Computational Electromagnetics
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1D Finite Element Method
Lecture – 10.7
Generating System of Equations for 1D FEM

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So, now we will look at how to discretize this equation, now that we have got it and solve it ok.

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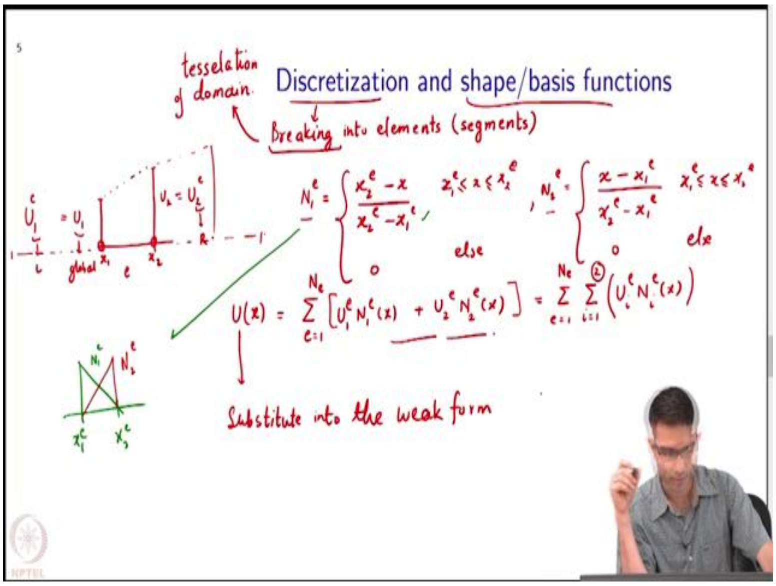

So, for discretization the first thing that we have to that we have already seen as discretization involves splitting it into elements right. So, this is breaking up into elements, the old word we had used for it was segments ok. So, but we are calling them elements now ok. In 1 D we can call them segments in 2 D what do we call it? We call them elements also, but the act of breaking up a domain in to little-little triangles or squares or whatever is called tesselation. So, that is another word you will hear.

So, breaking up is a sort of a colloquial word, the more technical word you will see in the literature is tesselation of the domain ok. So, this we in 1 D we already know what to do? Break it up into segments and the second part is we have to now introduce the basis functions or the shape functions in this case ok. So, we already know what to do in terms of the basis functions, if I take one small segment over here right. So, this is some element e, these are the two nodes of this element over here ok.

The global unknowns are U_1 and U_2 on this let us say and what is the local name that we will give for U_1 here? So, it will be U_1^e right just e right and this guy will be U_2^e is that right yeah. So, here this again refers to left this here refers to right ok. So, I am writing this again so that you get used to this convention that we have and this of course, is global ok. So, with this together what do we have already discussed the basis functions N_1^e is N_1^e is going to be 1 at x_1 and 0 at x_2 right.

So, what was the form of this? So, $N_1^e = (x_2^e - x)/(x_2^e - x_1^e),\ x_1^e < x < x_2^e;\ N_1^e = 0,\ else$ we have already seen this, and similarly $N_2^e = (x - x_1^e)/(x_2^e - x_1^e),\ x_1^e < x < x_2^e;\ N_2^e = 0,\ else$

Putting this together I write the function inside over here, this U of x what do I write it as?

So, I can write this as $U_1^e N_1^e(x) + U_2^e N_2^e(x)$ right, but this see now left now pay a little bit of attention in the left hand side is $U(x)$ over what? The entire domain right; so, this entire domain which is existing all the way from let us say here to here. The term on the right hand side is non-zero in the eth segment. So, how do I in write a general expression for U?

Student: (Refer Time: 04:54).

Student: Multiply by.

Multiply by.

Student: Pulse.

Pulse we do not need to multiply by pulse because N_1^e already has the pulse notion inside it, because I by definition I have made it 0 outside. So, N_1^e 's are already they already have the pulse inside it, now I want you to give me a function U; where I put in the value of x and it goes to the correct segment and constructs it as a linear combination of 2 shape functions. So, what remains to be done here?

Student: Summation.

Summation exactly right.

Student: Nodes.

I have to sum overall elements not nodes right once I. $U(x) = \sum_{e=1}^{N_e} [U_1^e N_1^e(x) + U_2^e N_2^e(x)] = \sum_{e=1}^{N_e} \sum_{i=1}^{2} U_i^e N_i^e(x)$ So, every term in this summation is non zero only in it is a own element it does not spill over into the other element right it is like pulse basis function. So, it automatically you give me the correct give me any x you want of these N_e terms only one expression will be non-zero in the segment of interest and that is the term that I want.

When we are doing this for a some simple thing like a you know a first order basis function it does not really matter right. But, as I mentioned earlier we can have higher order shape functions, where for a segment you need let us say 2 nodes 3 nodes 4 nodes whatever right those many number of nodes are required to specify a higher order polynomial, then to write out this first expression over here becomes tedious right.

The summation is over every element, now inside each element this number 2 that I have over here this can vary depending on how much accuracy I want in the basis function ok. But this is more a implementation or coding issue to sort of give you more compactness in that if it is not very clear you can leave it for now ok. So, this is pretty much as far as this goes, this expression as you can see supposing I have another segment over here, it automatically takes care of everything right. So now, that we have done this let us see what should what is next is now we have to take this guy and substitute into the weak form right.

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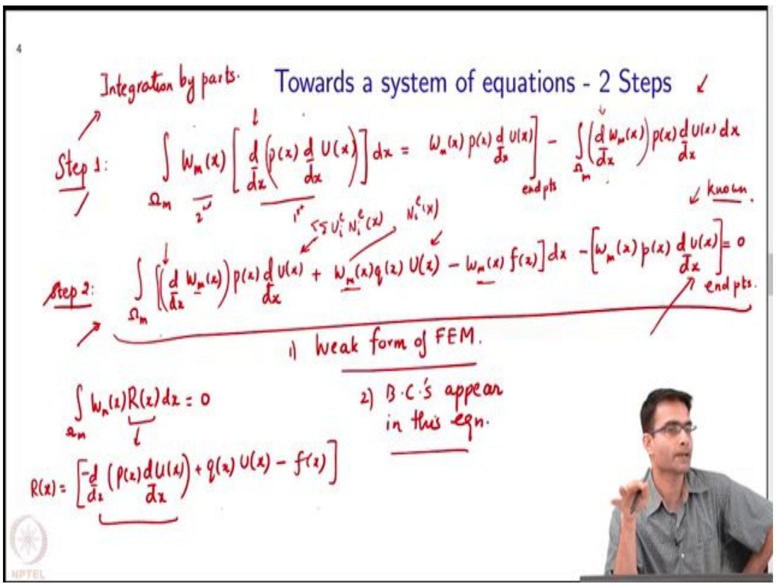

In this equation over here the previous equation over here, U was general this U right. Now, what I am going to do? Now that I have got this form of U, I substituted into this equation that I got alright. So, let us do that.

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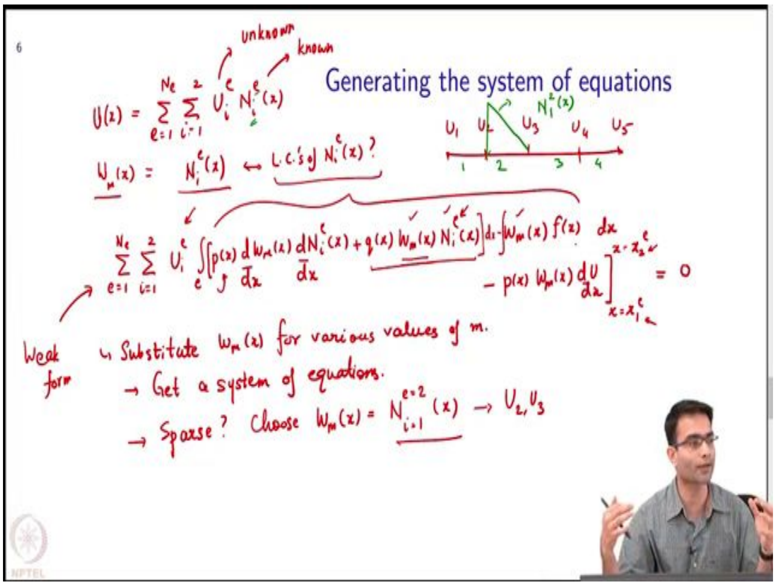

So, to generate the system of equations what will I what can I write it? So, first of all is

$$U(x)=\sum_{e=1}^{N_e}\sum_{i=1}^{2}U_i^e[\int_e[\frac{d}{dx}W_m(x)p(x)\frac{d}{dx}N_i^e(x)+W_m(x)q(x)N_i^e(x)-W_m(x)f(x)]dx-W_m(x)p(x)\frac{d}{dx}U]_{x_1^e}^{x_2^e}]=0$$

It is Galerkin's method so, shape functions and testing functions are the same. So, what should I choose $W_m(x)$ as N_i^e so, remember is the testing function fully known or does it have some unknowns inside it. If you go back to this equation, this is your step to look at this may big equation over here where do your unknowns going? Which terms carry the unknowns? What is the unknown in this equation? Is it W, p, U, q, f which is unknown U is unknown that is what we want to find out everything else is known.

So, therefore, W_m is also known I am going to choose it. I choose it to be equal to each of the basis functions that is Galerkin's method ok. So, let us come back to this now what so, what should I choose $W_m(x)$ as. So, you can see $U(x)$ what is unknown in $U(x)$ which is the unknown part in $U(x)$.

Student: (Refer Time: 09:54).

This guy right, this guy is unknown. This guy is fully known because all that I need to know to specify this other node location, which are known because I broke up the domain into known node locations. So, this so, what should $W_m(x)$ be? So, question is can $W_m(x)$ contain U_i ok? So, let us go back to this equation. So, you are substituting this your unknown over here which consists of $U_i^e N_i^e(x)$ this is what is being substituted for U. So, the unknown is here now $W_m(x)$ is appearing over here and here we said W should be known, there is no point in testing with the unknown something. So, what should $W_m(x)$ be?

If it contains a U_i^e inside it as you are asking a U_i^e then what will happened to U_i^e in this equation? It will become second order I will get a U_i^e squared and you know those kind of thing, but what is the nothing is achieved by doing it. So, what is left then? We cannot have n_1 and n_2 together in $W_m(x)$ is there what you are saying.

Student: (Refer Time: 11:18).

So let us see let us go one at a time. So, do you agree that U_i^e should not be in $W_m(x)$? Because it will be product of 2 unknowns then right. So, U_i^e should not be there then what is left? Unknown function N_i^e so, can I choose this to be $N_i^e(x)$, is there any conceptual problem with this alright. So, if you go back to this equation, if I choose this $W_m(x)$ to be something like $N_i^e(x)$ is there any problem? This equation will contain the unknowns U_i^e's will appear at various places and everything else will have to be done by integration right.

So, but you have raised an important point this choice of we said that in Galerkin's method the shape basis functions and the testing functions are the same, but we still have a little bit of freedom. Do we choose $N_i^e(x)$ or do we choose linear combinations of $N_i^e(x)$'s? So, I leave it at that when we actually take a concrete 1 D problem you will see which one to do both are correct yes ok. So, the question is will have to substitute this into the weak form e into $N_i^e(x)$ into i times is there what you are saying?

Student: (Refer Time: 12:55).

Yeah ok. So, let us actually write that down ok.

$$U(x) = \sum_{e=1}^{N_e} \sum_{i=1}^{2} U_i^e [\int_e [\tfrac{d}{dx} W_m(x) p(x) \tfrac{d}{dx} N_i^e(x) + W_m(x) q(x) N_i^e(x) - W_m(x) f(x)] dx - W_m(x) p(x) \tfrac{d}{dx} U]_{x_1^e}^{x_2^e}] = 0$$

So, all I have done is substitute U and I have put kept w as w ok. So, this is one equation and how many variables will appear over here? As many of the U i e's are there in this whole segment over here from here to here, there are various there is U_1, U_2, U_3, U_4, U_5 all of these guys are there, do they all appear in this equation? They all appear in this equation. So, what I have written are global numbers and these things over here these are local numbers, but there is a 1 to 1 correspondence between local and global and global and local ok. So, we can write it in either terms.

So, will you agree that this is one equation in 5 variables? One linear equation in 5 variables because over here is everything inside this bracket known f is given to you, q is given to you p is given to you N_i^e you chose W_m you chose. So, it may be looking complicated and ugly, but ever thing inside this is known to me. So, what is the very first trick that I had to learnt to do this integration? Quadrature right. So, quadrature is a very greatly simplifying thing because it does not need me to know the detailed analytical integration of whatever right.

I can increase the order of my quadrature rule and get arbitrarily good accuracy right. In fact, your N i es there going to they are linear polynomial. So, integrating them will in fact, give with quadrature rules will give exact answers as long as p and q and f are well behaved also ok. So, another common trick that is done is because this discretization is going to be chosen to be fine you can in fact, substitute p, q and f in terms of a pulse basis function. So, you can assume them to be piecewise constant in each segment so, then these guys also just become numbers.

For each segment we just put in the value or the midpoint of p, q and f. So, all you are left with is a polynomial to integrate, over here for the last term will be a linear term this is a linear term multiplied by this will be quadratic right. So, basically all polynomial terms is trivial to integrate them. So, what remains now of course, is I have got one equation in 5 variables and what would I do next? How will I get enough equations and enough variables? This guy $W_m(x)$ is in my hand I can put in different different w's get more and more equations ok.

Each $W_m(x)$ is non zero on the different segment. So, I will get different equations, but all equations will have some combination of this Us now. So, what we do is. So, substitute $W_m(x)$ for various values of m. Get a system of equations ok. Now, looking at this so, I mean we will take a detailed 1 D example where we will we will come up with the system of equations. So, is there any part of this procedure which is not clear? Let us clear let us make it fully clear now yes. $W_m(x)$ will be from 1 to 5 where m is some arbitrary thing m need not have a correlation with the elements or whatever I mean because there are how many elements there are 4 elements.

Student: (Refer Time: 19:23).

So, m cannot be element number, m will be something else such that I get enough equations in enough variables that is my objective only then I can solve it right. That is why I wrote that $W_m(x)$ can either be N_i^e or some linear combination of N_i^e's my basic objective is to get enough equations and enough variables. So, how to choose these W_m s we will look at in detail in the following slides ok, but if the basic idea here is enough equations in enough variables that is the objective ok. So, we will just put that aside for now.

Now, looking at this equation, we should be able to understand one very very key property of f e m why do I get a sparse system of equations, can you can you tell me why is it sparse? So, as a concrete example take $W_m(x)$ to be let us say we will just look at 1 equation ok. So, we will take $W_m(x)$ to be this function. So, this is element 1 2 3 4. So, this what is the notation for this shape function? N element number is 2 left or right.

Left is 1 at the left node so, it is therefore, 1 yes ok.

So, I am going to choose $W_m(x) = N_{i=1}^{e=2}(x)$ ok. I am testing the weak form with this shape function. Now, can we argue that the system of equations I get is sparse, let us look at the how should we do this.

Student: Second term (Refer Time: 22:46).

So, this second term if I look at let us start with the easy thing the second term the second term $W_m(x)$ what is the domain of $W_m(x)$? Is only element 2 because this N this guy I have chosen is non zero only over e=2 right. So, if I if this e, this e is summing over all the elements. So, if this e goes to element 1 3 or 4 what will happened?

Student: 0.

0; therefore, $U_1, , U_4, U_5$ will they appear in this they will not appear they get multiplied by 0. So, similarly you can see the only terms that are going to be non-zero in this in this equation will be what which Us will be non zero in this will appear in this?

Student: e equal to 2.

e equal to 2 numbered which e which Us? You are right e is equal to 2; e is equal to 2 has which Us in it?

Student: U_2 and U_3.

U_2 and U_3 yeah that will not be multiplied by?

Student: U (Refer Time: 24:00). So, that will be subjected.

Have I think I have missed a term over here have I? The last term does not have a U yeah the last term does not have a U.

Student: Square bracket (Refer Time: 24:12).

Square bracket actually yeah square bracket should not include the.

Student: Second term

The last term.

Student: Last.

Is what you are saying let me just check if I made a mistake in yeah you are right the yeah. So, this term you are saying should be like this and. So, this will be d x right the endpoints for each segment will be there right. So, they in fact, this endpoints will cancel except at the very endpoints of the.   .

Student: Whole domain.

The integration is the whole domain.

Student: Right.

Endpoint term is?

It is included in the summation actually. The well the I summation is appearing like this endpoints right in is integration by parts ok. So, anyway this term does not bother us, because W it does not contain any Us inside it in anyway right. So, the U's that will come in this equation are only going to be U_2 and U_3 right. So, you see that I did not get all 5 in this one equation.

But remember in our integral equations whenever I took anyone pulse basis function I got all I got a dense system of equations I have got n equations in n variables and each equation had all the variables. So, it was the dense system of equations. Here you can see what is saving as is the fact, that both the basis and the testing functions are sub domain that is the first thing. The second thing there is no Green's function which is non zero overall segments that was it that was the reason why I got all non zero terms here there is no global kind of a function that is connecting 1 segment to another segment.

So, only if there are shared edges or shared nodes is there any coupling. So, this gives us a. So, we can you can sort of extend this argument choose W_m to be different-different terms, you will get each equation will have only a subset of the total number of variables. So, that is how you get a overall sparse system of the equations.