Computational Electromagnetics
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Method of Moments
Lecture – 8.9
Volume Integral Equations: Solving - Part 1

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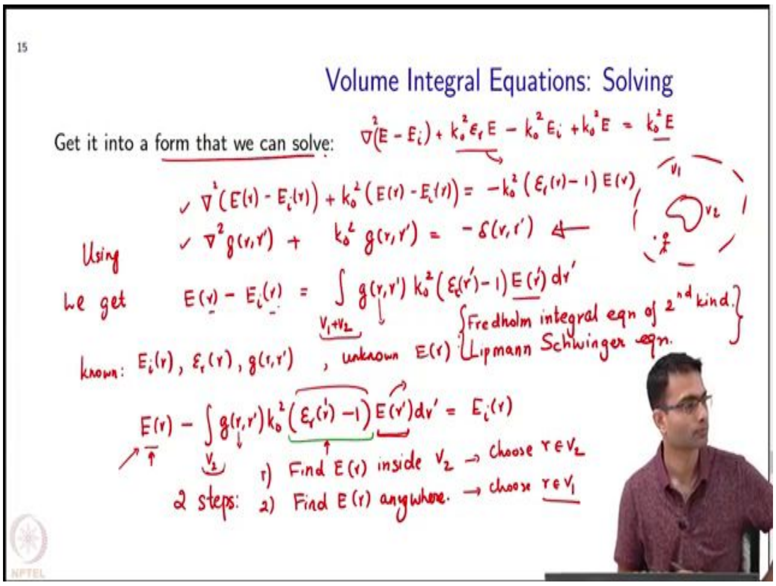

So, let us see how we can get it into the a form that we can solve right. So, let us just re write our equation over here. I had $\nabla^2(E - E_i) + k_0^2 \varepsilon_r E - k_0^2 E_i = 0$. Now the hint was that I want to get $E - E_i$ as the common variable both for ∇^2 and k_0^2 term that is the first constraint second constraint was what?

Student: ε_r.

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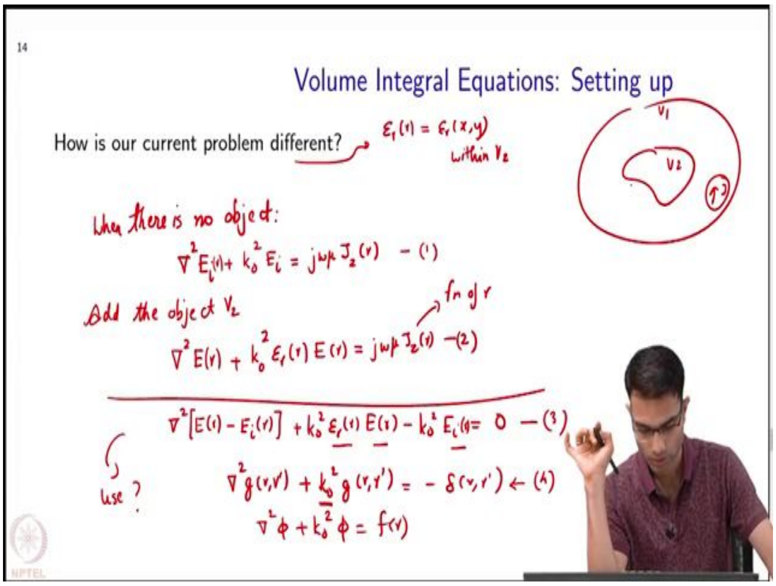

Right so, the second constraint was in this equation 4. This guy was what was it constant or a function of space.

Student: Constant.

Constant. So, these are the 2 things I have to sort of fit in, so, that I can use Green's function. So, what can I do over here? I want to get I want to remove this guy from the left hand side because it has a function of space ε_r is a function of space. So, what can I do, this move it to the right hand side I can do something like that, but if I move it to the right hand side I no longer have a E term to go along with k_0^2. So, very simply what I can do is let me add and subtract a common term. So, what if I add $k_0^2 E$ on both sides?

So, then this will become $\nabla^2(E - E_i) + k_0^2 \varepsilon_r E - k_0^2 E_i + k_0^2 E = k_0^2 E$. I can gather the k naught squared together. So, what will happen? I have $E(r) - E_i(r)$ right. So, I have got what I wanted. Now whatever is unwanted I should move to the other side right. So, the other side will be what. So, the other side this term will continue to be there and this guy moves to the right hand side.

So, I have a $k_0^2(\varepsilon_r(r) - 1)E(r)$.

$$\nabla^2(E(r) - E_i(r)) + k_0^2(E(r) - E_i(r)) = -k_0^2(\varepsilon_r(r) - 1)E(r)$$

So, now, this looks like a form that we can solve using our Green's function why simply because the operator is common $\nabla^2 + k_0^2$ has appeared on both sides.

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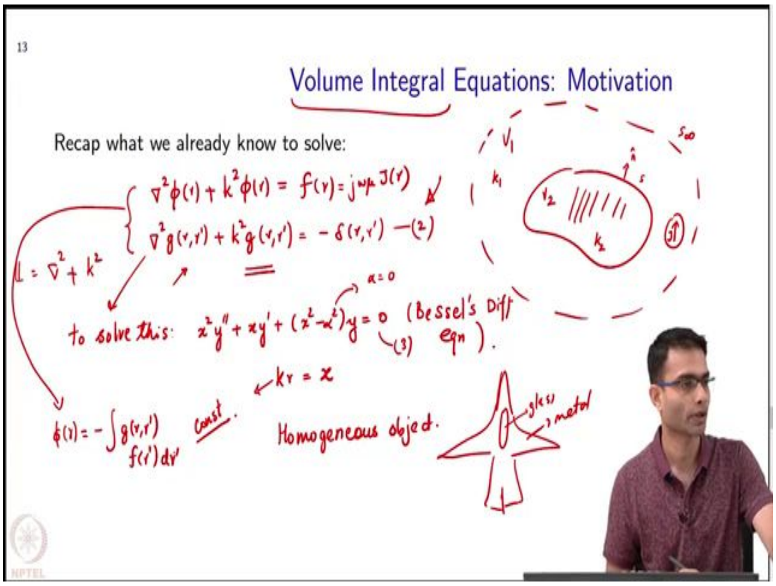

So, when we had yeah just wanted to re rewrite one thing over here. When I had these 2 equations, how did I write the solution? $\phi(r)$ was written as combining these 2 equations I can write ϕ in terms of f and g as the convolution right and that convolution was minus.

$$\phi(r) = \int g(r, r')f(r')dr'$$

So, now, we can make use of that for solving this equation right. So, using $\nabla^2 g(r, r') + k_0^2 g(r, r')$ we get. So, the variable has to be held the same. So, its $E(r) - E_i(r)$ right that is what is going to be equal to the convolution now of Green's function.

Student: Sir we do not know what E_i (Refer Time: 04:00).

We know what E_i is and the entire right hand side over here one of those minus signs got cancelled right. Is that fine? Let us this and what is the. So, the region the region of integration is going to be the full volume $V_1 + V_2$ that makes it more general.

Whereas so, this is what this is the expression that we get. Now looking at this expression does it have again does it have all the information about the problem inside it. So, let us write down what is known what is known in this problem? I know the incident field what else do I know?

Student: (Refer Time: 05:07).

Object shape, permittivity; I know which is all encoded into the relative permittivity right. I also know Green's function and now I can use the closed form Green's ex expression which I derived previously why because I am using this equation and I know the solution on this equation and what is unknown?

Student: Electric field.

Right the electric field this is what is unknown. Now looking at this what kind of a first of all is it an integral equation, yes or no? It is an integral equation why because the variable of a interest which is the unknown variable is it appearing inside the integral sign? Yes here it is, but which kind we have studied different kinds of integral equations what kind of differential equation integral equation is this?

Student: (Refer Time: 05:59).

Fredholm integral equation of the second kind. because the unknown is both inside and outside the integral sign right. So, we will just make a note of this. This is a Fredholm integral equation of second kind. This is what the math people call this equation if you study quantum mechanics and calculate scattering cross sections in quantum mechanics you get actually the exact same equation and the physicists the physics people call it by the name Lipmann Schwinger equation.

So, if you read a physics books you physics book you are likely to hear this word for it if you read a more double E or math book you will find Fredholm integral equation of second kind, but they both refer to the same object ok. So, let us now sort of massage this equation into a form that we can solve ok.

So, what I what you would do is? Standard technique gather all the unknowns on one side move as many knowns as possible to the right hand side right. So, turns out to be a very simple thing to do I just have to move the integral over here right. Now it's sort of convenient can I make the region of integration smaller than what I have shown right now? Right this was our V_2 this is our V_1.

So, I am integrating over $V_1 + V_2$ that is what I said the most general case can I make it just V_2 look at the integrand is the is the integrand non zero everywhere.

Student: (Refer Time: 08:00).

Look at this term right this term is non zero where.

Student: Only thing.

Only inside V_2. outside V_2 in the region V_1 its vacuum. So, what is relative permittivity of vacuum 1? So, this term this integral over here our life has become simple because automatically it is restricted only to the region of interest V_2. So.

Student: Its r'.

Its yes this is r' correct yeah. Now what am I finally interested in? I want to know the field somewhere outside here right that is where I want to find out the field. For example, this V_2 could represent an aircraft and I want to calculate its radar cross section or it could be some new Wi-Fi antenna I have designed I want to see how its radiation pattern of this right. So, I want to find the field everywhere. So, if I plug in a value of r belonging to V_1, can I what is the problem with this equation?

So, supposing I put r is equal to some point inside V_1 here. So, I will get an expression for $E(r) = E_i +$ this term over here. In this term over here I do not know one quantity which is I do not know the electric field inside V_2. So, I cannot actually use this equation right now. So, what is the first step I should do? If I know the field inside V_2 if I know E inside V_2 am I done yes, if I know E inside V_2 then the integral I know; E_i I know therefore, I know E everywhere.

So, we have already come across this trick earlier integral equations always solved in 2 steps. So, the 2 steps are in this case the first step is find $E(r)$ inside V_2, this was like our electrostatic example we had of the charge on the rod first find the charge then find the potential everywhere you cannot directly find the potential everywhere. So, first find E inside V_2 and then find $E(r)$ anywhere.

So, first step 1 I have to somehow get into formulating a system of equations and we have seen the framework for it and that framework is integral equations what is the framework for solving integral equations, we have we are talking about in this module? Method of moments. So, method of moments is what will use to solve this integral equation.

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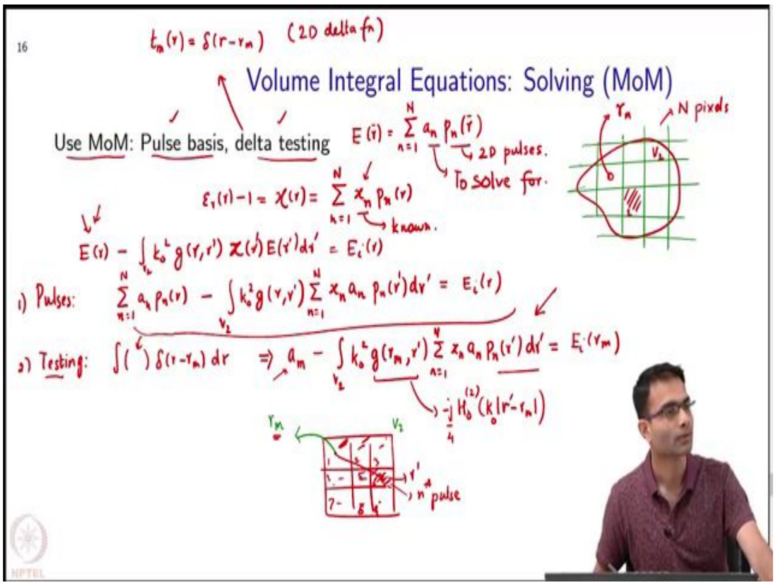

So, we will use the simplest method of moments that we know. So, far which is pulse basis delta testing that is what we are going to do except that now I have a 2 dimensional volume in which I have to find out pulse basis functions. In the surface integral formulation I had a line and I had pulse functions on that line. So, now, we are sort of becoming a little bit more general.

Student: Sir.

Yeah.

Student: So, how is it?

So we will let us let us look at little bit systematically in the first step, I will choose r belonging to V_2 for in this equation I will substitute r belonging to V_2 various points and solve it which we will discuss how to solve. At the end of it I know $E(r)$ completely inside V_2.

Now when I come to here and I choose r belonging to V_1 then you see this the domain of integration remains V_2 over here. So, I still need $E(r)$ inside V_2 even though I am putting r belonging to V_1 r belonging to V_1 goes in over here and in this r everything else remains the same that is going to be 0 outside V_2 because see the variable of integration.

Student: Integrating over V_2 right.

Yeah only over V_2.

Student: So, its not going (Refer Time: 12:16).

Inside V_2 this term does not go to 0 you are right this that is why we have made the region of integration only V_2 ok. So, that is our.

Student: (Refer Time: 12:17) it was initially we have to (Refer Time: 12:30).

It was in the originally the region of integration was $V_1 + V_2$ then we realise we do not need to also include V_1 because that integrand is itself go into 0 outside V_2 right. So, we made it a little bit simpler. Now coming back to our how we will actually solve this using the method of moments. So, pulse basis and delta testing. So, here is my object now I am just going to show you V_2 and now as we have discussed we are discretising the volume. So, if I am discretising the volume I need to sort of use a make a grid like this; The grid is not going to be exactly capturing the object ok.

So, this is just a cartoon representation as I make the size of the pixels now these are called pixels as I make the size of the pixel smaller and smaller better is the approximation of the surface. Later on you can go become fancy and choose non rectangular units you can make triangles you can make whatever you want, but the basic idea let us try to understand. So, having shown you the grid over here pulse basis means whatever is the variable of interest is expressed in the pulse basis function. So, what is the variable electric field?

Student: Inside.

Inside the inside V_2 right. So, I will write $E(r)$. So, whatever I said I have divided this domain into N capital N pixels and this pulse basis function is 1 in the n th pulse and 0 everywhere else right. So, if I take for example, 1 if I take this to be the n th pulse. So, this I mean I am being a little lazy in just writing r actually it is the $\vec{r}$ right its a 2 dimensional problems.

So, its a $\vec{r}$ so, this $p_n(r)$ is 1 over here and 0 everywhere else right. So, its like a its a column of height 1, 0 everywhere else these are my 2D pulse basis function. So, remember that these are 2D pulses. So, the unknown was electric field now the new unknown is.

Student: a_n s.

a_n s right. So, I have to solve for these to solve for. Now if you look back at this equation over here, this is what I am discretising as pulses there is this term also over here which is going to appear everywhere. So, may as well discretise that also into pulses ok, it will become because when I do the integration the integration will split over all the pulses right.

So, I am going to write this $\varepsilon_r(r) - 1$ by the way this it gets irritating to write this every time. So, there is a new variable that you use for it you just call it $\chi(r) = \varepsilon_r(r) - 1$ pardon me.

Student: It is actually.

Its.

Student: Susceptibility.

This is not while yeah this is you can call it susceptibility yeah which is going to be also written in $\chi(r) = \sum_{n=1}^{N} \chi_n p_n(r)$. So, this is just a discrete representation of permittivity just remember that χ_n is known it is not unknown its given to me in the problem the permittivity of the object is given to me ok. So, let us.

Student: (Refer Time: 16:28).

It will amount to one and the same thing, but the fact is you know χ_n. So, there is no point in combining χ_n and a_n into 1 because you know part of it and you do not know another part. So, we will just keep it separate. So, let us just look at our equation once again. So, I had

$$E(r) - \int_{V_2} k_0^2 g(r, r')\chi(r')E(r')dr' = E_i(r)$$

That was our equation ok.

So, now, the first step is to put the pulses in right.

$$\sum_{n=1}^{N} a_n p_n(r) - \int_{V_2} k_0^2 g(r, r') \sum_{n=1}^{N} \chi_n a_n p_n(r')dr' = E_i(r)$$

So, far so, we have taken care of the first part the next part is delta testing. So, when I say testing, what do I am taking delta functions for the testing. So, what am I doing? I am multi taking the inner product of both sides with the testing function, the testing function is in this case $\delta(r - r_m)$ let us say.

Where m let us say is the centre of one of these pulses. So, we can call this as r_m. So, in testing what will I do I will take this left hand side and multiplied by $\delta(r - r_m)$ and integrate over dr. So, the question is this $E(r)$ over here is it $E(r)$ or $E(r')$ is that a question it is going to be.

Student: But that is now you are measuring V_2 right.

This is being integrated in V_2 yes, but its outside the integral sign right. When we derived this let us go back over here we started with this equation and use the Green's function to solve it. So, when I took the difference and integrated over here r is constant for this integral. So, this is r and this is r ok. So, that is why this is continues to be r and not r' inside the integration this is r'. Getting back to the testing function, I have to take the apply this testing to both the left hand side and the right hand side right. So, this whole expression I can put inside over here that is the meaning of doing testing. So, what will that become?

So, take the $\delta(r - r_m)$ this is my testing function the m th testing function ok. So, I am taking one testing function which is an impulse located at r_m first term over here what happens to the first term.

Student: This delta function also 2D right.

Its a 2D delta function what happens to the first term let us go step by step I was summation over N pulses which pulse survives?

Student: r_m.

The mth pulse will survive what is the magnitude?

Student: a_m.

a_m right the summation goes away I am left with a_m because, the delta function annihilates all, but 1 pulse. Going in further this is where we have to do it a little bit carefully what will happen over here. So, will the summation remain or will it go away. So, the question is in for which coordinates is this delta function defined primed or un primed?

Student: Unprimed.

Un primed coordinates; inside the integral which is the quantity which is the function of un primed?

Student: (Refer Time: 20:52).

The summation and all are all over.

Student: Primed.

Primed coordinates. So, what is the unprimed quantity here?

Student: g.

Only g. So, what will happen to this integral? The integral will remain because this is integral over primed coordinates the function which is of un primed coordinates is only one g. So, what will happened to that g?

Student: $g(r_m)$.

$g(r_m)$ exactly and the integral and the summation will they go away no. So, I will get $\int_{V_2} k_0^2 g(r_m, r') \sum_{n=1}^{N} \chi_n a_n p_n(r') dr' = E_i(r_m)$ excellent. So, this is what pulse basis and delta testing has done to our equation. Can I solve this equation? It looks a little complicated, but in principle its clear there are knowns and there are unknowns I can solve this equation over here ok. So, let us see there are sort of 2 possibilities over here. Remember that g over here has a its a Green's function, it blows up at $r' = r_m$ right. So, that something that we have to be careful about, but before we get into those details is let us be conceptually very clear what is happening.

So, I will just take a 3 by 3 case. So, let us call this to be this is my r_m ok. So, what is this integral doing this is an integral over entire V_2 right this whole thing is my V_2 and what am I doing? I am running over this entire region keeping r_m constant. So, for example, r' first let us say belongs to this let us say this is the nth pulse. So, then $g(r, r')$ what was this function this was $(-j/4)H_0^{(2)}(k|r - r'|)$ right that was the definition of Green's function take the 2 points and take the distance between them that is what it was. So, then what remains? So, when I am doing this integration I am r_m is being held constant this distance is what is varying over where over this n th pulse correct.

So, I am holding r_m constant and integrating over this then this then this over all of these pulses 1 by 1 that is the meaning of the left hand side the second term over here. When I integrate over this pulse what is the coefficient involved? Unknown coefficient involved is. So, if this is pulse 1 2 3 4 5 6 7 8 9. So, when I do the integral over here which of the a s will appear?

Student: a_1.

a_1 right when I come here?

Student: a_2.

$a_2, a_3, a_4, \ldots, a_9$. So, this term over here that I have written is since its a summation I am going to get a_1 something plus a_2 something all the way up to a_9 something right and which term will come over here? Here r_m is chosen to be r_1. So, this will just be.

Student: a_1.

a_1. So, I will have 1 equation how many variables?

Student: 9.

9 variables a_1 to a_9 are my variables. I have got 1 equation 9 variables. So, the procedure would be repeat for all 9, I get a 9 by 9 equation to solve for ok. So, that is the strategy that we will follow is it clear? What is happening over here?