Computational Electromagnetics
Prof. Uday Khankhoje
Department of Electrical Engineering (EE)
Indian Institute of Technology, Madras

1D Finite Element Method
Lecture – 10.10
ID Wave Equation: Basis and testing functions

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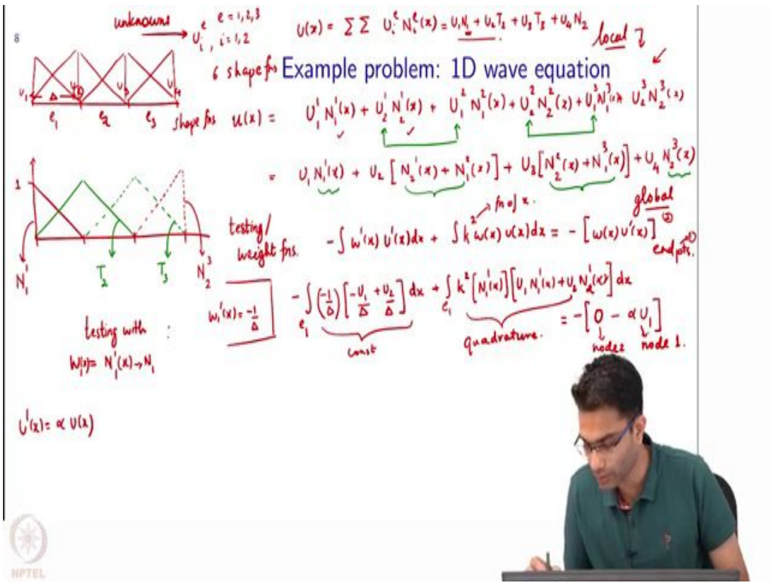

Now, let us come to that. So, to keep things very concrete what I am going to do is I am going to assume 3 segments just 3 segments and we will build our system of equations for 3 segments. For these 3 segments let us so what are what will call them is this is element 1, element 2, element 3 and this is node 1, node 2, node 3 and node 4 ok. So, what are my shape functions like? How many shape functions you have for this? 6 shape functions ok. So, what do they look like?

So, this is my U_i; i is equal to. So, this is what will be we put the we put it like this right i and e; $i = 1 \; to \; 2$ and $i = 1, 2, 3$. So, total of 6 shape functions ok. So, with this in mind how do I write the field? I write the field u of x in terms of these basis functions right. So, it is a summation and this time will explicitly open up the summation ok. So, the first unknown let us write the unknown and then the shape function. So, first unknown let us get rid of this summation. So, is U_1^{1} multiplied by $N_1^{1}(x)$.

Student: N.

Next is going to be U so, 1 on top and 2 below $N_1^{2}(x)$ right what else. Next I go to $U_1^{2}N_1^{2}$ now that you get the hang of it you can write it quickly and plus $U_1^{3}N_1^{3}+U_2^{3}N_2^{3}$ all of this is the function of x this is how would write it.

Now of these variables some variables are the same right these unknowns some of these unknown. So, these are unknowns right of these unknowns some the unknown are the same which are they?

Student: U 2 1.

$U_1^{2}N_1^{2}$ what else?

Student: Is there.

U_2^{2} and U_1^{3} right. So, the common variables let us just indicate them they are the same and they are equal to U_2 and U_3 respectively ok. So, what you could do is just a way of rewriting this if I wanted to write it is in terms of moving from local to global, if I wanted to rewrite it I could write like this $U_1\, N_1^{1}(x)+U_2\, N_1^{2}(x)+U_3 N_1^{2}(x)$. Have I made a mistake somewhere? Yeah this should be N_2^{2} actually yeah plus N_1^{3} correct. So, this was the first one was local the other one was global. U_4 this should be.

Local and global what unknown numbers now these two functions over here that I have in these green brackets over here, if I want to plot them over here let us let us try to plot the functions that I have shown highlighted in green over here. So, some of you see where this is going. So, you draw it little bit bigger here 1 2 3 and 4. So, the first one $N_1(x)$ which way it is opening is that left or right $N_1^{1}(x)$ this first term over here?

Student: Left.

Left right; so, this is what it looks like one at this point and this thing this is my $N_1^{\ 1}(x)$. Correct agree. What about N the second term in the green bracket what does it look like? It is a sum of

Student: 2.

Two functions, two triangles, one opening to the right one opening to the left; so, it is going to look like correct. I am going to call this let us call this T let us call a T_2. The next term in the brackets again it's going to look just like this one except it will be shifted to by 1 node right this is going to my I am going to call this T_3 and finally, the fourth one over here is going to be my usual $N_2^{\ 3}$ that is right opening third segment.

So this guy ok. So, how many unknowns did I have in the global in the in the overall problem U_1, U_2, U_3, U_4 ok. So, the reason that I did this combination is because remembering I have 6 shape functions if I do my testing with all 6 shape functions what I will get I will get 6 equations for each one of them one for each of them, but how many variables?

Student: 4.

4 variables right it will turn out and you can work this out.

It will turn out that two of these equations are.

Student: Redundant.

Redundant they are just formed by taking a linear combination of some of the other equations. So, to avoid having to do that what I am what I have done is I have combined adjacent shape functions into for example, say T_2 and T_3 ok. So, these above over here these are my shape functions and these guys that I have drawn over here these are what I will make into my testing functions or weight functions. This is still Galerkin's method because the shape functions and the weight functions are the same. What I am giving you as a bit of short cut one short cut compared to what most of the books have.

Most of the books will make you go through 6 equations, 4 variables and then eliminate and come back to 4 what we are doing is directly arriving at 4 because we can anticipate what is going to happen ok. So, these are the 4 testing functions; 4 testing functions when I apply to my equation I will get 4 equations 4 variables ok.

So, in general this is a strategy what is how did I mean if you look at it how did we combined this? Wherever a node were shared I combined the shape function that is what I did, the end points could not be shared. So, they remain as they are ok. So, now, let us let us go step by step let us do testing now with respect to each of these basis functions ok.

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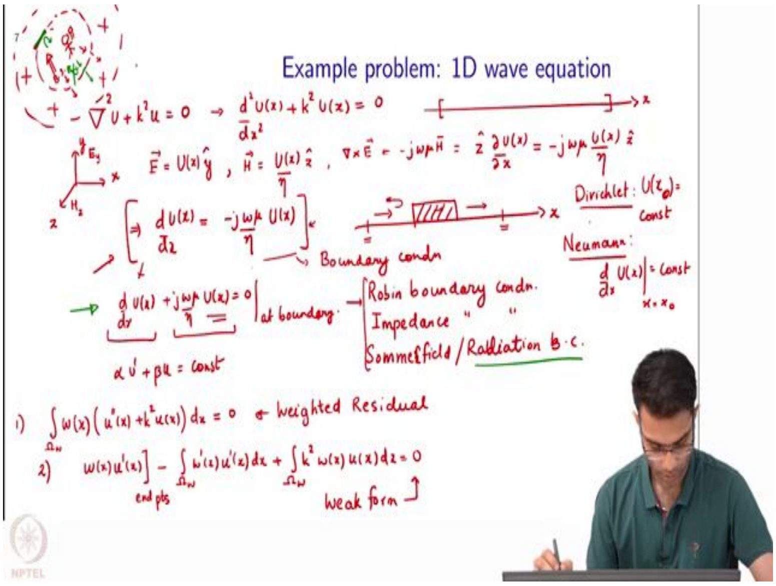

So, let us look back at our equation over here. So, so, we will let us write this down over here. So, the first term is $w'u'(x)$ minus. This is the minus and plus $k_0{}^2 w(x)u(x)$ is equal to the end points term right. So, this is equal to minus $-w(x)u(x)$. Why did I move this term to the right hand side?

Because we should know this $u'(x)$ is where I am going to impose my boundary conditions which I derived and it's a $w(x)$ of course, is known. Now without doing this end so, this should be done for every W that I take I will have this end point contribution. Can you look at this and say which all n points will appear in the final equation ok?

We will come to it as you start calculating it will come to it ok. So, let us now let us do testing with. So, the first weighting function which is equal to $N_1{}^1$ that is why w_1 that is what I am going to do it. U continues to be so, $U(x)$ we will continue to write it as in this form right $U_i^e N_i^e(x)$ that is that is what I will write it as this is also equal to in very short compact notation what is this $U_1N_1 + U_2T_2 + U_3T_3 + U_4N_4$.

Actually N yeah fine we just call it N_2N_1 and N_2T_2 and T_3. So, what do we what do we get when we test with N_1 which is shorthand notation which is N_1 ok. So, yeah.

Let us make it k^2.

Student: Because ε is.

Yeah, there might be an ε inside. So, that will be absorbed inside k^2. So, we will just leave it as k^2; k^2 is in general position dependent.

Student: Sorry.

Yeah otherwise there is an empty space.

Student: Then we should not.

Then we should include correct you are right yeah good.

Student: Its not be content they will be more in which.

Yeah I mean if epsilon is one everywhere it's a very uninteresting problem it is free space that I am stimulating right , but in our in our case they might be an object in between and that k will epsilon mu inside it we just keeping it a k squared fellow. So, this is a function of x. So, what happens to the first equation? So, what is going to be W' and U? So, what is W_1' I mean constant equal to what? Minus 1 by.

Student: Nodes.

No the nodes are not equispaced, I mean they are not spaced by 1. Remember the definition of N_1 right it was what was it $(x_2 - x)/(x_2 - x_1)$.

Student: X.

So, if I call this spacing to be some delta right; so, then $W_1'(x)$ is going to be equal to.

Student: Minus 1 by delta.

$-1/\Delta$ all right what about $U'(x)$? In this and this integral is over which segment now is integral is going to be only over segment e_1 all other segments are 0.

Because W itself is 0 outside it right. So, that derivatives also not define outside. So, $U'(x)$ this is my U over here. So, which all terms will appear in U' or you can look at this form look at this form of U of x now you have to evaluate $U'(x)$. So, what will happen over here? So, let us write it down minus integral of over e_1 $W'(x)$ is we said $-1/\Delta\ U'(x)$.

So, I have to take the derivative of this term and this term because both of them are non-zero in e_1 right. These are the terms will not appear because they are not in the support. Δ is the width of segment this Δ which are marked over here. Derivative.

Student: Derivative.

I have $W'(x)$ right.

Student: Oh.

This is W dash.

Student: One thing.

Yeah.

Student: The lengths need not be the same.

It need not be the same lengths that I have shown are equispaced over here, but they need not be the same. So, instead of delta, I will write delta 1 the minor details. $U'(x)$ what do I write in $U'(x)$, what is the derivative of $N_1{}^1(x) - U_1/\Delta + U_2/\Delta\ dx$.

Student: Delta.

Other terms are not going to appear then I come here $k^2 W(x)U(x)$. So, what is this going to be?

Student: $W(x)$ is N_1.

$W(x)$ is N_1 right. So, in my in this notation. So, $W(x)$ is $N_1 + N_1{}^2(x)$ actually I should I should write it little bit more careful over here it is the product of $N_1{}^1$ with the other two terms. So, this is going to be.

Student: U 1.

$U_1 N_1{}^1(x) + U_2 N_2{}^1(x)$.

Student: 2 1.

2 1 x sorry.

Student: 2 1 2.

Correct dx and the end points term ok. So, what can I say about the end points at which at which points are my evaluating this? At U_1 and U_2 is where I have to evaluate this ok. So, the first is. So, this is going to be at node 1 minus node 2 that is what it is going to be. So, what is the value at node 2 this is node 2 at node 2: what is the value of $W(x)$ at node 2? $W(x)$ is my $N_1(x)$.

Student: 0.

0. So, I have minus 0 what is the value of $W(x)$ at node 1? 1. What is the value of $U^{'}(x)$ at node 1 the trick question?

Student: Minus 1.

Minus trick question.

Student: Sir U_2 will be 0 there fine.

U_2 is not going to appear. So, all of you what you are doing is you are looking at the form of $U(x)$ you are talking the derivative and telling you what it is for getting that why did I move this term to the right hand side it should be known.

Student: Yeah.

So, known is coming from where, not from the form I have assumed for $U(x)$, but from the boundary condition. So, what is my boundary condition let us go back to it. Boundary condition is $U^{'}(x)$ right. So, $U^{'}(x) =-j\omega$ let us just call this whole term some α ok. So, it is equal to $\alpha U(x)$. I have to use this otherwise I am not imposing boundary condition. So, so, $W(x)$ at the boundary is 1 and $U^{'}(x)$ is going to be $\alpha \times 1$.

Student: So, at what (Refer Time: 18:34).

So, boundary point node number 2 this is node 2 that has no contribution. Node 1 $U^{'} = \alpha U(x)$. What is $U(x)$ at node 1 yeah, but what is that?

Student: U_1.

U_1 with an α, $-\alpha U$ I have to substitute here this is an end point substitution, I have to substitute the value of $U(x)$ at node 1 and I by definition the value of $U(x)$ at node 1 is U_1 and this is the α that came from the boundary condition right. So, where did the where did the free with the wave length and all of those parameters where did that go?

Student: And from there right.

It is got observed into alpha right the frequency and all that. So, the incident frequency is appearing somewhere in this equation all right. So, its not its not gone, so, this as come from node 1. So, what does this look like? This equation that I have written over here. Can I solve it? I mean can I simplify it rather what is let us go term by term what is this expression over here in terms of x is it a function of x? No right. So, this is a constant with.

There is a U_1 and a U_2 there right an integrated over dx. So, just the length of the segment will come. What about the second term? So, this is also integrated over e_1 what kind of what polynomial is this? is it constant linear quadratic?

Student: Quadratic.

Quadratic right N_1 is linear and N_1 and N_2 are linear. So, this is a quadratic function what and what are the variables $U_1 U_2$ anything else is a variable k, k is a function of x right. So, if my discretization is fine enough I can assume k to be approximately constant within that. So, point piecewise constant approximation I can make for or if k is something more complicated there is no problem what will I do for this term?

Student: U.

Use a quadrature rule in general right. So, this can be evaluated by quadrature right fine and from here what will happen U_1 will be taken back to the left hand side. So, there is a little bit of a there is a little bit of an inaccuracy and what I guess what I have said so far in evaluating the boundary condition I have written it as $\alpha \times U_1$, but there I need to modify this a little bit ok.

So, let us erase this one now this 0 term is correct this alpha U_1 is slight needs a little bit from modification ok. So, let us just look into our modification before you go further. So, let me erase that for now this boundary condition may intuition is correct right.

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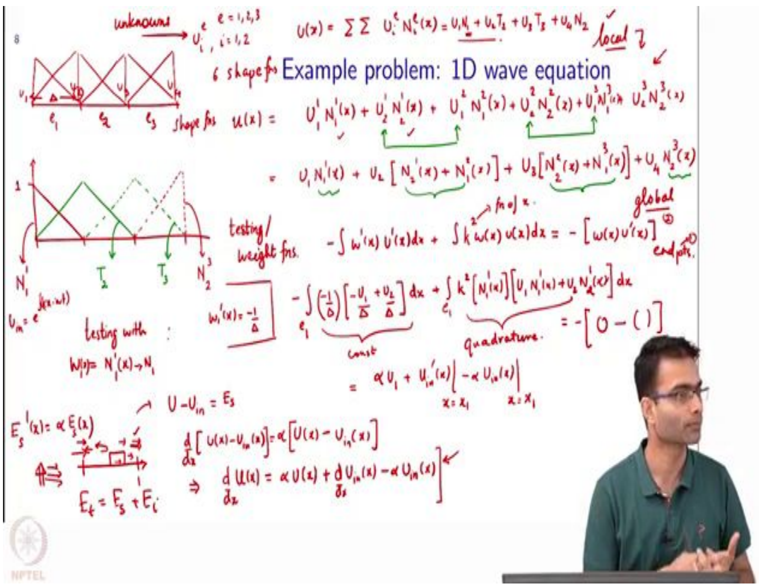

So, what is just saying that add this let us say this is this is the boundary the wave is going in this direction and at this point I am imposing this boundary condition. Similarly, at the left boundary same thing is happening and imposing this is boundary condition. Now this boundary condition applies to what all does it apply to let us draw our object over here ok.

So, the same antenna problem over here as human being over here right; so, this guy is sending out let us say and incident field let us redraw it over here ok. So, I have my antenna over here now what it is doing it is sending out of field like this let us say 1D problem some trying to draw it in 1D. I have my object over here these are the 2 boundaries of my simulation domain fine what is happening is that this wave is entering inside over here.

Getting scattered over here going through over here and then going off over here this is physically what is happening. Now the intuition for the boundary condition that I had given you was that the phase should not reflect back from this hypothetical boundary correct. At this point what is the total field there is a scattered field that is coming from the object and some part of the incident field both of them are outgoing and both of them should not be reflected back that is what we want ok. So, here they seems to be no problem.

But when I come here what is happening, incident field is actually going in the opposite direction from outgoing. The scattered field is; the scattered field is the part I should not allow to come back in to the simulation domain right incident field I do not care about its given by a mathematical expression. So, this boundary condition that I am saying that, no reflection from the boundary it should be applied to the total field or the scattered field?

Scattered field should not reflect back supposing I delete the object then there is no boundary condition to impose incident field can enter from one and exit from the other and that is consisted, but scattered field is always going out we are assuming that the object is fully contained inside the simulation domain.

Student: Which is fine that is all.

So, what is fine?

Student: Rename the scattered field is an outgoing itself.

So, exactly this scattered field is outgoing right. So, here this scattered field is outgoing here this scattered field is outgoing. So, the boundary condition should take care of scattered field not total field. Now what are the variables that we have chosen over here what are they total field. So, should we impose the boundary condition on total field or scattered field?

We should not be imposing it on U we should be imposing it on scattered field right. So, this boundary condition over here was let us. Let me write it as this E total sorry E scattered; E scattered is proportional to this is the correct boundary condition, this is what simulates a wave not coming back going on forever free space. So, in terms of U what will I write this as? So, E scattered so, we know that E total is E scattered plus E incident right.

So, all I have to do is since my variable is E total. So, E scattered will be $U - U_{inc}$ right. So, this is equal to E_{scat} ok. So, now, let us use the boundary condition. So, the boundary condition is saying

$$d/dx(U(x) - U_{inc}(x)) = \alpha\,(U(x) - U_{inc}(x))$$

$U_{inc}(x)$ I we will assume is given to us you know it's a plane wave or whatever I know the functional form. So, when I simplify this once more what do I get d/dx finally, see in this equation you see I want a condition on $U^{'}(x)$ ok.

So, this is the important thing that we have to keep in mind. So, the right hand side which I evaluate now which now that I have done a little bit more carefully node 2 still gives me 0 as before and node 1 over here this is what I will replace over here ok. So, what will I get I will get a minus some term over here let us a minus, minus which gets combined into what will thatterm be is equal to will simply be. So, what is this term evaluated at node 1 right. So,

$$\alpha U_1 + U_{inc}^{'}(x = x_1) - \alpha U_{inc}(x = x_1)$$

Student: Derivative of the incident being right.

Yeah.

Student: So, I just particular point this will be.

Yeah incident field is going to be given to you we will assume that.

Student: Fine, but what about the derivative.

That is also so, we will calculate it you know. Supposing I give you $U(x)$ is the plane wave.

Student: There are point.

No, we are not computing the incident field it is given to me in functional form for example, U_{inc} can be $e^{j(kx-\omega t)}$ or some other complicated function because it is like then given in the problem the current source is given to you can calculate what field it produces in the absence of the object that is given to you fine.