Computational Electromagnetics
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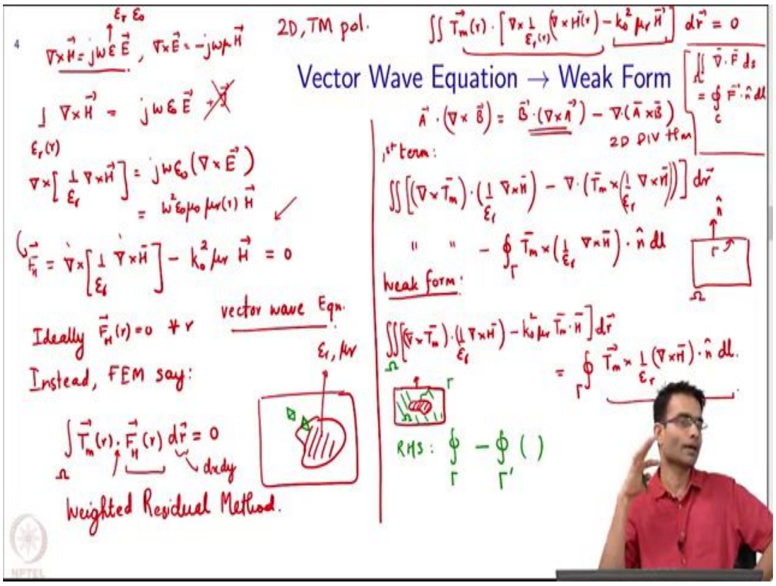
2D Finite Element Method
Lecture – 11.02
Converting to Weak Form (2D FEM)

(Refer Slide Time: 00:14)



So now what we will do is we look at how do we actually we have looked at the shape functions, the next thing we have to see is the start with the wave equation go to the weak form that is the strategy of FEM right. So, what is the let us see what is the equation that we can use. Maxwell's equations everything starts from Maxwell's equation right. So, the first equation is $\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$, $\nabla \times \vec{E} = -j\omega\mu\vec{H}$ all right. So, what we will do is like in the previous case we can consider 1 polarization, the other polarization the same kind of logic will follow right. So, we were doing TM polarization in surface integral. So, H in the plane and E_z so, if I want to continue to you talk about 2D TM polarization that is what I want to continue to talk about for definiteness. So, what do I need to talk about? H, H in the plane that is what I will solve for because that is the thing I can express as a vector right E is a scalar there is no point of vector over there ok. So, what I will do is now.

Student: (Refer Time: 01:37) H_x, H_y, E_z know.

Yeah, H_x, H_y, E_z so, let us take this equation for example, right. So, $\nabla \times \vec{H}$ we have done this before in order to use the other curl equation I needed to remove something from the right hand side of the first equation and that was a.

Student: $\varepsilon_r(r)$.

$\varepsilon_r(r)$ right this epsilon over here was $\varepsilon(r) = \varepsilon_0 \varepsilon_r(r)$ and $\varepsilon_r(r)$ is a function of space. So, I can bring this over here $\varepsilon_r(r)$ is a function of space this is going to be equal to $\frac{1}{\varepsilon_r(r)} \nabla \times \vec{H} = j\omega\varepsilon_0 \vec{E}$. Next step so, I am ignoring the J term for now, it is easy to add it in next step would be to take.

Student: Curl.

Curl on both sides. So, what happens is where I replace this and I am going to get $\nabla \times [\frac{1}{\varepsilon_r(r)} \nabla \times \vec{H}] = j\omega\varepsilon_0 (\nabla \times \vec{E}) = \omega^2 \varepsilon_0 \mu_0 \mu_r(r) \vec{H}$ right. So, I have eliminated the electric field and I have got 1 vector wave equation in my unknown which is H right. So, I can just combine this into 1 equation just write this as $\vec{F_H} = \nabla \times [\frac{1}{\varepsilon_r(r)} \nabla \times \vec{H}] - k_0{}^2 \mu_r(r) \vec{H} = 0$ right this is my vector wave equation.

So, typically what is given to me in this problem is; ε_r, μ_r as a function of space is given to me may be some incident field is given to me object is given find out the fields everywhere ok. So, this is the vector wave equation ok. Now, given some kind of a domain over here with some object over here right, this is my ε_r, μ_r ok. So, what is what do we have to do in the FEM, we will be breaking this whole thing into little triangles right all over and then doing.

So, the wave equation is $\vec{F_H} = 0$ right. So, that is also what was the term we had used in 1D, the residual when I took everything onto 1 side I called it the residue right. So, ideal world I want $\vec{F_H}(r) = 0 \;\; \forall\, r$ that is the solution of the wave equation not possible. So, what is the next best thing I settle for in a average weighted sense right.

So, instead FEM says weighted residual method integrate over domain. So, take some m^{th} testing function, now I must take a dot product because this $\vec{F_H}$ is actually a vector right. So, this is the starting point for the FEM weighted residual method by the way any differential equation this is the approach you will do from the residual in force in the average weighted sense.

So, subsequently we will see what we need to do to convert it into the weak form this is fine ok. So, we will continue with our discussion of 2D vector FEM and what we have so far is the wave equation right. Wave equation one thing I wanted to mention that when I took this equation over here there was there is also actually a J term over here, which I have ignored for now depending on the problem you will need to add it in ok.

So, in this discussion we are assuming there is no current supplied if there were a current then it would also appear in this $\vec{F}_H$ equation ok. So, right so the residual will also include this J term just to keep in mind that in general it should be there. regardless the philosophy of FEM is the same a weighted residual method I take my testing functions and impose this residual to be 0 in an average sense ok. So, now, let us let us open this up and see what the weak form is obtained from here.

Before we do that we should have an idea of what we are going to do. So, what was the key trick in 1D that allowed us to convert this to weak form integration by parts and we needed that because.

Student: (Refer Time: 07:25).

There is a double derivative right you can see $\vec{F}_H$ over here there is a ∇ and there is another ∇ over here $\nabla \times \nabla$ this sort of a double derivative term over here and we do not want to deal with double derivatives given the chance. So, we would like to convert it into single derivatives right. So, integration by parts is what we will have to do, but in two dimensions. So, that is something if I if I tell you integration by parts in two dimensions I do not think it rings a bell immediately. So, we will see there is a very simple what you called expression from vector calculus that that will help us there.

So, let us write this out I just write this on the top over here

$$\int_{\Omega} \vec{T}_m(r).\vec{F}_H(r)d\vec{r} = 0$$

$$\iint \vec{T}_m(r).[\nabla \times \tfrac{1}{\varepsilon_r(r)}(\nabla \times \vec{H}(r)) - k_0{}^2\mu_r\vec{H}]d\vec{r} = 0$$

Remember ε_r is a function of space otherwise the problem is not interesting ok. So, I have written dot dr since we are considering a 2D problem this is a two dimensional differential. For example, this could be a dx, dy, dz ok. So, I am, but I am writing it in short notation as $d\vec{r}$ ok.

So, now is the part where we will have to apply integration by parts. So, let us let us look at a very simple identity this first part of the left hand side ok, this is what we are going to look at the second part is easy. So, what does this look like; this looks like $\vec{A}.(\nabla \times \vec{B}) = \vec{B}.(\nabla \times \vec{A}) - \nabla.(\vec{A} \times \vec{B})$.

$$\vec{A} = \vec{T}_m(r)$$

$$\vec{B} = \tfrac{1}{\varepsilon_r(r)}(\nabla \times \vec{H})$$

So, there is the simple way to rewrite this. Ok these are standard identities in vector calculus ok, next what do we do we will substitute this identity inside the first term of the integral. So, what happens of we will just take the first term, what happens of the first term. So, let us write out the first term.

$$\iint[(\nabla \times \vec{T}_m).(\tfrac{1}{\varepsilon_r}\nabla \times \vec{H}) - \nabla.(\vec{T}_m \times (\tfrac{1}{\varepsilon_r}\nabla \times \vec{H}))]d\vec{r}$$

Is that fine nothing very surprising over here. So, have I escaped the second derivative, is there any second derivative in the first term negative second term is there. So, what do we do?

Student: But.

But if we if I integrate I mean how do I integrate; does it look like some theorem or vector calculus you have already seen.

Student: Yeah divergence.

$\nabla.$ something.

Student: (Refer Time: 12:10).

And a surface integral over it, so that is like which theorem.

Student: 2D divergence.

2D divergence theorem exactly so, 2D divergence theorem right. So, this term will remain as it is minus so the second the double integral will become a what integral?

Student: Line.

Line integral right the divergence theorem helps us to reduce one dimension. So, outgoing flux becomes a.

Student: Line.

Line or surface depending on what so if I in 3D a volume integral divergence over entire volume becomes a flux integral over the surface. In 2D a surface integral will become a line integral and how much flux is going out of the line ok. So, $\nabla.$ something is replaced by this whole expression. So, $\oint_{\Gamma} \vec{T}_m \times (\frac{1}{\varepsilon_r} \nabla \times \vec{H}).\hat{n}dl$ ok. So, what is my this is if this is my computational domain Ω is the is the whole interior thing and Γ is the surface or the line in this case.

So, have I now escaped let us say double derivative gone right the first term does not have it the second term by using the 2D divergence theorem that is also gone right. So, for those of you who are having trouble we call in this 2D divergence theorem I will just write it over here. So, $\iint_{\Omega} \vec{\nabla}.\vec{F}dS = \oint_{C} \vec{F}.\hat{n}dl$.

So, I mean it looks slightly complicated all this, but it is a not very difficult. So, putting it all together what does my weak form look like, first term will remain as is right.

$$\int_{\Omega}\int [(\nabla \times \vec{T}_m).(\frac{1}{\varepsilon_r}\nabla \times \vec{H}) - k_0^2\mu_r\vec{H}]d\vec{r} = \oint_{\Gamma} \vec{T}_m \times (\frac{1}{\varepsilon_r}\nabla \times \vec{H}).\hat{n}dl$$

So, this is the 2D weak form question.

So, should gamma be the overall boundary of this medium or the boundary of the triangle. Overall boundary of this medium because if you look at this entire slide have I made any use of the triangle anywhere? No, I mean I have just indicated by figure, but the triangle does not appear because I have kept H as H whatever H may be later I will write H in terms of triangles ok, but as this goes the this divergence theorem talks about the overall boundary. So, that is why this boundary term is coming just on the boundary. Now, and this does not in any way force me to have H to be you know sub domain basis functions or entire domain basis H is H whatever it is so a very general.

Student: Did this equation we do not have (Refer Time: 16:57) entire the tessellation.

We do not have the tessellation here right. We just used basic identities from vector calculus; we do not have H the tessellation so far.

Student: Still (Refer Time: 17:07).

Whole domain everything so, it makes sense to do all of the vector calculus simplifications first and then go to tessellation basis function and so on. Because I mean that way you reduce your the complexity of your problem ok. So, n hat will be the output now. I have just drawn a rectangle it can be any shape you want, depending on the problem fine. So, let me ask you supposing instead of such a computational domain I had a bit of a slightly different domain. So, I had the outermost triangle like this and I had some missing metallic you know some part where I know let us have metal piece I know the field inside is inside the metal is.

Student: 0.

0, there is no point in simulating and finding out it to be finding it out to be 0 right. So, supposing I want to exclude this hashed region from here, what would change in what I have done?

Student: The first minus.

Right, so this divergence theorem will get modified to exclude to have 2 line integrals. So, that you are integrating only over this green region right. So, if I call this Γ and I call this Γ', in that case the RHS will become $\oint_{\Gamma} () - \oint_{\Gamma'}$ and the expression remains the same ok. And the region gamma here will exclude this hash region right. So, this is a something that we should keep in mind for you know regions where you do not want to do any simulation where there is no use right.

Because any I mean the larger your computational domain the more price you will have to pay in terms of computation ok. So, this is our weak form this is this is fine. So, now, we come to the very crucial part which is boundary condition right. So, you can see that the carrying the trick forward from 1D the boundary condition is going to appear this right hand side term over here. That is where I will get to impose the boundary condition ok.

In the 1D case what was my right hand side term like; it was a du/dx. In the case of 2D it is not exactly du/dx, but it is $\nabla \times H$; $\nabla \times H$ has appeared over here. And now I have to figure out how do I approximate this, again without doing the math what do you expect will happen. In the 1D case I saw du/dx, I replaced it by some constant times u because that best simulated how a plane wave behaves those constant depended on the medium and all of that. So, do you think I mean the physics will not dramatically change when I go from 1D to 2D, so what do you intuitively expect to happen.

Student: Constants.

I should get some constants and H that is the form I should expect. Pardon me.

Student: It is square some.

Yeah I mean, but linear in H is not some derivative in H right. There will be some bunch of some linear expression in H is what I expect ok. So, it is good to do this check before you do the derivation so that, you know your intuition matches your math.