Computational Electromagnetics
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Introduction to the Finite Element Method
Lecture - 10.04
2D Basis Functions

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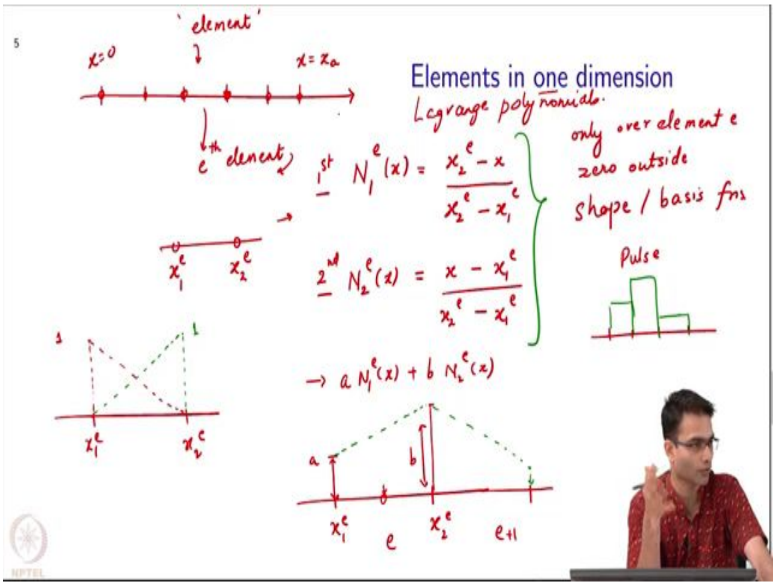

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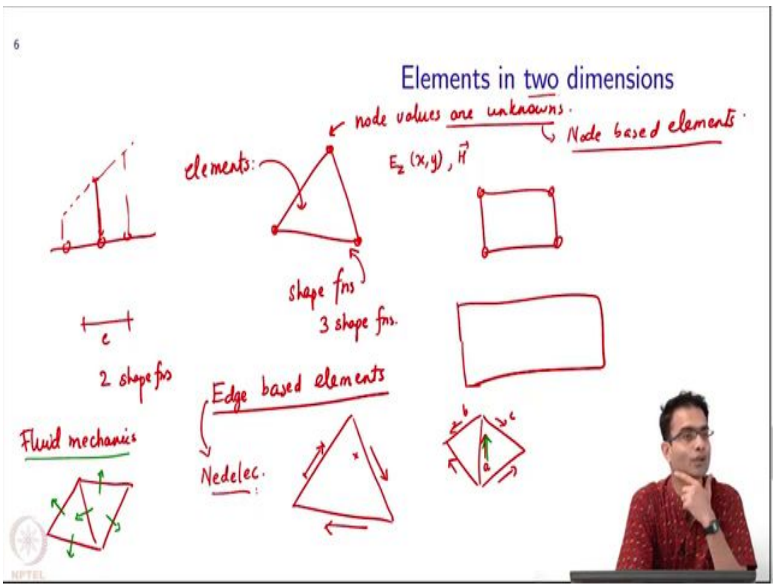

Now, we can extend this idea in two dimensions and the answer I mean the geometry again is very nice ok. So, what we will do is, in two dimensions, first of all I want to tell you in one dimension we had no choice the weight of discretize is line segments. In 2D however, we have a choice ok. So, the elements it can be like this triangle that is what you saw in the very beginning the aircraft whichever was broken, but these are the only choice.

Student: No.

Now, it I can have a square or I can have a rectangular any number of shapes I can have right. So, each element then has inside it some shape functions ok. So, we will stop calling them basis functions now we will we start calling them shape functions so that when you read the literature it is smooth right. So, for example, when I took the 1D line segment over here this had 2 shape functions right.

Now, when I take a triangle like this right the bare minimum would be 3 shape functions, then we will discuss how there are 3 shape functions right. So, one attached to each node basically right. Now when I go over here this I can you conceivably have four shape functions over here. So, what we are doing is, we are bringing in more and more flexibility into modeling the unknown by having these kinds of shape functions over here. So, that is one thing I wanted to tell you in the case of 2 D.

Now, you one thing you notice that here I am making the unknowns to be the values of the function at these nodes, that is the straightforward extension what I did in 1D, in 1D my unknowns were these nodes from that I define these shape functions right everything is good. I can continue that idea here and say nodes right nodes are the node values are unknowns and we will see in the next slide how to construct the shape functions.

But you get the intuitive idea nodes are the unknowns. Can they be some other kind of unknown? Well centre point ok, but that is also an unknown. So, let me sort of posses question in another way, Maxwell's equations are essentially vector equations right and so, my unknowns are actually vectors, but I am trying to express them in terms of scalar nodes.

So, what happens to the direction then? So, if I had something like you know E_z , I can say this is in the xy plane. So, E_z can take E_z is the function of x y. So, there is a value of E_z is itself a scalar, but what if I wanted to do represent a true vector field? Because for example, in this case H is in plane; in the TM polarization H is a vector in plane, supposing I wanted to represent the H vector in plane, will the nodes we useful for me? Node is not going to help me how do I get direction out of it? It gives me a value it does not give me direction right. So, these are what we looked at so, for these are node based elements.

It is also possible to define what are called edge based elements. So, edge based elements and we will we will look at both node based and edge based elements, the element shape remains the same so, triangle ok. Now the unknown over here becomes the value of a vector along a certain edge ok. So, it might seem a little abstract now, but will build it explicitly everything will work it out. But basic I mean the idea you should keep in mind is if my unknown now is the magnitude of this vector. So, if it is the magnitude of this vector I am at least able to impose a direction along some edge ok.

And similarly for these 3 edges and at some point over here if I want to find out the field, it is going to be a linear combination of something based on these 3 vectors. So, I am able to associate at each point in space, a direction ok. So, again this is a new concept. So, we will we will studied in detail, but you should know that there are node based elements and edge based elements. Node based elements typically have limited accuracy compared to edge based elements, edge based elements required to be very careful about vectors and directions ok.

Student: What you call supporting?

So very good question; so, he has the asking what happens supposing I have 2 elements like this ok, it is the same as in the case of 1D I had 2 elements the value of this node over here did not matter, because supposing the shape function took its value over here then it can go down to here and go up to whatever value over here, because this node value was constant. Similarly in this the edge is common to both elements. So, the edge direction is consistent between both the elements.

So, it's not that there is one direction on the left element and another direction on the right element its a shared edge, earlier you had a shared node now you have a shared edge. So, along that shared edge the direction of the field is unique, there is no flipping happening across so.

Student: Both the calculations we within the side.

Will be considered because the unknown is common between both edges. So, it's built into the system of equations that you will get only one direction for each edge. So, this guy will have its own other edges pointed anywhere, combined in the combined way when I look at all the elements of the system every edge will have a unique direction. So, that is that is what is giving a lot of power in expressing the field in other vector over any funny looking object ok.

Student: Sir good.

You mean the for example, if I call this edge a and edge b and edge c.

Student: Ok.

You are saying that between a and b, there is a big jump in direction.

Student: yeah.

That the equations will take care off; because when I impose this weighted residual condition.

Student: Oh that.

It will make it consists in such that there is no finally, we are trying to obey Maxwell’s equations. Maxwell’s equations will not allow a sudden flip between the field at 2 adjacent points unless there was some material or something like that. So, that is built into how we formulate the system of equations, we need not vary about it. So, at least as far as electromagnetics course these are the kinds of basis functions we use, but electromagnetics are not the electromagnetic community is not the first community to do finite element in fact, the first community were fluid mechanics people, mechanical engineer, aerospace engineers.

They also use what you call edge based elements, there edge based elements are slightly different. I will show you what their. So, fluid mech fluid mechanics this is just sort of general knowledge. Their edges I mean they do not like edges what they are concerned about is, like fluid flow they are concerned about how much fluid is flowing across a certain interface. So, their basis elements shape functions look like this actually, these are the common or conserved quantities between elements.

A vector which is normal to the edge, that is shared between 2 elements. So, if this water flowing across in the interface, how much comes from the left that much goes into the right. So, that conserve quantity is this green arrow over here. In the electromagnetics case this arrow, it is ensuring which property of Maxwell's boundary conditions.

Student: Tangential.

Tangential continuity; we have building it into the way we design the elements itself. So, you get it for free. Similarly you do a different construction you get the fluid mechanics case, the normal component is conserved. So, you build it into the solution, you do not have to do something extra to get this property yeah ok. So, this edge based elements also are they are named after the scientist who discovered it Nedelec ok, but first we will start with node based elements.

So, number of variables currently is 3 for the low for the most for the simplest node based or edge based element, per element there are 3 unknowns and those 3 are of course, shared across.

Student: Sir.

If there are n elements in the domain how many unknowns are there?

Student: 3.

No; they will not be 3 unknowns because many of the nodes are shared right. So, similarly in the case of 1 D and how many nodes are a 1 2 3 4 5 6 7 sorry 6; 6 nodes are there. So, unknowns are 6 elements are 5 5 elements have.

Student: 6.

$5 \times 2 = 10$ you would thing 10 nodes per element, but of those 1 2 3 4 are shared. So, $10 - 4 = 6$, which is all my. So, similar accounting happens in the case of two dimensions ok.

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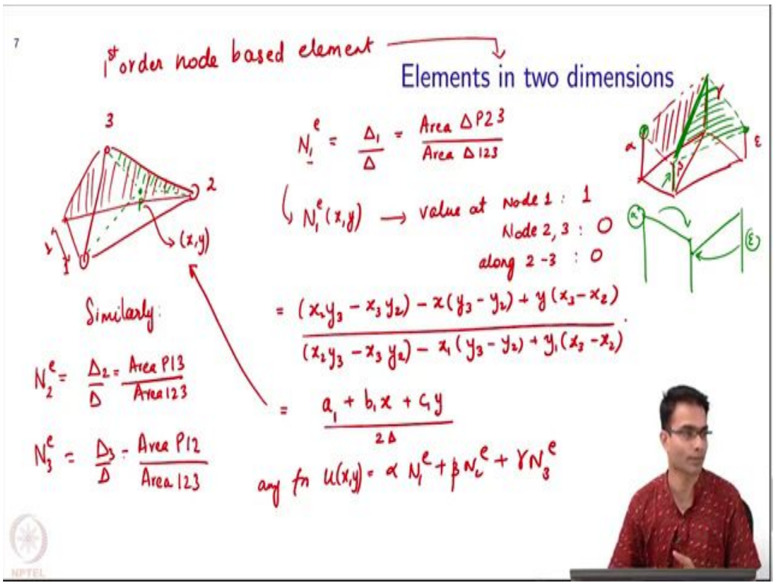

So, now let us actually explicitly construct the first node based element. So, this is the first order node based element in 2D ok. So, I take my triangle ok. So, I am going to call this node 1 node 2 and node 3.

So, what I am going to do? So, I am going to tell you the final answer and you will wonder how did people come up with it, but you will see when you see it, it is very elegant. What I am going to do I am going to define some arbitrary point p inside this triangle ok. Remember everything all this entire discussion that is happening about the shape functions are limited to the interior of this or the element, this triangle not outside the triangle.

Let us connect this, hypothetical point to the 3 nodes this is my point P ok. Now I define my first shape function as the area of triangle 1. So, what is triangle 1? It is the area of triangle P_{23} divided by area of triangle Δ_{123} this is the definition ok. So, that is why you notice that its N 1. So, I exclude 1 and therefore, is the area of P_{23} so that means, this area.

So, now let us let us look at some of the properties of this function over here; so, $N_1{}^e(x,y)$ as a function of x y. It is going to be a function of x y because whereas, where is the x y going to come from? It is the coordinates of this point P. The node locations are fixed P can roam around anywhere inside, as P roams around my function $N_1^e(x, y)$ is now a 2 dimensional function. The coordinates will come in over here ok. What is its value at node 1?

Student: 1.

At node 2 and 3 0 at the triangle fall flat; so, this is like that Lagrange polynomial of the first order case, it has its value 1 at its own location and 0 at the other node. Similarly this function has its value 1 at a certain node, 0 everyone else ok. Not just node 2 3 along 2 to 3.

Student: 0.

Still 0 right. So, its like its like a tent, right the tent has some height over here if I think about it looks something like this right it has its value 1 over here this much is 1 and it goes to 0 at these 2 points and it stays 0 along this whole edge until that reaches 2 ok. Now what kind of a functional form do you think this $N_1^e(x, y)$ takes? Is it constant linear quadratic what is it?

Student: (Refer Time: 13:43).

It should be linear in I mean in x and y. So, we can use whatever recent high school Heron's formula whatever it is right. So, $x_2 y_3 - x_3 y_2$, I will just write down the explicitly area of triangle formula. That is just formula of area of triangle. So, there are a whole bunch of constants over here right. Once I have got my geometry, I have made triangles out of it. So, the node locations N_1^e, N_2^e, N_3^e these are fixed so, these all numbers.

So, I can reduce boil this down to $a_1 + b_1 x + c_1 y$ divided by, because the denominator is twice the area. So, we can see that this is a nice one dimensional function in x and y. Similarly I can define what should I define my N_2^e as? Triangle 2 by triangle which is area of P_{13} by area Δ_{123} ok. and N_3^e is going to be similarly area of P_{21} ok.

So, all of these 3 shape functions are linear functions ok. So, what is the geometric picture you can think of how, what am I doing to my function? So, you all agree that this visualization that I have shown you over here like a tent with height 1 at node 1 and 0 by the time it reaches node 2 3 and the line connecting 2 3. Now if I take any function any function say

$$U(x) = \alpha N_1^{\ e} + \beta N_2^{\ e} + \gamma N_3^{\ e}$$

right; so, not x, but x y. What is going to look like? If you wanted to visualize this what does it look like.

Student: Plane over that.

A plane like a like a tent again like a tent and the height of the stand at 1 2 and 3 is alpha beta gamma right. So, it is going to be if I am going to use my fantastic drawing skills this would be alpha then gamma and beta and it is a plane that is at suspended by these heights over here ok. So, what do you what will you get when you go to adjacent elements. So, this is 1 triangle for example, there will be another triangle over here.

Let us say this has some value ε. So, when I consider the second triangle what will be the height of I mean what will it look like?

Student: It has been (Refer Time: 17:43).

It will go from here like this right another tent over here, but its value along this will be conserved. it will be the same value there will be no these 2 tents will come and meet at the same line there will be no jump over here, because of the way we have constructed it. Whatever is happening at alpha has more influence on this because the way the shape function corresponding to this becomes 0 by the time you come over here, similarly whatever is happening over here has no influence on this.

So, this from this line is detected purely by β and γ, which is common to both of these elements. So, I have got a smooth function continuous I mean it's a continuous function its not differentiable because we can see its like 2 lines meeting here. So, differentiability is not there. Even if they do not meet up.

Student: (Refer Time: 18:33) same within convert to your same contract.

Same line.

Student: Same line same line.

The common edge.

Student: The common edge and.

Yeah.

Student: Here if it is does not work.

No that will not happen right supposing your. So, let us take a view along this ok. So, this is your one edge comes in like this, the other edge if it comes like this.

Student: That the third can.

The third has no influence on this line right. So, this was my I am looking from here this direction, this was my height α and this was my height ε.

If these 2 did not meet at the same point, there is no way for you to fix it because whatever is happening here has no role to play over here, whatever is happening here as no rule to play over here. The value of the height of this line over here is fixed purely by β and γ. So, you get continuity for free, β and γ are shared nodes.

Student: Yeah.

For the 2 elements. So, they shared they do not have a separate value. So, you can see that it's a very nice continuity property is built into the basis function. So, therefore, the solution that I get it also obeys I mean I am projecting my solution on this basis it will have these nice properties to reveal.