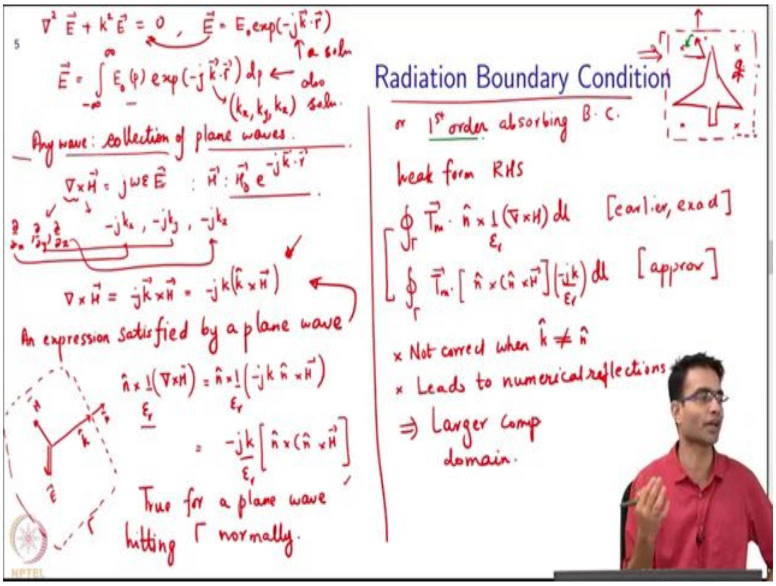
Computational Electromagnetics
Prof. Uday Khankhoje
Department of Electrical Engineering
Indian Institute of Technology, Madras

2D Finite Element Method
Lecture -11.03
Radiation Boundary Condition

(Refer Slide Time: 00:14)



Let us look at the Radiation Boundary Conditions now. Now before I get into that one general principle we will derive. So, suppose I have an unbound medium. Now we know that the solution to Maxwell's equation. So, this unbound medium follows the Helmholtz equation right and we know that the solution to this is a plane wave solution right. So, what is the solution what is the kind of solution that I get E is E naught. I am ignoring the time dependency we have already taken care of that.

So, this is the plane wave solution over here and you can see that if I and if I plug this into this I mean this is coming from the solution to this Helmholtz equation right, it satisfies it identically right, everyone is convinced. The first term when I take del square I am going to get a $k_x^2 + k_y^2 + k_z^2$ which is equal to the second term over here there is a $-jkx$ on at one more derivative will become minus $-k_x^2 - k_y^2 - k_z^2$ and there is a k^2 over here right. So, it will cancel off I will get 0.

So, you agree that this is a solution now let me ask you the following question. Is this also a solution? So, I call E is equal to I make this E_0 now a function of k. So, what have I done? I have made the amplitude to be dependent on the wave vector does this work? So some change I will have to make over here, but let us see if I substitute this in here what happens?

Student: (Refer Time: 02:38).

What is the first term going to do? ∇^2 will it give me see ∇^2 is only going to act on the spatial terms the spatial terms is only.

Student: e^{-jkr}.

e^{-jkr}. So, what will I get from here I will get a minus?

Student: $-k_x^2 - k_y^2 - k_z^2$.

$-k_x^2 - k_y^2 - k_z^2$ correct what about the second term?

Student: k^2.

k^2. So, does it cancel off then?

Student: (Refer Time: 03:15) simply very $-k_x^2 - k_y^2 - k_z^2$.

Yeah. So, the first term is going to give me a $-k_x^2 - k_y^2 - k_z^2$ square I am writing this k as.

Student: k x k y k z.

k_x k_y k_z right vector. So, the first term is giving me a $-k_x^2 - k_y^2 - k_z^2$ and what happens to the second term?

Student: Minus j square by.

Yeah. So, the slight the slight problem over here is that I am using k as both a variable of integration and the constant k in the equation. So, maybe that is a bad idea. So, what should I let us let us not use k let us call this by some other thing let us call this let us say p let us call this p. Now can I say that this satisfies right?

Student: (Refer Time: 04:23).

k^2 right. So, this integral over different values of p does not do anything. So, if this is a solution, this is also a solution right. So, what is this what is an interpretation of this general equation that I have?

Student: Superposition.

Superposition of different plane waves that is what it is right. Amplitude is varying that I am; it is like a like a Fourier transform idea. Take the superposition of different plane waves right. So, the general idea behind this is that any wave that is travelling in free space I can write as a collection of plane waves ok. So.

Student: So, this is like separated on the all the different spatial frequency.

Yeah, spatial frequency not omega t that this that is the same.

Student: (Refer Time: 05:34).

Yeah.

Student: Some function in space separating them.

Separating them.

So, this will be this is not exactly needed to derive this boundary condition, but it is a good intuition to have ok. So, we will just keep this in the background ok. So, now, let us let us look at our let us go back to our wave that is travelling in a general direction $\hat{k}$ ok. So, in a, let us say a homogeneous medium, we know that a plane wave is traveling right. So, when I do something like $\nabla \times H$. So, what is $\nabla \times H$ is equal to? $-j\omega\mu$ sorry.

Sorry yeah. So, this is minus. So, this is $j\omega\mu$ no entire εE vector right this is what I have ok. Now H you know is of what form?

Student: H naught.

$H_0 e^{-jk.r}$ we know this. What kind of terms does curl of H have? So, curl over here will have what kind of terms d/dx.

Student: d/dy.

d/dy.

Student: d/dz.

And d/dz ok. So, when I do this expression over here when I try to evaluate $\nabla \times H$ and H is of this form. So, whenever it encounters a $\partial/\partial x$ what will happen?

Student: $-jk_x$.

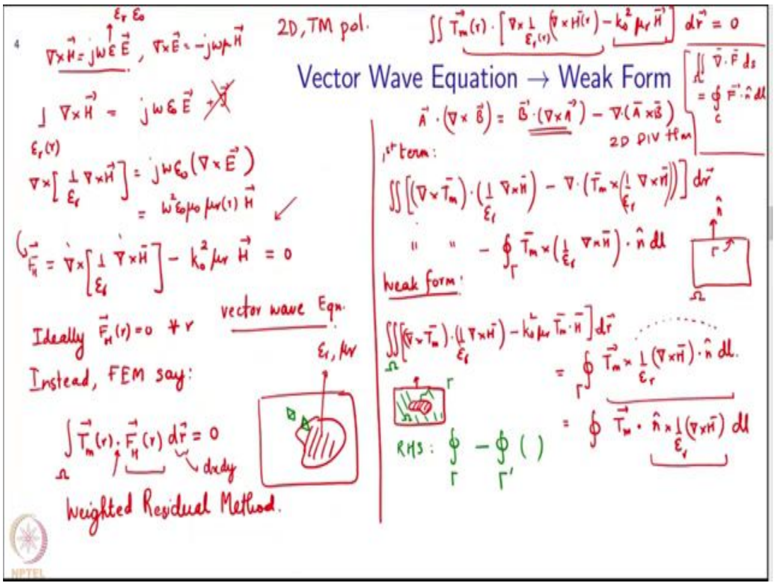
$-jk_x$ will come out from here wherever there is dx part. Similarly there will be a $-jk_y$ and a $-jk_z$ whenever these derivatives are encountered. So, you can you can do this simplification that the act of putting del cross is for a plane wave equivalent to this ok. Many of you may have seen this simplification in your undergrad EM course that for a plane wave I can replace this $\nabla \times$ by just a $-jk$ vector I just gave you the intuition for it, but you can drive it rigorously yourself ok.

Student: Yes sir.

This further I can write as minus $-jk(\hat{k} \times \vec{H})$ ok. So, you can straight away see what we are going to do. This is the expression that a plane wave satisfies right. So, an expression satisfied by a plane wave right. So, what is the left hand side? $\nabla \times H$ there is a derivative involved, what is the right hand side? Is it linear in H it is linear in H there are no special derivatives or anything right there is some there are some other constants over there. So, this is the basic the same idea here in 1D which I had.

The expression obeyed by a plane wave du/dx was proportional to u exactly the same situation is going to happen over here $\nabla \times H$ proportional to H of course, there are more vectors and all over here, but this is the basic idea that we will use to derive the boundary condition ok. So, let us draw a vector right. So, this is my k vector. So, as long as so, the wave is travelling over here my E and H are going to be in orthogonal directions for a plane wave right. So, at any point if you ask me this expression is obeyed ok. Now, let us go back and see what is the kind of term that we have to worry about right.

(Refer Slide Time: 09:55)



So, the kind of term we have to worry about is here is dotted part over here TM is anyway going to be my choice what am I left with is this expression right. Now this expression can also be rewritten in the following way. So, I have $a \times b \cdot c$ right can I if I swap the order of that cross and the dot what happens?

Student: Negative sign.

Negative sign; so, this expression can also be written as; Can I write it like this dot? So, dot right 2 minus signs and I get the same expression. So, it would have first become $\oint - T_m \cdot 1/\varepsilon_r \nabla \times H \cdot \hat{n}\, dl$ and $a \times b = - b \times a$. So, the n comes to the middle alright. So, the expression that I really have to worry about is this guy over here. So, what is this let us rewrite this expression over here. So, I have $\hat{n} \times 1/\varepsilon_r \nabla \times H$ this is the term that I want to get an approximation to agreed the rest I can take care of.

So, what can I do over here? So, what I say now this is the so called a leap of faith if I assume that my boundary over here is this such that the plane wave is hitting this at a normal incidence then which direction is $\hat{n}$? This dotted line is my boundary ok. So, this is my wave is inside this region over here. So, which way is my n hat?

Student: (Refer Time: 12:04).

So, what I am saying is that I have this dotted line which is the computational domain this is the computational domain I have a plane wave that is traveling in this direction hitting the hitting the boundary normally which direction is n hat?

Right so, $\hat{n}$ is also along in this special case only in this special case. So, in that case I can replace this for this special case only for this special case what can I write this as I can write this as $\hat{n} \times 1/\varepsilon_r$ and replace the rest from this guy. So, $-jk\hat{n} \times \vec{H}$; so, this is true for a plane wave hitting Γ. So, this was my Γ normally why because I simply replaced my unit vector $\hat{k}$ t by the unit normal vector $\hat{n}$. So, no one can object to this right.

Student: Sum.

No, this ε_r is for a homogeneous medium I mean epsilon r is not a function of space over here why because if ε_r is not a function of space then I can safely assume that the solution is a plane wave correct. We know that for a homogeneous medium Maxwell's equation gives a plane wave solution of this form $e^{j\vec{k}\cdot\vec{r}}$.

Student: So, this ε is at the boundary

This ε is in the neighborhood of the boundary.

Student: That means, the vacuum.

It may vacuum or it may be some homogeneous medium like water or something let us say you have a radar wave travelling under water and antenna under water. So, this is the homogeneous epsilon of water right whatever it is, but what I what I am talking about is first of all I need to have a plane wave and next of all the plane wave should be hitting the boundary normally. So, if you notice that I mean when we do FEM calculations we will have the scattering object and surrounded by some amount of vacuum and then terminate the computational domain.

Student: It is terminated the (Refer Time: 14:37).

There is some amount of air.

Student: So, that is $\hat{k}$ and $\hat{n}$ are (Refer Time: 14:42).

No what we do is. So, let us say this is my aircraft over here. Now the question is where should I draw my computational domain should I just make it tightly fitting with the aircraft or should I expand it a bit right. So, the computationally inefficient, but easier thing to do is to make my domain like this.

So, when I do this I have I have to discretize all of this free space region and that is a price I pay and we will come later on what are the advantages and disadvantages of doing. But when I do this now when I look anywhere close to the boundary because its free space Maxwell's equation should be satisfied at each point in space free space or homogeneous space I will get a plane wave.

So, plane solution is valid further if this plane wave were hitting the boundary normally then this expression holds right. So, this is what is called your either you call the radiation boundary condition or 1st order.

Student: So, a boundary you mean that is outermost.

No boundary I mean the outermost boundary.

Student: (Refer Time: 15:54).

Outermost boundary absorbing boundary condition ok.

Student: (Refer Time: 16:05) ε_r why are we.

Why are we keeping $1/\varepsilon_r$ there?

Student: No why are we.

Yeah. So, why? So, the question is why are we keeping this $1/\varepsilon_r$?

Student: No why are we concerned with the r (Refer Time: 16:18).

Well because if ε_r were a function of space then technically speaking I may not get a plane wave like solution over there. So, the approximation of replacing $\nabla \times H$ by $k \times H$ will not be valid right. So, what I am here. So, the big leap of faith is going to be that everywhere in my right hand side this expression over here. So, in the weak form of RHS is going to be replaced by.

So, what did I have earlier I had Tm dot n hat 1 by epsilon r curl of H what else d l this is what I had earlier and this is the exact form. Now what I am replacing it by I am replacing this by T_m dot this whole expression. So, $\oint_{\Gamma} \vec{T_m} \cdot [\hat{n} \times (\hat{n} \times \vec{H})](-jk/\varepsilon_r)dl$. So, is this always correct?

Student: No.

Answer is no, when is it not correct.

Student: Because it not correct (Refer Time: 17:56).

So, the first thing is not correct when the direction of propagation of the wave is not equal to $\hat{n}$. For example, if you look at this aircraft over here let us say the incident wave is coming from the left hand side like this it's possible that you know it hits this aircraft over here and the wave goes off in this direction right. So, now, $\hat{n}$ is here and the wave is hitting non normally, but what approximation am I doing.

I am assuming as though it everything is hitting normally. So,, but I am still using this expression. So, this is the price that I am paying for the convenience of a very simple boundary condition this expression that I have written on the over here is a very simple to implement. The problem is, its exact only at normal incidents.

Student: Save (Refer Time: 18:49).

So, if I now make the computational domain larger and larger right. So, this becomes more and more accurate because waves will more or less begin to reach normally think of a very large circle right whatever reaches that will reach only normally assuming a finite object. So, whatever hits non-normally what will happen to it?

Because this boundary condition is not exact, there will be a mathematical reflection right. So, what will happen is this hits it like this and due to the approximation some way will get reflected vacuum. This is inevitable if I use this. So, this is what is. So, that is why I called it a.

Student: 1st order.

1st order absorbing boundary condition ok. It is called absorbing boundary condition because it appears as what is absorbing everything that is going out.

Student: Nothing is reflecting.

Nothing is reflecting back therefore, it is called absorbing boundary condition right. So, not correct when this $\hat{n}$ is not equal to $\hat{k}$. What else is not correct about it? Well this is the main thing. So, it leads to numerical reflection therefore, larger computational domain. So people have built. So, let us say if you wanted to improve this k we would not do it here, but what would what would you try to do if wanted to improve this?

Student: Change the.

Change the.

Student: Shape size.

Change the shape, size that is always getting reflections at normal incidence. That becomes tricky to do if I am particularly writing a general purpose solver I do not know apriori what is the shape of the origin that is going to go in. So, for example, what are we doing we are imposing that the reflection go to 0 at 1 angle a second order boundary condition will allow you to make the reflection go to 0 at 2 angles, 3rd order will allow you to do it at 3 angles and so on. The mathematical form will become more and more complicated ok.

And when you look at the mathematical form of higher order boundary conditions you will realize let us stick to 1st order boundary conditions and pay the price by making my domain a little bit larger ok. So, this is very very commonly used boundary condition ok. So, your first code that you write this implement this get it working on top of that then you would try to implement higher order boundary condition no.

So, the word absorbing is slightly misleading what is happening is we are simulating the condition for a wave to go off unreflected that was our original motivation because this expression was the expression obeyed by a plane wave that is going unhindered and when I draw a mathematical boundary then nothing will come back because that is the equation that physics has given me.

Student: So, it is.

So, it is as though supposing I had a perfect absorber there something that absorbs and does not reflect that would do the same thing. So, that is why some people called it absorbing boundary condition.

Student: They will say that obeyed and you do not know what is outside and nothing came back.

Nothing came back. So, it is like right if I was sitting inside the domain supposing I was standing here for this observer what happened a wave went off never came back. What is the one way of thinking about it was absorbed right. So hence, the word absorbing boundary condition.