Computational Electromagnetics
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Method of Moments
Lecture – 8.5
Surface Integral Equations: Evaluating the Integrals – Part 1

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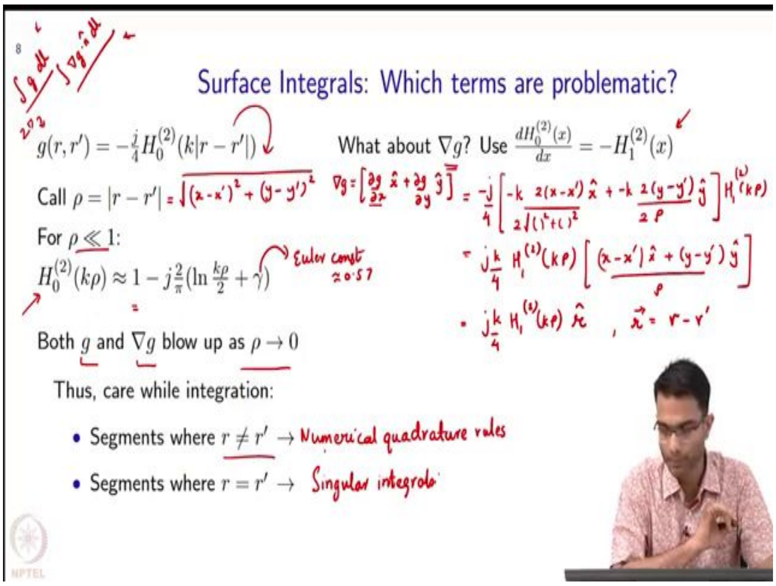

$$-(j/4)[-k(\tfrac{2(x-x')}{2\rho})\hat{x} - k(\tfrac{2(y-y')}{2\rho})\hat{y}]H_1^{(2)}(k\rho)$$

Now, let us go ahead let us look let us take a little deeper look at these integrals that I have to calculate, remember the 2 integrals I had to worry about was $\int g dl$ and $\int \nabla g.\hat{n}$ yeah. These are the two integrals that I am always thinking about over each segment ok. So, we spent an entire module talking about the Green's function and we came up with the 2D Green's function as here right. So, it is $-(j/4)H_0^{(2)}$ right.

So, this one everyone remembers the one term that we did not calculate so, far is what is ∇g ok? So, just to make the notation a little bit easier this $|r-r'|$ that appears everywhere it is a scalar is just a distance is the distance between 2 vectors r and r'. So, I will just call it ρ and $\rho = \sqrt{(x-x')^2 + (y-y')^2}$. Now to get at ∇g I need to take the derivative of g with respect to I can do it in polar coordinates or I can do it in Cartesian coordinates. In this particular case it turns out that using Cartesian coordinates gives us a simpler answer.

So, here is again piece of information fact its given to you that the derivative of a $H_0^{(2)}$ gives you $H_1^{(2)}$ this is sort of given to you right there are many many properties of Bessel functions which we will use as we go and this is one of them. So, now, if I want to evaluate ∇g what do I do? I will just use this property right. So, what is ∇g now?

So, I can so, $\nabla g = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y}$ using the definition right simple.

$$\nabla g = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} =-(j/4)[-k(\frac{2(x-x')}{2\rho})\hat{x} - k(\frac{2(y-y')}{2\rho})\hat{y}]H_1^{(2)}(k\rho)$$

$$\Rightarrow \nabla g = (jk/4)[\frac{(x-x')\hat{x}+(y-y')\hat{y}}{\rho}]H_1^{(2)}(k\rho)$$

So, what does this vector look like? So, I have $[\frac{(x-x')\hat{x}+(y-y')\hat{y}}{\rho}]$. What is the norm of this vector length of this vector? It is one; it is a unit vector right and what is the direction of this vector, $r-r'$?

So, very simply this just becomes $\nabla g = (jk/4)H_1^{(2)}(k\rho)\hat{\zeta}$ and $\zeta = r - r'$ (script r in the video). So, this just simplifies the notation right. So, not I mean we started with $g \, \alpha \, H_0^{(2)}$ and I got $\nabla g \, \alpha \, H_1^{(2)}$ not very complicated right ok.

Now, when we are evaluating these two integrals over here as long as g and $\nabla g.\hat{n}$ as long as these terms are finite there is no problem I can use gauss quadrature rule numerical integration. The trouble will happen when these quantities begin to well we know that Green's functions have a singularity at the origin; origin in this case means when $r = r'$. That is the singularity.

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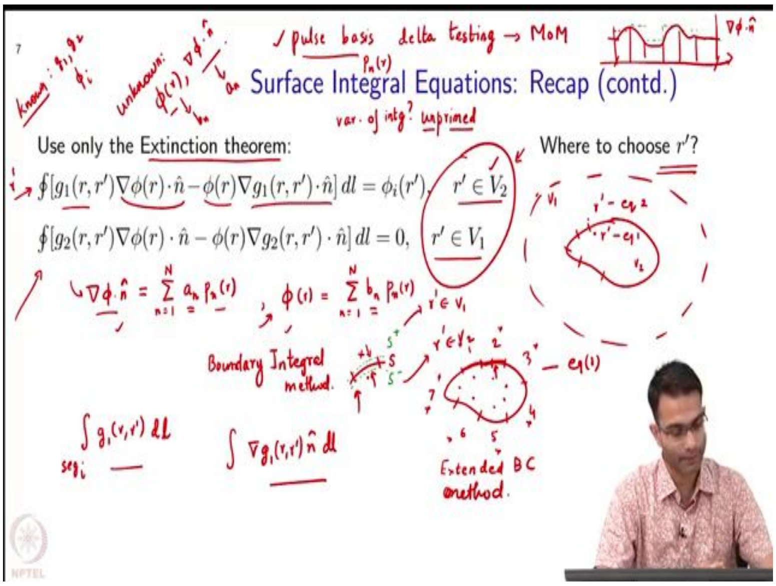

Now, when I am doing this if we go back to this when I am doing this integration over here what am I holding fixed? I am keeping fixed r' is fixed right and where is r going over the looping over the entire boundary. r' is also on the boundary r' is looping over the boundary. So, there will be at least one segment where r is going to be equal to r' right. So, at those points what will happen? So, this is for example, how the Hankel function looks like for very small ρ and this is something you have already seen.

So, it has a real part which is well behaved and as an imaginary part that looks like $log(\rho)$ and what is the value of $log(\rho)$ as $\rho \to 0$? $-\infty$ right. So, this is going to be the problematic integral over here. So, this was for $H_0^{(2)}$. Do you think $H_1^{(2)}$ will be any better behaved? no $H_1^{(2)}$ also has a singularity sitting at the origin ok.

Student: (Refer Time: 06:47).

Right so, both g and ∇g they blow up as $\rho \to 0$ right. So, therefore, in our integration strategy we have to use 2 different techniques. So, when $r \neq r'$ that is the happy case there is nothing to worry about that I simply use your numerical quadrature.

Student: (Refer Time: 07:19).

Right so, this γ over here is a constant its a Euler constant some roughly 0.57 or something ok. So, where do I get these properties from? I get these properties from the handbook of Bessel functions this is extensively documented all these special functions have very well documented properties graphs and all I will include a reference to it right. So, when r is not equal to r' I will use numerical quadrature rules no problem segments where $r = r'$ those are what are called singular integrals.

These have to be done carefully and we will look at these next how to evaluate these integrals those of you who have done a course on complex analysis will find some of this familiar if not it does not matter ok. So, this is what we will look at, but is the general idea clear. Always keep in mind that our basic motivation is to solve these two equations in the process of solving these two equations some terms are easy some terms are not easy. Let us take one more step back before we go into more details once we have solved these equations I want to find out finally, the field anywhere in space that is my objective.

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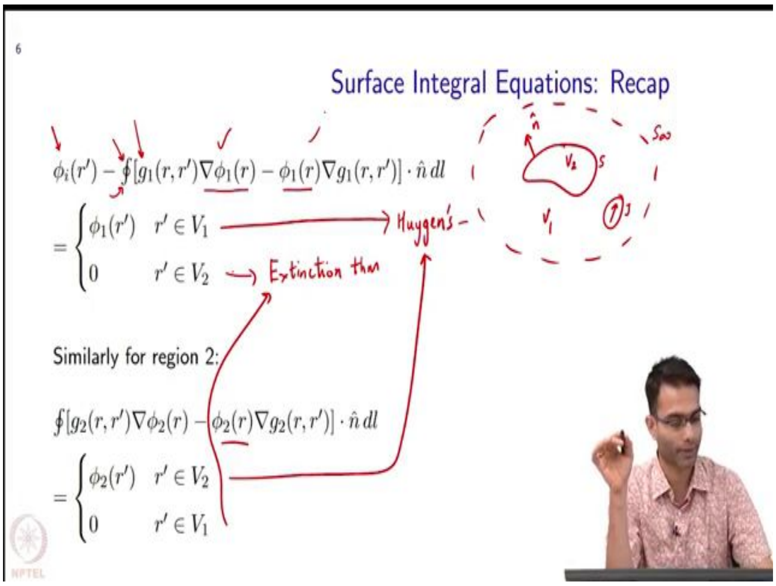

Then I go back even 1 more slide I use Huygens principle to put back these terms which I have evaluated and I can find the field everywhere. So, do not lose sight of the overall picture right we are zooming in and in more and more into details, but remember that is what we want to get at. So, solving the equations and these are the problematic terms.

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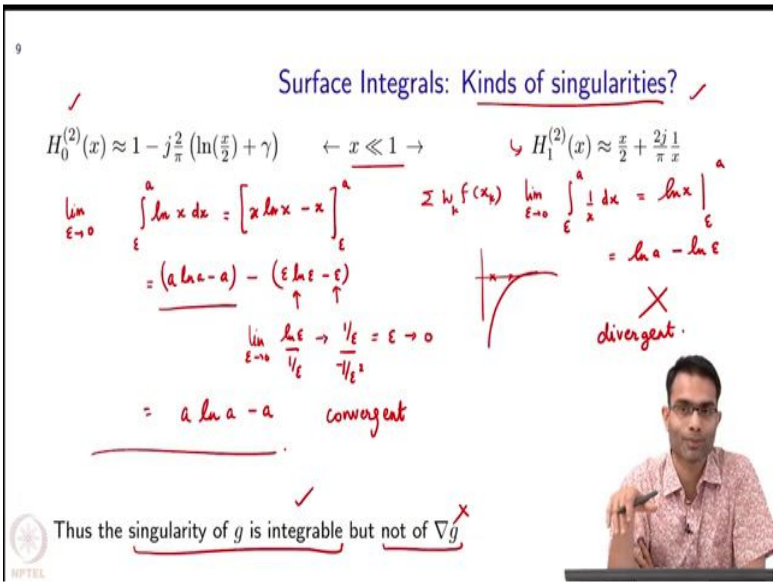

Now, let us look a little bit more. So, what are the 2 kinds of singularities that we have as x becomes very small $H_0^{(2)}$ I have told you looks like this $log(x)$ and $H_1^{(2)}$ also has an asymptotic form over here which goes the problematic term is 1/x right. So, as x tends to 0 1/x also blows up right. Now you can see that these both these functions they tend to infinity in slightly different ways right, they tend to infinity in slightly different ways.

So, let us let us let us look at how this manifests mathematically. So, imagine that I have an $\int_{\varepsilon}^{a} log(x)dx$ and I am going to eventually take the limit $\varepsilon \to 0$ ok. Now is this before we evaluate this what do you think I have a I have a singularity at the origin and I am including it in the integration I am taking the limit $\varepsilon \to 0$.

What do you expect should happen? It should blow up that is what intuitively you would think and for this case what do I have limit $\varepsilon \to 0$ $\int_{\varepsilon}^{a}(1/x)dx$ ok. If these integrals exist trivially then there is no problem I can go back to my original procedure evaluate the integral over all segments get my system of equations and I am there. But before we do that let us zoom in and see is it possible. So, what is the integral of $log(x)$?

$xlog(x) - x$ ok, evaluate it at ε and a right, if you want to check just take a derivative you will get $(x/x) + log(x) - 1 = log(x)$. So, this is going to give me $alog(a) - a - \varepsilon log(\varepsilon) + \varepsilon$.

So, $\varepsilon \to 0$ $alog(a) - a - \varepsilon$ there is no problem what about $-\varepsilon log(\varepsilon)$ how do I evaluate this? Lhopitals rule right. So, I will do $lim_{\varepsilon \to 0} \varepsilon log(\varepsilon) = lim_{\varepsilon \to 0}(log(\varepsilon)/(1/\varepsilon))$. So now, it is in the what form?

Student: (Refer Time: 11:51).

Well in this case infinity by infinity from I can do it in either way right. So, this will become $lim_{\varepsilon \to 0}[(1/\varepsilon)/(1/\varepsilon^2)] = 0$.

So, I integrated a function which had a singularity and what did I get I got only this guy. $alog(a) - a$. So, this singularity is what is called integrable all right. So, the singularity of g is integrable. So, when I go back into my evaluating those method of moments terms wherever there was a g term there is no problem even if the argument $\rho \to 0$ because you can see the integral is finite even though the function is blowing up its like a delta function; delta function is blowing up, but its integral is finite

So, that is a singularity, but say integratable singularity right. Now on the other hand over here I have this other this second term over here which is blowing up as 1/x. Now when I go to evaluate this what do I get $[log(x)]_{\varepsilon}^{a}$ ok, gives equal to $log(a) - log(\varepsilon)$ does this limit exist? This limit does not exist right. So, this term over here blows up. So, I have a function a singularity which is going as $log(x)$ what is blowing up at the origin even its integral does not go ok.

So, it is like integrating a double delta function, I can integrate it once I mean like if I integrate it once I do not get anything finite, but if I integrate it twice. So, when I integrate it with this once I got $log(x)$ if I integrate this once more I will get something finite integral of $log(x)$ is finite. So, this singularity over here is in certain some sense a stronger singularity then $log(x)$ right. So, this integral I will call it a convergent integral and this is a divergent integral. So, that is what we have to do. So, the singularity of grad g is not integrable this is integral. So, we have identified we are zoomed in zoomed in and found out that the main problematic character is this $H_1^{(2)}$ any questions about this?

Student: What if we approximate this by series (Refer Time: 14:28).

What if we approximate this by a larger series right. So, if you approximate it by a larger series you will get more and more terms right. So, you will higher powers of x you will get, but the point is that you cannot ignore the first term; the first term will always be there right. So, each of those terms should converge to a finite number and that will not happen right. So, if the first term itself does not converge there is no point going for that now its not going to converge.

Yes. So, the question is that if I integrate this $H_1^{(2)}$ using a quadrature rule will I have the same problem answer is yes because what you are trying to do is in some sense you know you have some integral like this and you know that you will take the value of the function at some finite number of points. If you use the quadrature rule you will get some answer because its numbers you will put you know what is the quadrature rule saying summation $\sum_k w_k f(x_k)$ as long as you chose $x_k \neq 0$ right.

You chose it to be some finite number you will get some answer to this, but theoretically this integral is divergent. So, that is actually even scarier because you will get some answer for it and you think it exists, but under underneath the hood it does not exist there is a special technique of modifying quadrature rules when you know that there is a singularity inside it is called singularity extraction ok. But, as it is if you use a quadrature rule you will get a misleading answer because, theoretically you can see that this integral is divergent you should not you, if you get an answer you should know that it should be suspicious.