Computational Electromagnetics
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Introduction to the Finite Element Method
Lecture – 10.02
Basic framework of FEM

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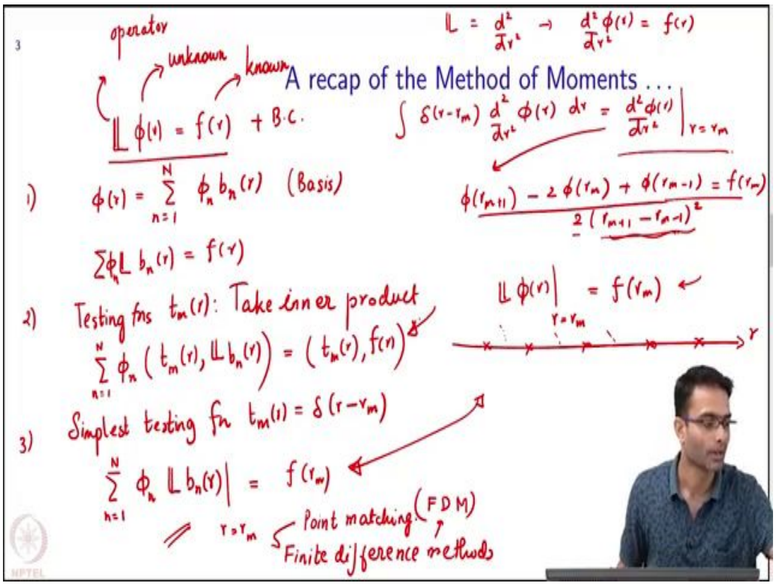

So, let us get into some of the details of this Finite Element Method and as I had mentioned much earlier, the conceptual way we worked through it is actually very similar to what we already discussed this so, what so called method of moments; so, let us just recap that. So, what was we had $L\phi(r) = f(r)$. If you remember that was our operator equation on which we have studied the method of moments right.

So, the recap over here, what was L was an operator right so, far the operators that we had seen were integral equation operator they had a integral sign inside it ok. $\phi(r)$ was what the unknown which I wanted, and $f(r)$ is unknown forcing function along with some boundary conditions ok. So, we that the all the boundary conditions always need to be specified in this case right. So, we will assume the same sort of formulating equation over here as a starting point, what did we do when we went to the method of moments, what was sort of step 1?

Student: Basis function.

Basis functions right. So, I said $\phi(r)$; I said I can write in this form a basis a set of basis functions right, and then what did I say? So, I substituted this inside, so I got and this $\phi(r) = \sum_{n=1}^{N} \phi_n b_n(r)$, $\sum_{n=1}^{N} \phi_n Lb_n(r) = f(r)$.

Student: Equal to (Refer Time: 02:12).

Right; so, we could do that, but basically step 2 was what, basis is one first step.

Student: Testing, right?

Testing right; so, I had so these were basis functions and the next step was testing functions $t_m(r)$. So, what did I do? I took a inner product right; so, take inner product. So, what happened? So, this summation stayed outside right; so, this is $\sum_{n=1}^{N} \phi_n$ is a number that is constant and here is where the testing function came, that is the left hand side and right hand side simply became the inner product of right. So, this part is sort of familiar to all of you right; then, what was the simplest kind of testing function that we considered?

Student: Delta.

Delta right; so, then if you take the simplest testing function we; so, that is $t_m(r) = \delta(r - r_m)$ that is how I wrote it right. So, this inner product and all of this stuff what does it boil down to; this inner product collapses, the integral collapses I am left with $\sum_{n=1}^{N} \phi_n Lb_n(r)|_{r=r_m} = f(r)$.

So, example is for example, $L = d^2/dr^2$ ok. So, then this operator equation simply is $\frac{d^2}{dr^2}\phi(r) = f(r)$ right; now you can see why I have written this in this fashion right. So, when I multiply this by some testing function, so $\delta(r - r_m)$ and then that is the meaning of testing; right the left hand side? Right so, this will become evaluated at $r = r_m$ that what it means that is the only values that survives inside the integral right so,.

So, this was your testing function over here and the method we had I mean name for this was given as point matching what point matching was the name we have given right point matching ok. You could stop at this point and this gives rise to a popular set of approximate methods which are called finite difference methods; so, this also leads to finite ok. So, what do I mean by finite difference methods, you can look at this example over here right.

So, here let us say that I do not want to calculate any analytical derivatives and I have this you know second derivative of phi evaluated at one point. But I want to write this only in terms of function values, in terms of finite differences, then what will I write this as I mean, this would be then become something like $\frac{\phi(r_{m-1})-2\phi(r_m)+\phi(r_{m+1})}{2(r_{m+1}-r_{m-1})^2}=f(r_m)$ right.

So, like this you can repeat it for various values of r_m and you will get a set of equations for every $f(r_m)$ on the right hand side you get a set of equations. So, that is known by the name of finite difference methods. We will look at these in detail when we come to the third module of the this course ok, but I am just identifying a branch of point which happens right over. So, right; so, what happens is the same framework that we have for the method of moments is what we will use going forward for finite element method question;

Yeah as you assume they were separation is one that is yeah good to be more precise. Yeah you are saying you asking, what should be the separation between the node points? Yeah so, you have to be a little bit careful over here you should be careful to make sure that you do not by default set this just 2 and forget this term over here it matters ok.

Because, overall your I mean your scaling overall whatever you do, it should be done in a consistent way because as you seen in integral equation methods before, as I make my discretization finer and finer I get more and more accurate answers. So, that discretization translates into the difference distance between nodes. So, I am approximating a derivative by a finite difference right. So, that the denominator matters; so, you have to be careful about it.

People who do not include this factor here in the denominator they included in the numerator of the on the right hand side. This is just by a I mean the idea behind this equation is finite difference approximation of a derivative; forward difference, central difference, backward different those are ways of approximating a derivative you applied two times you get this expression ok. But yeah, we will not go more into this because we will have a dedicated module for studying finite difference methods ok.

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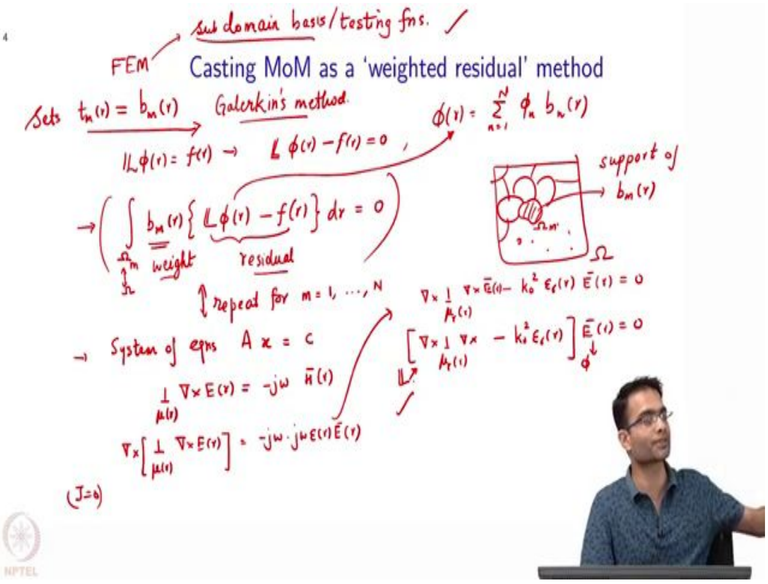

So, this is the same framework that we are going to use and coming more towards the FEM right; what FEM does is it sets the testing function to be equal to the basis function ok. So, that is the most I mean it is not absolutely the golden rule, but this is the by further most common way for doing FEM is you choose the testing and basis functions to be the same ok. And, this is also called we had come across this word earlier Galerkin's method ok

So, just one thing I want to point out from the previous slide when I look at this equation over here I can also rewrite I mean I can rewrite this whole equation in a even simpler format; like this which is $L\phi(r)|_{r=r_m} = f(r_m)$.

So, in some sense what am I saying with this equation is that I am enforcing this equation at a few discrete points $r_1, r_2, r_3, \ldots, r_m$; correct right. So, if I had my whole axes like this r, I am only enforcing it at some you know regular grid of points over here, but I am not saying anything about what happens at intermediate points over here, here, here and so on right. So, the natural question is that you know is there available to get little bit more accuracy ok. So, what we can do with this is this same equation that I had $L\phi(r) = f(r)$ what I will do is let me rewrite this in the form like this $L\phi(r) - f(r) = 0$ ok.

Previously, we were enforcing this at just one point; now, what we say is to get better accuracy I am going to use a testing function other than a delta function and that is the same as a basis function that we here. So, what is that basically mean? It means that an integral over so, $b_m(r)$ this is testing right, this is same as step 2 earlier testing with a basis with the testing function, in this case the testing function is the same as a basis function right. So, I will continue to write $\phi(r) = \sum_{n=1}^{N} \phi_n b_n(r)$ ok. What is the domain that we have, let us say some this is a computational domain, let us call it Ω ok.

Now, throughout FEM one of the sort of key features is that these basis functions or testing functions they are sub domain functions; means, they are non-zero only over some small region so, sub-domain basis and testing functions ok. So, in principle this integral is over the entire thing, but it does matter because b_m is non-zero only over some small region right. So, wait so, let us say that this is the region where this b_m is 0 so, we call this non-zero sorry this sigma m it. So, this is we called it the support of b m of r; support means, the region in space where it is non-zero outside it is 0 ok. So, in that first picture where we saw that aircraft which was sort of broken up into triangles, each triangle is like this one sigma m omega m that is the idea.

Student: (Refer Time: 14:22).

Will be composed of little-little such things ok; so now, we started we by saying that the earlier method was enforcing this delta testing it was enforcing the differential equation at a few discrete points. Now, what am I can we give a interpretation to this equation? So, this is the differential equation I want to enforce, so this is I can give view this as a residual if this is non-zero; that means, the differential equation is not being fully obeyed correct, and I can view this as a weight for this overall integral.

So, now, I am saying I am not insisting that this residual be 0 at every point in space, but in an average weighted sense over this domain Ω_m enforce it to 0 ok. This is very different from saying that $L\phi(r) = f(r)$ for every r that is too strong a condition. So, we are weakening it we are saying you can I will allow you too have it go to non-zero values in a average sense. For example, if $b_m(r)$ was 1 like a pulse function, then I am I saying the average residual is 0 that would be the meaning of this equation right, but in general $b_m(r)$ need not me just a pulse function can be something more complicated.

So, the reason I am giving a slightly different interpretation is because the FEM community it looks; it is like giving a interpretation to the method of moments equations which we had already derived. So, here when we have derived the method of moments equation over here, we did not have this weighted residual interpretation right, we just said ϕ is projected on basis b then I do testing along t_m that was my interpretation we were happy with it.

The finite element people they arrived at the same equation via slightly different route and their interpretation is weighted residual. The equations are same this is a interpretation given is different ok, but if you read papers and text books on the fem they will call it a weighted residual method; when you read that you should not get confused, you should realize; it is basically MOM Method of Moments right. The same thing is being called by different names and different frames; the concept is the same basis function, testing function that is all.

Student: (Refer Time: 16:54).

Yeah we are enforcing it to be 0 in a weighted sense.

Student: So, if we break Ω, if we not (Refer Time: 17:01).

It will be because b_m is 0 outside Ω_m, it does not matter you can as well call this omega it does not matter ok. So, now, your phi over here is coming from this expansion in the basis function so, how many unknowns in this equation?

Student: n.

n equations, how many n-equations and how many equations, how many equations is this?

Student: (Refer Time: 17:41).

I mean this is just one equation for a given choice of b_m.

Student: yeah.

This is one equation. So, the basic frame work of FEM is break up your whole domain into such you know.

Student: n (Refer Time: 17:55).

Such a exactly whatever dot dot dot right. So, repeat for m is equal to 1 to N and you get a system of equations, $Ax = c$ and where your x's are what? x is going to be a column vector of $\phi_1, \phi_2, \ldots, \phi_n$. That is my unknown every testing function gives me one equation and I have n such distinct functions; so, that is that is the sort of philosophy that we use.

How do we choose these basis functions? So, the finite element method needs that these basis functions they should have certain properties of continuity and differentiability which we will come to later ok. So, just to tell you that you cannot choose these; you cannot choose these b m's in some random fashion, they should have some nice mathematical properties ok, continuity and differentiability are some of the popular ones that these functions should have.

Now, so far I have been writing this is as a very abstract you know L operator kind of thing, but we will not want to see something more concrete. So, what could be an example of this L operator? So very simple example, Maxwell's equations the first two Maxwell's equations we know we can eliminate one variable like E and H I can eliminate H and just write it in terms of E right. So, for example, what was your $\nabla \times E$, $\nabla \times E = -j\omega\mu H(r)$ and μ is always function of r in general ok. Now, if I wanted to use the next equation what should I do?

Student: (Refer Time: 20:04).

If I take a curl over here on the left hand side can I get use the curl on the right hand side? No, because I know the $\nabla \times H$ I do not know $\nabla \times \mu H$ right.

Student: (Refer Time: 20:14).

So, what I can do is?

Student: (Refer Time: 20:17).

I can exactly, I will just $\nabla \times [\frac{1}{\mu(r)} \nabla \times E] = -j\omega \nabla \times H$; $\nabla \times H$ is?

Student: (Refer Time: 20:39).

$\nabla \times H = j\omega\varepsilon(r)E$; assume J is 0 its not much of a complication ok. So, let us write it actually need not be it is easy to deal with it right. So, this whole thing can be simplified as $\nabla \times [\frac{1}{\mu(r)} \nabla \times - k_0{}^2 \varepsilon(r)]E = 0$.

So, what you have now on the left hand side is actually your operator L right; so, operator $L = \nabla \times [\frac{1}{\mu(r)} \nabla \times - k_0{}^2 \varepsilon(r)]$, I can write it like this yeah.

Student: (Refer Time: 22:03).

So, this is an example of my L operator and ϕ is E ok, if I had a current term where do it go; this J term it is a non-zero where do it go it would be my f on the right hand side right because there would be no electric field or anything it would come on the right hand side. So, we have seen examples of non-zero on the right hand side, but the main difference that you can observe in this equation from what we already studied is what there is no integral equation in the operator; I mean there is no integral sign in the operation, it is purely means of what? Derivatives right

Student: (Refer Time: 22:51) 0 right.

Yeah it will come in the place of 0 on the right hand side yeah. In fact, yeah it would be $-j\omega J$ on the right hand side if current one also. So, this is sort to the sum it up this is your basic setup of finite of element method is that; first of all the basis functions are sub-domain that is a key point next your operator is a differential operator therefore, there is no Green's function involved in its solution. The moment I got it integral equation I need it I mean Green's function will appear there as a as a solution ok. And the sparse part is not clear from this, but we will come to it as we take up examples ok.

And so, this is the basic framework with which we will sort of go ahead and study the finite element method little more ok. So, subsequent modules we will look at what are the examples and cases for this $b_m(r)$, how will we arrive at a sparse system of equations, how do we solve it with 1 D and 2 D examples ok; any questions on this so far.

See notice that, in almost not almost in every single new method that we come across, we always just start from Maxwell's equations and either take it through a differential route or a integral route and then different methods come from them. Starting point all you need to know is Maxwell’s equations little bit of vector calculus.