new integration limits will be this green triangle, because we have done that mapping.

So, I am doing this change of variable from x y to u v right. So, the limits of integration all those three nodes they move to these green nodes. So, the limits of integration will be this green triangle fine and everything is in terms of u and v fine. And the reason this is extremely powerful is now I have a closed form expression for calculating the matrix elements.

Imagine you have a code where there are 1 million elements. And if you have a closed form expression you just substitute it then and you get the thing, there is no added calculation needed over here there is no quadrature rule also needed over here right. So, in more complicated problems you may have to use a quadrature rule for each of these million entries right. So, that is a significant if you have to do it 1 or 2 times it would not matter but if you have to do multiplied by 1 million the time adds up.

Like in your integral equation approach there you had to calculate the matrix coefficients by using quadrature rule. The higher the order of quadrature you use the better accuracy you got right. So, that tells you that you know if you wanted a very accurate answer you would have to spend a lot of time in evaluating the matrix itself, then you would spend time in inverting that matrix. Here you are getting a closed form expression for the entries itself. So, to build the matrices fast the matrices itself fast to solve it is also fast. So, these are like double advantage of FEM methods.