Computational Electromagnetics
Prof. Uday Khankhoje
Department of Electrical Engineering (EE)
Indian Institute of Technology, Madras

2D Finite Element Method
Lecture - 11.08
Matrix assembly - Part 2

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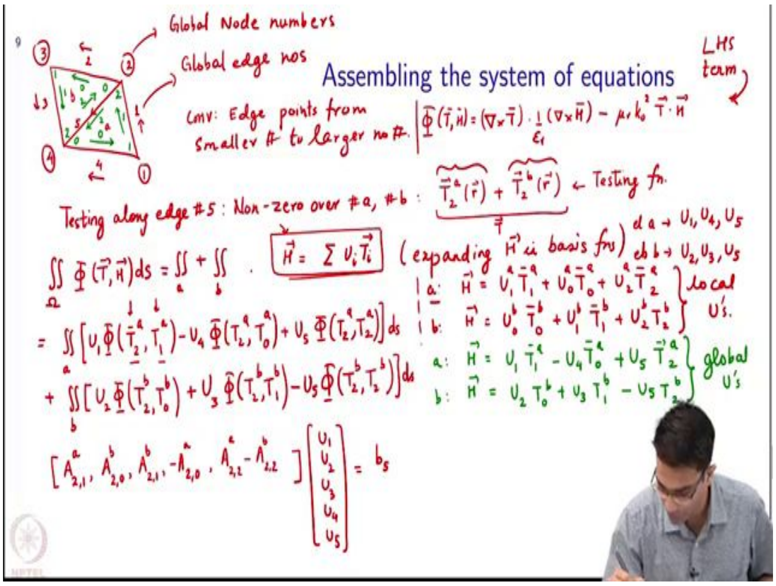

So, this is my testing function let us start with the equation that I get. So, what is my function going to be like? So, over omega right. This is what is going to be the left hand side right, where T is this whole thing over here that everything over here. The first thing you can notice is that this integral is over 2 elements. So, I can break it up into integral over #a plus integral over #b. Why? Because this T which I have over here is non zero over both right.

So, I can write this as $\iint_a + \iint_b$. So, now, so far we have only kept $\vec{H}$ as $\vec{H}$ now we finally have to substitute $\vec{H}$ in terms of these basis functions right. So, what is $\vec{H}$ using basis functions $\vec{H}$ is at any point inside any triangle $\vec{H}$ is a linear combination of the $\vec{T}$'s and the weights are the unknown U 's right. So, that is how I will write this. So, I can write $\Sigma U_i \vec{T}_i$. So, if I just write $\Sigma U_i \vec{T}_i$ I am referring to global.

Because there is no sub superscript referring to which element or whatever right ok. So, this is expanding. So, $\vec{H}$ now we will take one by one ok. So, the first integral is over element a and $\vec{H}$ is going to be in terms of the three local testing; three local basis function of element a. So, the first term will; the first term, but we will write it in terms of global U 's because I want equations in the global U 's that is what I am solving for.

So, first term what will be the first term be. So, what are the, for element #a, there are how many what are the U 's that are coming into play over here, for element #a? U_1 .

Student: U_4 .

U_4 and.

Student: U_2 .

For element a.

Student: So, global nodes right.

This is edge based. We are not talking about global nodes we have to talk about global edges.

Student: Also edge.

The unknowns are the tangential components of the fields along the edges right so, U_1 .

Student: U_4 .

U_4 and U_5 ; U_5 is the common one element b.

Student: U_3 .

$U_3 U_2 U_5$ these are the U 's that will go into this equation in every element ok. So, when I am integrating over element a. So, the first term that I am going to get is let us say U_1 the term involving U_1 . What do I write next? I want you to complete this bracket over here. So, the term corresponding to, so which, in this $\vec{T}$ the testing function which $\vec{T}$ will come into play?

Student: $\vec{T}_2$ yes.

Yes $\vec{T}_2$. Well $\vec{T}_2$ which one a or b?

Student: T 2 a .

$\vec{T}_2^{\ a}$, let us forget the r, it is there everywhere right and U_1 has come. So, which is the which from the $\vec{H}$ term which one will come over here now? $\vec{T}^{\ a}$ has to be next. Remember I am expanding $\vec{H}$ according to this. $\vec{T}^{\ a}$ is the basis function for element #a. Remember the shape function that we had shown in the very beginning, right? Do you want me to show you the basis function again?

Student: No.

Fine. So, what should this be?

Student: So, there is a single basis function for a single type.

For #a, there is a single basis function within an element for a given edge. So, we are inside element #a. Element #a has U_1 with $\vec{T}_1$ a remember edge number. So, the local edge number is $\vec{T}_1$ is 1 in green right. $\vec{T}_1^{\ a}$, $\vec{T}_2^{\ a}$ and $\vec{T}_0^{\ a}$ these are the 3 things right. So, when I have U_1 which testing which basis function would it have been?

Student: $\vec{T}_1$.

You have three choices 0 1 2 which of those is going to be.

So, it is going to be 1 because of this guy right that is the one that is going to get U_1. So, let us may be let us write it like this. In element #a, H is going to be written as $U_1\vec{T}_1^{\ a}$.

Plus $U_1\vec{T}_2^{\ a}$ what?

Student: 0.

0 plus $U_5\vec{T}_2^{\ a}$. In #b, $\vec{H}$ is going to be equal to. So, we will start with $U_2\vec{T}_0^{\ b} + U_3\vec{T}_1^{\ b} + U_5\vec{T}_2^{\ b}$.

Actually let me let me write this once again. There is a slight subtlety over here let us write this a little bit carefully. Let us write everything in terms of local edges first ok. So, local edges will be every simple U_1^a, U_0^a, U_2^a this is local right. So, local edge and similarly this will be $\vec{T}_0^{\,b} U_0^b$.

Student: 2.

2 ok. So, this is in terms of local. Now #a, if I want to write. Now I want to replace the local Us by global. So, what will I write for the first one for the first one U_1^a I can just write this as $U_1 \vec{T}_1^{\,a}$, second one is s $U_0^a \, \vec{T}_0^{\,a}$. So, that is corresponding to U which one?

Student: 4.

$U_4^a \, \vec{T}_0^{\,a}$ next is finally, $U_5^a \, \vec{T}_2^{\,a}$ correct. So, is that is the last term correct $\vec{T}_2^{\,a}$; $\vec{T}_2^{\,a}$ is pointing from in element a from?

Student: 2 to 4.

2 to 4 correct. So, here is 1's let us see what is 0 looks like there is small mistake over here also the convention is from smaller to larger right.

Student: The inside we were just going from 0 or 2 2.

No we were going from smaller local node number to larger local node number right. So, those are the conventions, right. Now what is the convention?

Student: So, once the apply the convention you're going to so, both global and.

Yeah I have assumed the convention.

Student: I hope you yeah apply a assume a convention will apply for both the local and the rural.

It is my choice.

Student: We can change it.

We can change it later.

Student: Every using the same.

Yeah I am using the same. So, edge global edge number 5 points from 2 to 4 right. So, the direction of this, is this way. now is the part. So, was this U_5 correct as smaller to larger?

Student: No now.

Yes, that has also changed now.

Student: So, 2 0.

Yeah ok. So, then we will let us go back to our original convention which was just cyclic right this is what we had earlier yeah. So, now, the convention is consistent. So, that is the local edge convention local edge convention was just going in a cyclic way.

My global edge convention is smaller node to larger node ok. So, the direction of 5 is from 2 to 4.

Global 4 global 2 to global 4.

Student: That will be plus smaller.

So, this will become a plus very good and in element b this will become. So, ${U_0}^b$ is U_2 right $U_2 {\vec{T}_0}^b + U_3 {\vec{T}_1}^b$ here I will get a $-U_5$?

Student: $-U_5$.

Consistent because the local Us and the global Us have to match.

Student: The U_4.

U_4 yeah.

Student: The sign change sign is direction opposite for the global and.

For U_4 let us see for U_4.

Student: Yeah.

Yeah correct. So, for U_4 this should be a minus sign. Are the other ones alright?

Student: No.

No, testing functions are what they are.

Student: No the reasoning.

So, we are doing that going from when we start writing the system of equations the $\vec{H}$ that I substitute has to be unambiguous in terms of the Us that is why I am making taking great means to ensure that the signs are correct over here. So, U_1 is fine U_4 is in the opposite direction. So, what I am doing now is I am comparing local edge and global edge are they pointing in the same way U_1 is correct U_4 there is a minus sign U_5 is in the same direction in element a. When I come to element b 2 and 0 same direction, 3 and 1 same direction, 5 opposite direction that is why that is why I got a minus sign good point.

Student: Why are we taking different conventions for global and local you can use the same.

You can use the same, but you will not escape this minus sign issue when you you can assume any other convention you want you will run into some minuses that have to be fixed ok. So, there is no escaping the minus signs.

Student: There is that the numbering will changes.

Numbering changes ok. So the first term coming back to this left hand side over a is $U_1\Phi$. So, is this one correct what we have written $U_1\Phi(T_2{}^a, T_1{}^a)$; $\vec{T}_2{}^a$ why because that is the testing function and $\vec{T}_1{}^a$ right that is coming from the first term over here plus next term is going to be what will the next term be?

Student: These are b.

No I have not finished a.

Student: $-U_4$.

$-U_4\Phi$ this remains the same $\vec{T}_2{}^a$ and what do I have?

Student: T 0.

Student: T 0.

$\vec{T}_0$ very good and then I have.

Student: Plus U_5.

$+U_5\vec{T}_2{}^a$ what will this be?

Student: $\vec{T}_2{}^a$.

$\vec{T}_2{}^a$. Now this the first guy over here this is what I was using as my testing function over a because b this second term over here is 0 over element a. So, that is why I did not contribute now I have switched to the integral over element b first variable what will it be?

Student: U_2.

U_2 what is the testing part now?

Student: $\vec{T}_2{}^b$.

$\vec{T}_2{}^b$ what is next?

Student: $\vec{T}_b{}^0$.

$\vec{T}_b{}^2$ plus next term .

Student: U 3.

$U_3\Phi(\vec{T}_2{}^b, \vec{T}_1{}^b) - U_5(\vec{T}_2{}^b, \vec{T}_2{}^b)$.

So, this gave me the left hand side. So, left hand side I have got one equation in how many variables?

Student: 5.

5 variables right. So, this happened on element when I took edge number 5 because edge number 5 is common to both the elements. Supposing I had taken global edge number 1 how many variables, what all variables would have appeared? Global edge number 1, what all variables would have come?

Student: U_1.

U_1.

Student: U_4.

U_4, U_5 there is no U_0.

Student: Edges.

Yeah edges ok. So, this is the maximum that an edge can be shared by is; obviously, by 2 elements right. So, the maximum number of non-zero entries in any equation even if there are 10000 elements in this any edge can only be shared by 2. So, the maximum number of variables that can appear in an equation is 5 right 1 common plus 2 from the other 2 from the other 5 right. So, the spareness of this matrix the bandwidth of this sparse matrix cannot exceed 5.

Take any edge even if there are a million elements each equation will have at best 5, if I am on the border boundary edge how many equations will appear? 3. So, the bandwidth of this sparse matrix can never exceed 5 regardless of the size of the mesh. That is the real power of FEM.

So, there is a little bit of a convenient way of writing this now let us try to write this in a way that is easier to code ok. So, I am trying to write $Ax = b$. So, I am just trying to write down one equation ok. So, I will write U_1 in this problem there are 5 U's. I have to write the corresponding a's. So, the first term what will what I can write this as is.

So,let us look at the U_1 term there is only U_1 term right. So, this integral is over #a and it is the coupling between what and what. So, there is 2 here and there is 1 over here right. So, this I can write as 2 comma 1. So, what is this saying? So, superscript a refers to element #a. Within element #a, you are taking the dot product of local basis 1 and local basis 2 that is the meaning that is the that is all there is.

Then the next is U_2. So, U_2 term is going to come from A I will write it as integral over b and 1 comma 0 next comes U_3. So, that is going to be $A^b{}_{2,1}$ next comes U_4 right. So, what will this be?

Student: Minus.

$-A^b{}_{2,2}$.

I will just put a comma to make sure that it is not mixed up and finally, is U_5; U_5 can be written as over $-A^b{}_{2,2}$.

Student: Local dimension right.

Both are local right both this and this are local because I have the I have mentioned 2 1 or 0 2 1 that is only 0 1 or 2 those are the local edge numbers I could have written these whole equation purely in terms of global edges also. So, it is my choice there are various ways of doing it, but the reason I have written this in this way is because this is telling you how you would actually code it.

You would have a function which takes as input three things which element which local edge numbers it will do the integral and you will populate your capital A term over here. So, this is your one row and this is going to be equal to b_1, b_2, b_3, b_4, b_5 .

Edge 5 is in the interior right. So, actually, of these this is corresponding to I mean I will not, I will not get all of these over here this was testing along edge number 5. So, this will simply be equal to b 5 yeah one equation is what I have written. The same form the left hand side is the same in total field formula in total and scattered field formulation.

Student: Only the b will change.

The b is where it will change if it were a scattered field formulation then as we discussed in the previous module, we may need to expand the number of Us and add one more equation corresponding to the difference of the Us being equal to incident field ok. But let us keep it simple let us just think about total field formulation for now ok. So, where will I get a non-zero right hand side from what is the possibility? What are the possibilities?

When can the b s be non-zero? So, its edge number 1 2 3 and 4 they are on the boundary right this is the very trivial cartoon like example where there are only 2 elements. So, there are so many boundary edges 1 2 3 and 4 all boundary edges only 5 is interior. In the real world scenario it is going to be the other way most edges are interior edges the number of boundary edges is very small. So, in this case all the terms that will give you non zero quantities on the right hand side will come from 1 2 3 4 ok.

And we have already discussed what is the form to write this thing. So, this is how you build a system of equations in your 2D vector FEM.

Student: Sir why for shared edge become 0.

Why could?

Student: Why could the shared edge v will become 0.

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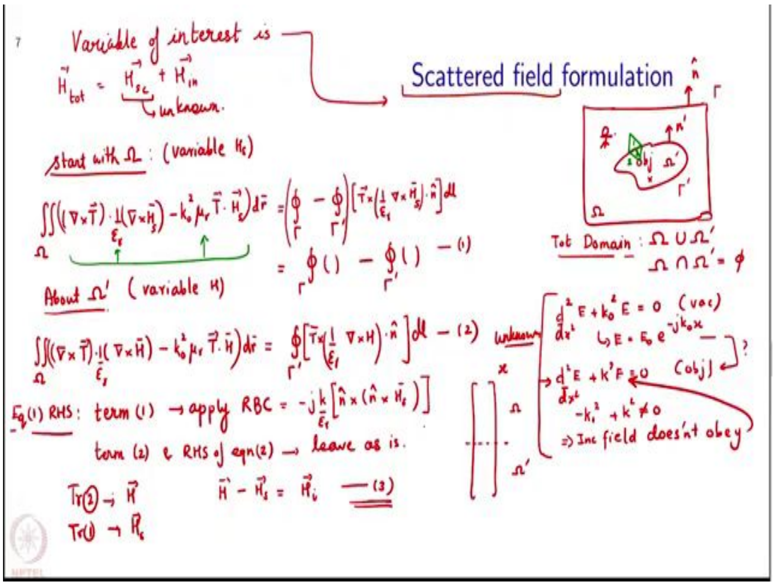

Right. So, why for the shared edge v will become 0 is because well let us go back to let us go back to yeah here.

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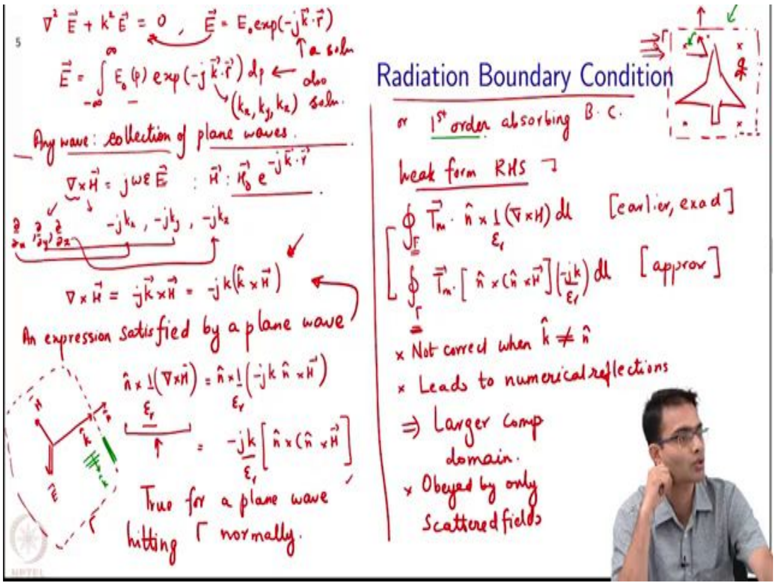

Or let us actually keep it simpler let us just look at here the weak form of right hand side here this is only over the edges on the boundary.

Student: Right.

Right. So, for an edge on the boundary what we had taken was $\vec{T}_5$ was in the interior now the $\vec{T}_5$ in the interior: what is that $\vec{T}_5$ on the boundary $\vec{T}_1$? Is it tangential or purely normal?

Student: Purely normal.

Purely normal right; so, that purely normal will this dot product with this whole expression over will end up being 0. So, the contribution to the right hand side will only come from those testing functions which are associated with that edge. Interior fellows will just given identical identically 0 dot product ok. So, you would work you have to work out the algebra little bit to see that to be the case.

Student: So, in sparse such well Us so, a lot of elements of the (Refer Time: 22:05).

Yeah in sparse the right-hand side terms being non zero only come from the boundary edges everything else gives up being 0 ok. So, yeah a lot of careful work has to be done to keep taking care of the minus signs. These minus sign business does not bother us if you were to do a node based FEM for 2D ok, but the trouble is that there are certain known inaccuracies with node based FEM.

I mentioned one problem which is called internal resonances. So, you get some spurious solutions which are not physical, but mathematically they end up spoiling the problem. So, you think that you have got a solution, but it is actually not and this vector based FEM kills it.