**Computational Electromagnetics**
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**2D Finite Element Method**
**Lecture – 11.01**
**2D FEM Shape Functions**

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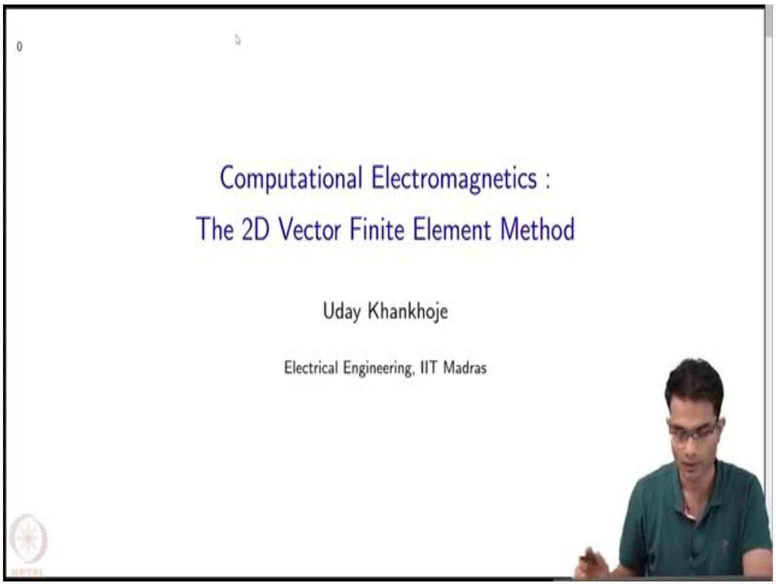


So, now so, this module is going to look at 2D FEM which is basically an extension of whatever we have looked at in 1D.

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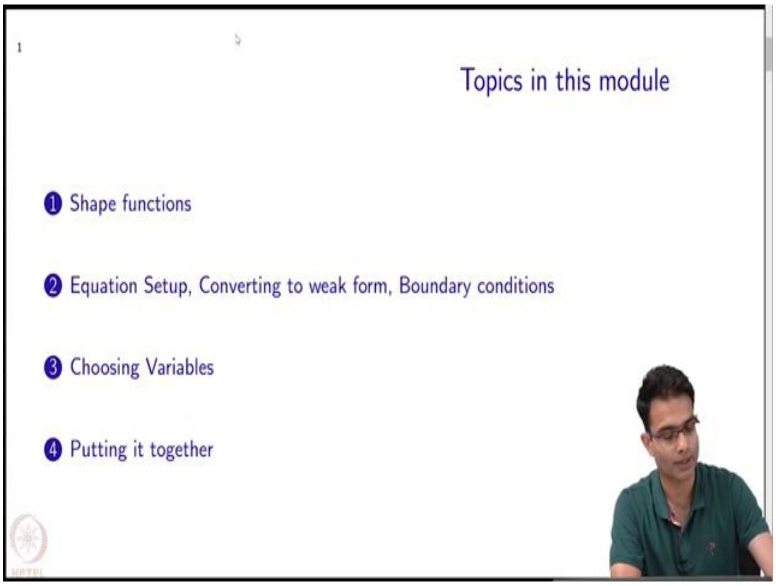


So, this is the sort of flow that we will go through something sort of significant over here is we are going to directly look at vector finite elements; what I have shown you so for are node based in 1D and I have showed you the shape functions for node based in 2D ok.

So, what we will do is we will directly jump to vector based a finite elements because you can it is easy for you to follow the logic of 1D to 2D scalar. So, let us do something more interesting 2D vector or the other word for it is edge based in that text it is refer to as node based and edge based ok. So, let us look at the kind of shape functions that we have ok.

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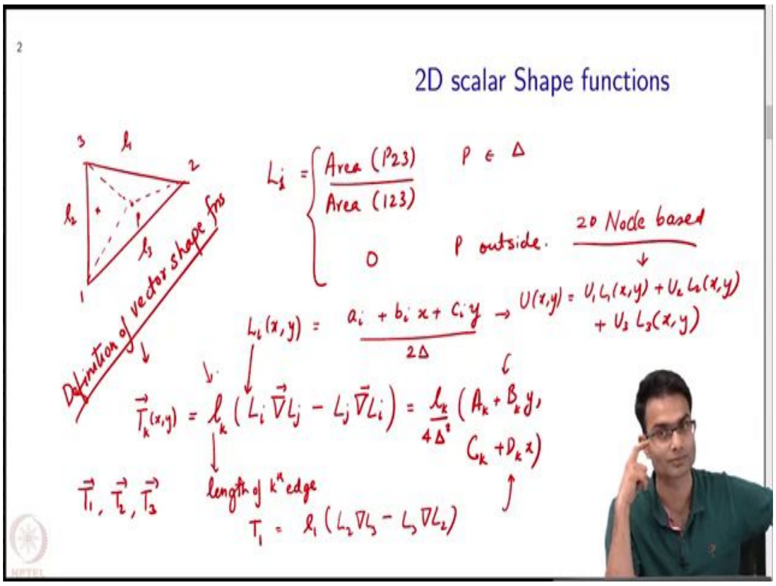


So, what was the kind of scalar shaped function that we had because just to refresh your memory if I had nodes 1 2 3 and some point P in between right. So, I had defined $L_i = Area(P23)/Area(123),\ P \in \Delta;\ \ L_i = 0,\ else$

And for example, $L_i(x,y)$ it had turned out to be a linear function right. So, we had for example, $(a_i + b_i x + c_i y)/2\Delta$ right that is what we had defined as the shape function.

So, this was node based and finally, with this so, what did we right? We said that the field anywhere inside $U(x,y) = U_1 L_1(x,y) + U_2 L_2(x,y) + U_3 L_3(x,y)$ right. So, the 2D node based thing is very very simple right. So, let us just write this over here this is node based 2D ok. So, any value if you take inside I can write it as a linear combination of $U_1, U_2, U_3$ and these are the linear functions.

So, now the question is how do I try to construct vector functions out of this. So, again here is the situation where the answer is a very beautiful geometric answer. So, I will write it out and then we will see what its properties are. So, I am going to define a vector function T. So, it is a vector function; that means, at any point in space it will have a magnitude and direction right unlike this function $L_1, L_2, L_3$ which at any point this has a magnitude the scalar function right.

So, this $\vec{T}_k(x,y) = l_k(L_i \vec{\nabla} L_j - L_j \vec{\nabla} L_i)$ . So, k can be 1 2 3 fine. So, notice where the vectors are coming in from right I am taking a gradient of this right. So, before we you get into any details what degree do you expect this to be?

Student: Sir linear only.

It will be linear only because, when I am taking the gradient.

Student: It is.

There first order they will go to 0 and then I will be left with $L_i$ 's right. So, again I will there are some very nice geometric features because of which all terms cancel of and your left with this expression. So, $\vec{T}_k = \frac{l_k}{4\Delta^2}(A_k + B_k y, C_k + D_k x)$

So, this is a definition of vector shape functions. So, you will have $T_1, T_2, T_3$ .

L is the same as above, the same L i which I defined in the node case I used over here right. Yes, I am using that to built the next thing ok. So, it is interesting to note that in the x component there is no x, in the y component there is no y and that just happens because, of the way in which this thing is designed these you know the x dependent terms cancel off ok. So, writing this expression like this you do not get much intuition of what is it actually look like?

So, why would you use node based or edge based right. So, we will come to that towards a little bit later, but basic idea is Maxwell's equations are about vector fields. So, by putting the solution in terms of scalar functions you lose a little bit of accuracy ok. So, vector based formulations are much more accurate. There is problem called internal resonances which happens in the case of node based formulation, where you get some spurious solutions.

Solutions which physically do not exist, but they appear when you try to solve the system of equations using node based those go away when you used edge based elements right.

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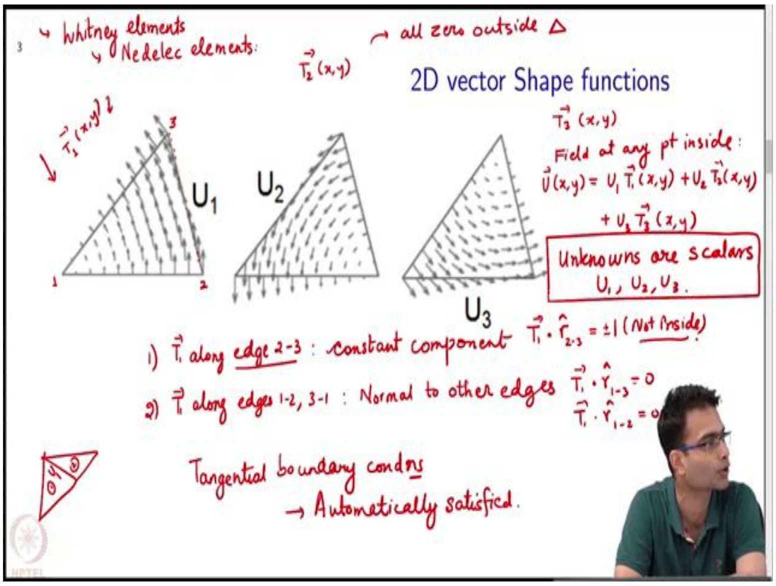


So, this is what these basis functions look like ok. So, for example, this is your $T_1$ this is your $T_2$ and this is your $T_3$ ok. So, as I said at any point in space x y there will be a vector so, therefore, a magnitude and a direction ok.

So, I am writing the field at any point as a linear combination of these 3 vector fields. So, before we go into that let us look at some of the properties 2 main properties. So, the first property over here is if I look at for example, any one edge over here let us look at $T_1$. So, remember this was 1 and in our convention this was 2 and 3 ok. So, when I look at for example, $T_1$ right in this example.

You can I mean you can from the figure you can sort of see that the component of $T_1$ along edge 2-3 what is it look like?

Student: (Refer Time: 08:25).

So, $T_1$ along edge 2-3 component of $T_1$ well for one is changing direction ones is starting out on the right then swinging to the left hand side. So, yeah the figure is slightly not very I mean which not possible to determine this from the figure, but if you do the calculation it turns out that the component of a vector field along the edge is constant. So, for what I mean is $\vec{T}_1.\hat{r}_{2-3}$, it is?

Student: Whole vector field.

The whole vector field.

Student: (Refer Time: 09:22).

So, I am evaluating at this, this, this, this points and any point over here right and any point along this dotted things who are take the component I get 1.

Student: Inside.

No on the edge only on the edge not the inside. That's the first thing. What about T 1 along the other edges. So, we will edge is 1-2 and 3-1 what do you think it looks like?

Student: Normal.

Normal right; it is purely normal there is no tangential component ok. So, what do you think might be the advantage of something like this yeah. So, $T_1$ does not at all corrupt the tangential component of the field along the other 2 edges. So, $T_1$ takes care of the tangential component of edge 1 by edge 1 I mean edge 2-3 right.

And $T_2$ for example, over here $T_2$ talks only about I mean $T_2$ s contribution along edge 2 is 1 along other edges 0 right. So, here for example, I should write this as 1-3 is equal to 0. $\vec{T}_1.\hat{r}_{1-2} = 0$ this above thing may be plus or minus 1 depending on the.

Student: Yeah.

Convention ok, we have to be careful about that. So, when I do this now I can write down field at any point inside how can I write it now? So, $\vec{U}(x,y) = U_1\vec{T}_1(x,y) + U_2\vec{T}_2(x,y) + U_3\vec{T}_3(x,y)$ linear combination.

That is how I am going to write it. So, I have, so, why does this what is the advantage of doing this my unknowns are the tangential components of the field right, if I go along any 1 of the edges what will I get I will get 1 or minus 1. So, I just multiply that by $U_1, U_2, U_3$ and these are my unknowns now in the problem that I have formulated. I said I wanted to do a vector based formulation; that means, my unknowns wanted needed to be vectors.

So, I wanted my unknowns to be vectors. So, what I have done is unknown vectors I have written as scalar unknown multiplied by a known vector field that is easier to solve that everything being unknown right. So, unknowns now are scalars what are the scalars $U_1, U_2, U_3$ ok, any other advantage that you can see of having done this? Hint is to think of more than one element yeah. So, these are all 0 outside. So, something about common edges what is what is a nice property of these things.

Take automatically take care of tangential boundary conditions for example, now supposing I have 1 triangle over here and another triangle over here.

Student: Continuous sir.

Right so, we know Maxwell's equation say that the fields the tangential fields that any boundary are the same unless there is some pure surface current let us ignore that for the we can deal with it later. So, if I have let us say you know free space from here or some homogenous or even heterogeneous material it does not matter at any boundary interface tangential fields are satisfied. So, if I take the basis functions inside this first element 1, let us say an element 2 along this edge if I assign the variable to be $U_1$ then that $U_1$ is common for element 1 and its common for element 2 because that that value $U_1$ is shared is it like what we had in the node based 1D FEM.

The value at the node is common to both elements why create 2 variables 1 variable right. So, boundary tangential boundary conditions automatically satisfied ok. So, this is I mean people went through a lot of effort to make these to construct these so as to have these properties. It is not just to happy coincidence that it happened the design these basis functions to have these properties because, if you are getting boundary conditions being satisfied for free a solution is bound to be more accurate ok.

So, these basis functions they go by various names in the literature right. So, I will just write down some of names they are to called Whitney elements by one community and another community calls them Nedelec elements.

Student: (Refer Time: 15:21) problem.

Yeah. So, the question is that if I have 2 adjacent 2 adjacent what do you call.

Student: Triangle.

Triangle elements then what about the direction of these arrows over here yeah; so, that will come to when you when we get down to implementation we have to make sure that along a particular edge there is a consistency in which direction the vector field is going yeah. So, this all gets taken care of under local to global conventions right. So, in FEM lot of time is spent on making sure that when I go from local to global and global to local everything is consistent ok. In fact, that is a source of a lot of coding mistakes also yeah.

So, we have to make sure that there is a consistent direction of the field along a given edge it cannot be that 1 element is saying this way the other element is saying this way because the unknown $U_1$ is common to both and there is only a unique direction to the field ok. So, everyone's comfortable with the shape functions. So, this is probably the first time you are looking at a vector field where which you have constructed right.

So, as I mentioned earlier the advantage of this is that you get much better accuracy in representing a solution and for example, the scattering problem which we did in the integral formulation are unknown there was $E_z$ TM polarization. The same problem if I want to solve now my unknown can be H in the plane $H_x$ $H_y$ will be written in terms of these vectors. So, I need not work with $E_z$ I can work with H.

Student: (Refer Time: 16:58).

When I travel from 1 to 2.

Student: Yeah.

Yeah.

Student: Let us see that side the 0.

The tangential components of U of $T_2$ and $T_3$ are.

Student: Like when.

No when you are going from your saying I am working from node 1 to 2 ok.

And yeah take any edge and work along it

Student: Yeah.

There is only one guy with the tangential component.

Student: Yeah This is.

Yeah, everyone else has normal component. So, the tangential component is being only is only being given by that one component. So, this is in contrast to these fluid mechanics people where there constant component along of certain edges not the tangential part, but the normal thing. So, that allows them to conserve flows across a boundary in our case we do not have a flow alright. So, now, let us let us yeah question?

Student: Is it.

Previous slide?

Student: Yeah, because of the.

Why is there no x in the x component and no y in the y component that is just how it turns out once you do the algebra ok, I am writing down the final form, but if you I mean these $a_i$ 's small a small b small c they all have the cord coordinates of the triangle inside it right. So, they get involved in constructing this definition ok. So, you can work it out they all cancel off. You're left with this very simple form it is a little counter intuitive and surprising, but that is what it is ok, but it is a simple exercise in algebra to write it.