Computational Electromagnetics
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The 1D Finite Element Method
Lecture - 10.05
Weak form of 1D-FEM Part-1

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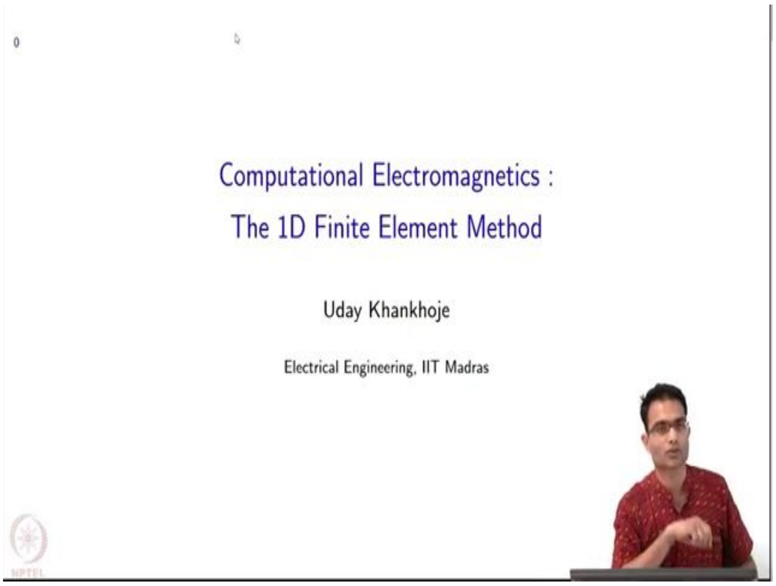

So, now having discussed the kinds of basis functions, we will look at we will try to understand this method in more detail and the simplest example is a 1D example ok.

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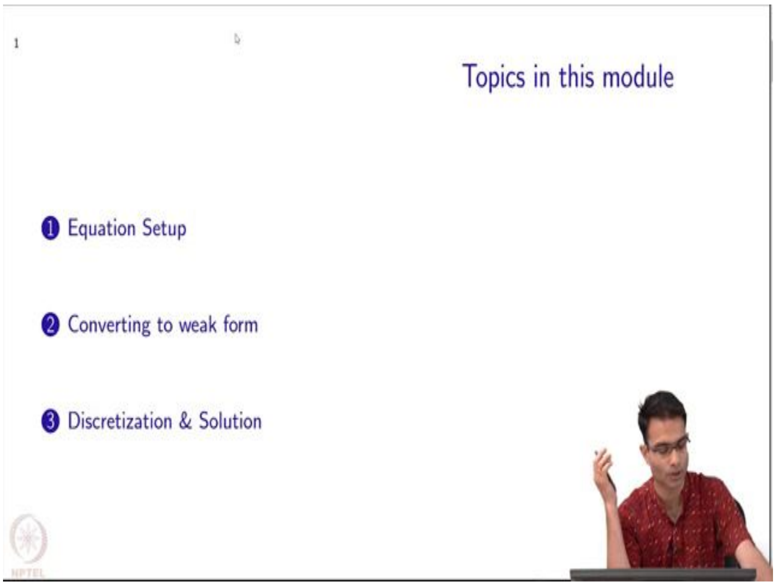

So, we will work through how do we you know setup the equations, convert it into something called a weak form, which is very characteristic to FEM, I will tell you what that is and then how do we solve it ok.

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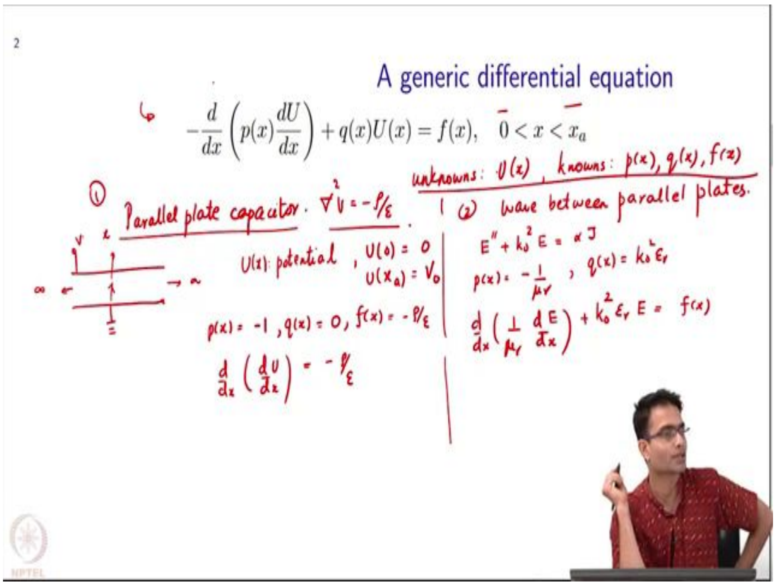

So, the equation setup, now on the board over here I have written very generic differential equation. So, in this differential equation your unknowns are what you want to find out are given into are is $U(x)$ and knowns are $p(x)$, $q(x)$ and $f(x)$ these are given to you ok. Now why I have chosen the differential equation like this to start off with?

So, first of all those of you who remember from your math courses, this differential equation looks very similar to the Sturm Liouville differential equation if you remember it. Not exactly there is there are some extra terms in that equation which are not over here, but I mean the left hand side structure looks similar and the power of that is it allows you to express almost any second order differential equation in this generic form ok. So, we will take I mean I will show you two examples.

So, for example, parallel plate capacitor, a dc problem. So, for a dc problem what is the you know you have a charge and you want to find out voltage that is your dc problem. So, that is also called as which equation? Given a charge find out the potential everywhere statics Laplace equation right that was your $\nabla^2 V = -\rho/\varepsilon$ right. So, this parallel plate capacitor which looks something like this that you know you have a capacitor infinite capacitor in both directions.

This is let us say a grounded and this is that some voltage you want to solve the problem like this. So, over here say $U(x)$ is the potential this is your x, fine and potential. What about the potential at the at this point? So, $U(0)$ is going to be 0 its a grounded plate and the upper plate is maintained at some voltage V_0.

And you want to find out how does the potential vary as I go from go through the capacitor, that is that is one something that you might want to calculate right. So, I know that the differential equation is this. So, can I can I fit into this generic form? I can right. So, what would I have to do? So, for example, what will be the value of $p(x)$?

Student: $p(x)$ will be -1.

$p(x)$ will be -1 correct then what is next? $q(x)$.

Student: 0.

$q(x)$ will be 0.

Student: 0.

And then $f(x)$ will be.

Student: $-\rho/\varepsilon$.

$-\rho/\varepsilon$ right so, then that will become

$$d/dx\,(dU/dx) \;=\; -\rho/\varepsilon$$

So, nothing new here just saying that this generic looking equation can capture a lot of problems in electromagnetic, that is why we sort of look at it ok. Then you also another problem the second problem I will take. So, one was parallel plate, the other is you know wave between parallel plates ok.

The equation for this you have already studied, it is the Helmholtz equation right a wave equation is the Helmholtz equation. And we know that the Helmholtz equation is something like you know $E'' + k_0^2 E$ is equal to; is equal to what? Some constant times the current we have seen this equation many times. So, how do I get this equation in this form?

Student: p square.

So, actually it would be tempting to set $p = -1$, but remember ε_r and μ_r need not be constants.

Student: (Refer Time: 05:27).

Need not be constants right. So, the same trick, that we had to do to use eliminate H from Maxwell's equations we have to take a $1/\mu_r$ right. So, this $p(x)$ will in fact be $-1/\mu_r$. So, the first term will become like this. Second term?

Student: k_0^2.

k_0^2 with one more thing, one more material property ε_r exactly. $\varepsilon_r E = f(x)$ whatever the current source so, right.

So, my $p(x)$ was this my $q(x)$ was $k_0^2\varepsilon_r$ ok. So, this is the sort of a general differential equation which is used many times fairly straight forward. One thing is that the boundary conditions are there, they have to be obeyed and respected at $x = 0$ and at $x = x_a$ and that will be important we will see how it goes.

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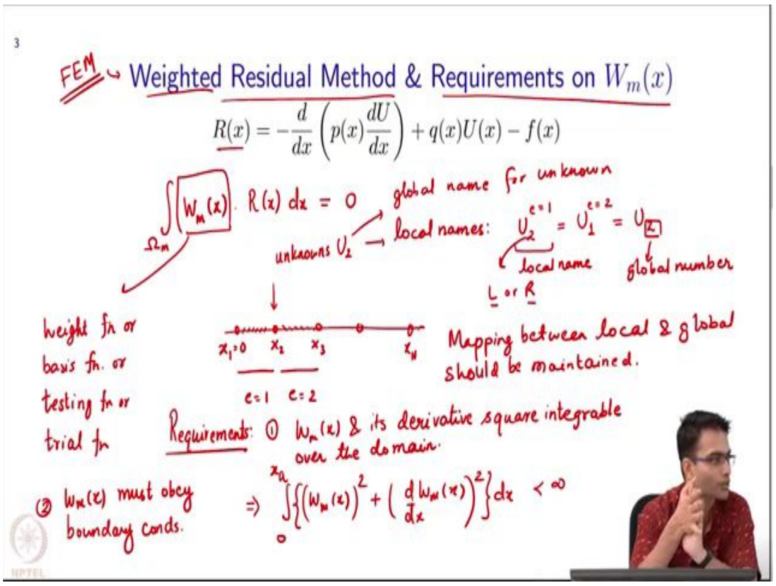

So, now let us put it into the FEM framework. So, FEM another word we had said was weighted residual method correct. So, first of all for weighted residual I must define the residual. So, this was my differential equation right. So, the residual is what? Left hand side minus right hand side is called residual. So, this residual $R(x)$ is defined as left hand side minus right hand side and then what do I want to do?

Weighted residual method; so, I have to take the integral of some weighting function I will call it $\int W_m R(x)dx$ and set it equal to 0 that becomes the weighted residual method ok. So, earlier I had call this as the basis function b_m I am just calling it W because W is coming from the word weighted ok. So, W for; so, this is your weight function or basis function or testing function.

These are all the different words used for it another word is you will find trial function. Everyone comes up with their own word for it they all mean the same thing ok. What is the domain of integration is the support of W_m. You know W_m for example, in the 1 D case is.

Student: Segment.

So, W_m is not a node it is the function defined on the segment on the mth segment right. So, it is a function and so, non zero only within a segment, zero everywhere else outside it that is how it looks like ok. So, let us sort of visualize this a little bit better. So, it is a 1 D problem. So, I should draw line, shape functions right, W will be the shape functions why because we are doing? We are going to do what is called Galerkin's method, where the basis functions and the testing function are the same right ok. So, let us call these like this right. So, this is my $x_1 = 0$ right. So, now the unknowns over here I can call it U_2 right. So, there is U_1, U_2, U_3 and so, one correct. So, U_2 is what I will call the global name for unknown correct there are you know. So, many nodes each one if there are n nodes each one is $U_1, U_2, U_3 \cdots U_N$ those are the unknowns right.

Now, in terms of local names what can I call it local names? So, for example, if this is the $e = 1, e = 2$ ok. So, I will call this over here U so, I am referring to this segment or this segment right. So, can I refer to like this? So, super script is $e = 1$. So, every element has two unknowns, the left node value and the right node value. Left node value I called as 1 right node value I called as 2.

So, its e is equal to 1 and for e is equal to 1 that is this guy which is at the left or right node?

Student: And x_2 is a right node

x_2 is the right node. So, I called at $U_2^{e=1}$. I could also call it $e = 2$. What will I call it in $e = 2$ this is the left node or right node?

Student: Left node.

Left node and this is all equal to sorry the subscript value. So, the same unknown is being called by three different names it, I mean it may seem unnecessary to do it, but it actually is a phenomenally helpful in coding, because you have a local naming convention and then you have a mapping to global naming convention.

When I do it on pen and paper for you know three nodes four nodes it seems very unnecessary, but when you write a code that should work for millions of elements its very very necessary ok. So, it will look a little silly at first. So, why we you just U_1, U_2 why have this U_1^e, U_2^e all of that, but it becomes very helpful to systematically form your system I mean form your equations.

Student: But why it is $e = 2$?

U_2 is the global name I am giving to it.

Student: That is fine.

No. So, this is a local name only a local name.

Student: there is $U_2^{e=1}$.

U_2.

Student: At $e = 1$.

No the U_2 two enough this two refers to left or right.

Student: That (Refer Time: 12:32).

And this refers to the global number.

Student: That subscript to is.

That is what the subscript two over here, if there is no element over here on top then it refers to the global number how many nodes are there?

Three nodes U_1, U_2, U_3, but if I am referring inside locally then this refers to l or r. So, better.

Student: There are (Refer Time: 12:54).

Yeah, if I have a 10000 elements, then I will have to be numbering them $U_1 \cdots U_{10000}$. But if I am maintaining a list of elements then within each element there only two U_1 or U_2 you could write it as U_1 and U_2. But then when you go to coded right U_1, U_2 becomes natural indexes for an array right yeah. So, this is something to keep in mind left or right ok.

So, these are this mapping between local and global should be maintained ok. So, typically there is some data structure that you will use to maintain this mapping. So, what is this mapping tell you? If you give me the global number you should be able to go and tell me what are the local numbers similarly if you give me the local numbers there is a mapping that will go and tell you the global number.

So, these are implementation issues when you implement your code you should have this functionality. So, now one thing I want to mention over here, about the choice of these W ok. So, over here we did not impose any requirements on W_m so, for right. So, what I spoke about, was the weighted residual method I will briefly mention what are the requirements on W_m ok.

Can W_m be anything for example, can Wm be a pulse basis function or something like that right. So, there are, at the very minimum, there are two requirements on W_m. So, requirements so, I mean W_m and its derivative these should be square integrable over the domain. So, I mean there are not two requirements this is requirement on U and its derivative.

So, implies that. So, what is; that means?. So, for example, if I take W_m square it and I take the derivative, this should be bounded ok. We are not going into the derivation of why this is required, but you can just take it as a statement of fact.

Student: Is not over Ω_m.

That is assumed because W_m will be 0 outside Ω_m. So, I may as well just write it for the entire domain. So, over this entire domain this W should be W and this derivative should be square integrable right.

Student: Reason being.

So, the reason; so, as I mentioned, we are not going into the reasoning, but the logic is similar to for example, representing a function using its Fourier series. The function they are needed to be square integrable and so on let so, similar idea. When this is satisfied, it is possible to prove that the FEM system of equations rigorously gives the solution. The system of equations that FEM gives will converge to the correct solution under this guarantee I mean under this requirement.

In addition I was actually right there are two conditions, this is the first requirement second is the easier requirement which is that Wmx must obey the boundary conditions. So, just as an example, supposing the problem had Dirichlet boundary conditions which said that x_1 and x_N. The final node over here x_N is 0. So, I am giving the boundary condition is at the solution should be $U(1) = 0,\ U(N) = 0$.

Can I use the pulse basis functions? I cannot I mean because the basis functions themselves I am not satisfying the boundary conditions its not enough that I just attach a weight next do to which is 0, but the basis function itself is not satisfying the boundary condition. On the other hand if I had a triangle starting from zero, then it is fine ok. So, let us just a small example, but these are the two main requirements we have to keep in mind.

Typically we use we are not in the business of constructing new basis functions. So, we are going to use well known basis functions, these requirements have already been satisfied we will not waste time in trying to check them, we will just use them.

We will force to be 0 right. So, it's not a wise choice of basis function to begin with fine. So, we will we will take up all this at this point. So, for clear? It can be whatever you wanted to be I mean galler can simply means that the testing function W and the basis function for U is the same. So, you can take a Lagrange interpolating polynomial like what we saw for both the basis and testing.

Student: Yeah. So, for that on this conditions are satisfied.

Yeah. For a Lagrange polynomial, yes, they are satisfying.