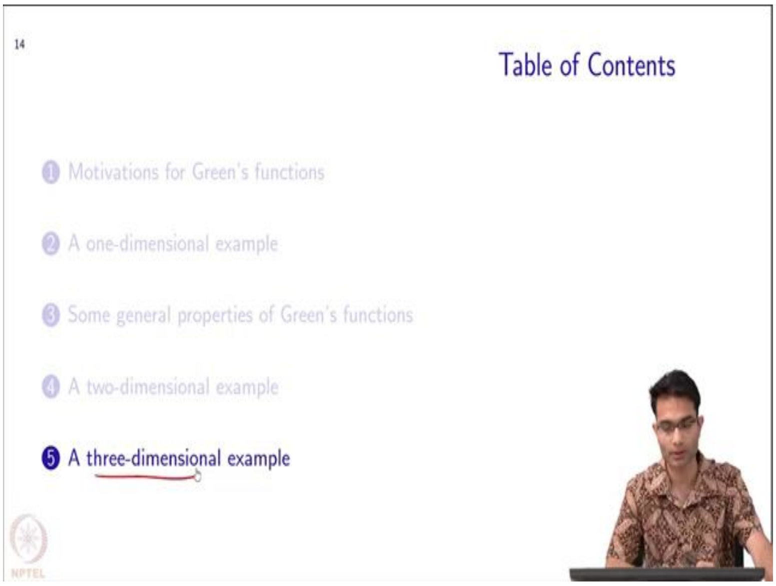
Computational Electromagnetics
Prof. Uday Khankhoje
Department of Electrical Engineering
Indian Institute of Technology, Madras

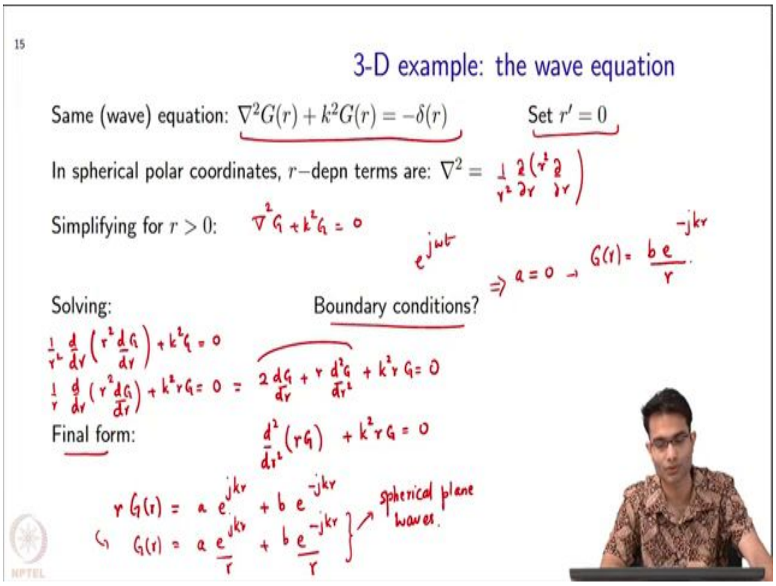
Introduction to greens Function
Lecture – 7.8
3-D Example

(Refer Slide Time: 00:14)



Alright, so now let us move to a 3-D example ok. Now, the 3-D example as it turns out will actually be easier to the math will be easier.

(Refer Slide Time: 00:22)



So, let us see what it is like ok. So, the wave equation remains exactly the same. The only thing to note now is that it's 3-D. So, of these one, two and three terms which terms will be different from before? Let us go first term in the first term be different, while it will be different because if I choose a coordinate system now what coordinate system should I choose for a 3-D problem?

Student: (Refer Time: 00:51).

3-D spherical coordinates that is the most natural because if I put my like if I do the same choice write it; if I put the point source the impulse at the origin then there is perfect spherical symmetry for the problem right. So then the rather than cylindrical coordinates spherical coordinates is going to be the most intuitive coordinate system.

So, ∇^2 will change right earlier it was a polar 2-D, now for 3-D spherical right. So, yeah I do not expect you to remember the form of ∇^2. So, ∇^2 has a lot of terms for θ dependence and ϕ dependence, we will just keep the r dependent terms and that is $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$. So, this we just look up and use it that is the operator right.

The second term remains as is; what about the third term? Third term we just have to take care to know to keep in mind that it is a 3-D delta function ok, earlier it was a 2-D delta function and even before that it was a 1-D delta function ok so, that is just one thing that we have to keep in mind. And we will just sort of repeat the recipe that we had previously we will choose $r > r_0$ such that what happens is that I get a $\nabla^2 G + k^2 G = 0$ ok. So, first I will solve this and then look at boundary conditions ok.

So, when I go to sort of solve this, let us put this in over here alright. So, these partial derivatives go away they become total derivatives because there is no θ, ϕ dependence, I have put the my impulse at the origin right. So, let us see so what I can do is. So, this is $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial G}{\partial r}) + k^2 G = 0$ ok; one small trick I will use I will just multiply r everywhere ok.

Now, I can the only thing that I can sort of simplify in this is the first term right. There is a derivative of something so I can open up this. So, two functions product of two functions and derivative right; so the first term will be so I can take the derivative of r square right. So, I will get $2\frac{dG}{dr}$ as the first term. Then.

Yes $r\frac{d^2G}{dr^2}$ and the second term remains as $k^2 rG$ is equal to 0 alright. Now any ideas on how to further simplify this. So, the hint is this that the we had looking at there are second order derivatives that is the maximum order of derivative, you see a two term over here. So, when you see a two term and the second derivative what kind of a trick could you play, could you write this whole expression as the second order derivative of something of some function. I want to try to write it like this $\frac{d^2}{dr^2}(.) + k^2 rG = 0$.

What might that something be, actually I made one mistake here there should be r yeah that one r got cancelled of yeah definitely a G has to be there.

What about simply rG? So, this will have G; $\frac{d^2(rG)}{dr^2} = r\frac{d^2G}{dr^2} + 2\frac{dG}{dr} + G\frac{d^2r}{dr^2} = r\frac{d^2G}{dr^2} + 2\frac{dG}{dr}$

Fine yeah. So, this manipulation has given me a equation that looks actually very familiar $\frac{d^2(rG)}{dr^2} + k^2 rG = 0$, what does this look like yeah even simpler than Helmholtz equation.

Student: Its.

It is the simple harmonic oscillator model right, I have second derivative of some function plus k square this thing. So, rG is equal to sin k kind of a or cos k that kind of a thing works the most general way of writing oscillatory function is so sin cos what can I generalize that too?

Student: e to the j.

e to the j right. So, I can write down in the final form over here $rG(r) = ae^{jkr} + be^{-jkr}$. So, how many solutions are possible independent solutions?

Student: Two.

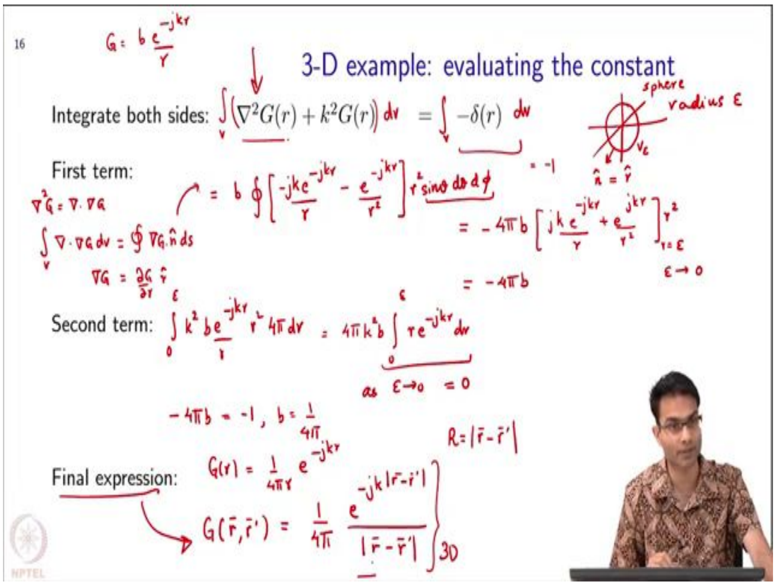
Two; So, I can this write this take the r over here and then I have a $G(r) = a(e^{jkr}/r) + b(e^{-jkr}/r)$ right. So, these kinds of functions are what are the called they are spherical plane waves unlike 1-D plane waves the amplitude of these functions it decays as you go away from at larger and larger r, which is physically what you expect, if I place a impulse at the origin if I go away amplitude should decay right.

So, again now we know what tricks to play now, right we can ask of these two terms can I eliminate at least one term by boundary conditions alright. So, the trick is the time convention. So, our time convention as before is $e^{j\omega t}$. So, which term can I get rid off?

Student: a term.

a term right, because a term will correspond to a incoming wave. So, a=0 and so the final form that I have is $G(r) = be^{-jkr}/r$ ok. Now what remains is the constant b ok, by now you know the game.

(Refer Slide Time: 08:48)



So, we take this differential equation and do a volume integral on both sides ok; even though I am putting one integral sign I do mean a three dimensional integration. So, what happens let us look at the first term this is the first term. I can play the same trick that I had played previously right. So, I should try to write this as the divergence of something right. So, I can write $\nabla^2 G = \nabla.(\nabla G)$ right. And then I have a over volume dV right so this will become what outward flux of ∇G.

Right; so this will become $\oint \nabla G.\hat{n}dS$ and what is my sort of problem set up? This is my x y z and I have a sphere right V_ε right, it is a sphere of radius ε, what is the outward normal ?

Student: r.

Which is equal to $\hat{r}$, because it centred at the origin right so, $\hat{n} = \hat{r}$. ∇G again this is only a r dependent problem so, this is only going to give me $\frac{\partial G}{\partial r}\hat{r}$. So, what do I get from the first term over here, I get. So, my constant b will be there over the surface ∇G right. So, what was my G? I will just write it over here, $G(r) = be^{-jkr}/r$ ok. So, I have to take the derivative of this guy with respect to r. So, first term will be it is a $\frac{\partial G}{\partial r} =- b(jke^{-jkr}/r + e^{-jkr}/r^2)$ What will dS be; what is d s surface?

Student: (Refer Time: 11:15).

r no, we have your not $rdrd\theta$ that is 2-D and 3-D?

$r^2 sin\theta d\theta d\phi$ that is the surface differential element; on the surface of the sphere that is the differential surface element ok. It is easy here because there is no θ, ϕ dependence. So, this integral over here is going to give me.

Student: 4π.

4π is what I am going to get out of it right, everything else inside the integrand is constant as a function of $d\theta d\phi$ alright. So, it is all going to just pop out over here. So, what will it pop out to? So, there will be there is a minus sign on both, both these terms of a minus sign so, I can take that out so I will put a minus I will get a $-4\pi b[r^2(jke^{-jkr}/r + e^{-jkr}/r^2)]_{r=\varepsilon}$ Which of these terms will survive as I put if I take $\varepsilon \to 0$? which of these terms will survive?

Student: Second term.

Only the second term because r^2 cancels of as $r = \varepsilon, \varepsilon \to 0$, what happens the first term goes to 0 second term survives and what do I get?

$-4\pi b$ very good so, that was the first term. What about the second term? Second term we will have to do this integral right so it is going to be $\int_0^{\varepsilon} k^2 b \frac{e^{-jkr}}{r} r^2 4\pi dr = 4\pi k^2 b \int_0^{\varepsilon} re^{-jkr} dr$ So, I am left with this and integration from 0 to epsilon is that clear. So, I can take out all of these constants 4 pi k squared d 0 to epsilon r e to the minus j k r d r so that is all that is survives.

So, you can work this out, this is a very simple integration to do. Like before I am not going to actually do it. It is very simple integral for you to do what do you intuitively expect this answer to be?

Student: 0.

0 right; how can you see this intuitively without actually doing the integration?

Student: Symmetry.

Hum, Symmetry is not going to help you here any other argument.

Student: Mod of the integrand is.

Yeah the mod of the integrand is.

Student: This is going to be r and a (Refer Time: 15:00) integrand from 0 to epsilon, but epsilon will be 0 (Refer Time: 15:03).

Yeah another we have seeing it is that the integrand is something which is, is it always finite? Is always finite and I am integrating over a volume. So, I am going to get some finite quantity multiplied by the volume at best right. And that volume is shrinking to 0, now if there was a delta function in here then I have to pay a little bit more attention, like if it were a log or something then I have to pay a little bit more attention, but here the integrand is finite integrating over a finite volume which I will set to 0.

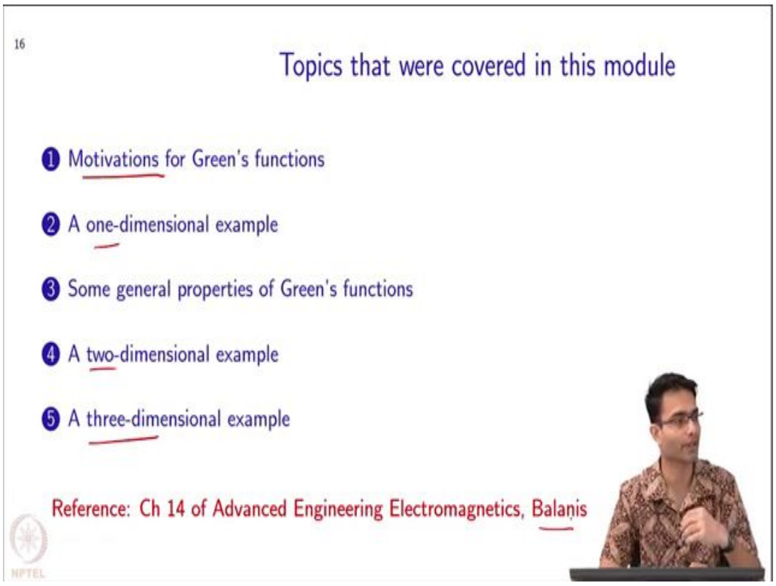
So, these kinds of tricks you should use and just say that as epsilon tends to 0 this is equal to 0 ok and you can manually verify by doing this integration right. And the third term is going to give us so 3-D delta function integrated over that volume I should get minus 1 ok. So, the final expression that I will get is $-4\pi b =-1 \Rightarrow b = 1/4\pi$. So, $G(r) = (1/4\pi r)e^{-jkr}$ that is it.

So, once again we will this was when I place the impulse at the origin, but if I want to write it in general what do I do? I will ask what is $G(r, r')$ so, now I should write in a little bit more careful way. So, I have a $G(r, r') = (1/4\pi)\frac{e^{-jk|r-r'|}}{|r-r'|}$; because I am interested only in the distance r here between the two.

So, in many books you will find that instead of writing it this was slightly cumbersome to write. So, when people use the shortcut that R; capital R is defined to be $|r - r'|$. So, you will find e^{-jkR}/R this is the final expression ok. So, you saw we started with the 1-D case of the string, which was which gave us a nice continuous function very easy to handle 2-D case there was a little bit more difficulty, because of the integration had the Green's function had a log term.

So, integrating it where to do it carefully and we came across a wave kind of function; which we had previously not seen before with a Hankel function or the bessel function. When we come to 3-D it got even simpler there was nothing fancy in the integration right and I got this spherical 3-D wave ok. So, if someone ask you what is the plane wave in 3D, do not write $e^{-j(kr-\omega t)}$ that is technically not right because it is goes up to infinity right. A 3-D plane wave is usually referred to as a spherical plane wave, a 2-D plane wave is a Hankel function. Now, 1-D plane wave is what we have always studied, $e^{-j(kr-\omega t)}$ alright.

(Refer Slide Time: 18:29)



So, what we have done is we started with some kind of basic motivation and went on to 1-D,2-D and 3-D. Chapter 14 of Balanis book gives a very nice introduction to Green's function in all three-dimensions so, that can we can take that as a reading material.

Student: The NPTEL (Refer Time: 18:49).

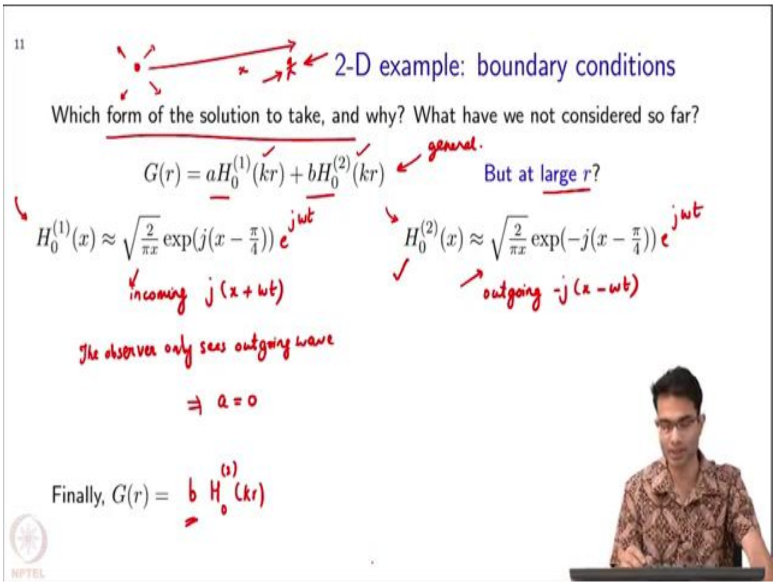
Why is there at; ok so.

Student: The functional form itself Changes.

Yeah, so ok why it; we can just spend some more time on this ok. So, the question is that when we look at 1-D, 2-D, 3-D why is there a different form appearing each time and mathematically you know the reason the reason is each time this guy over here, the del squared guy is what is changing its form, earlier supposing I had a 1-D plane wave, then it was just second derivative with respect to x.

Now, I have this $1/r, 1/r^2$ all of these different terms are coming. So, that is the changing the mathematical form of the equation itself right and that is what changing the final result ok. And as far as the physical intuition if you want for it then it is the way in which energy has to be distributed in multiple dimensions, which depends on how many dimensions there are. So, it is a wave it decays of to 0 as you go away from it its oscillatory, but how to manage this energy distribution in various dimensions gives rise to slightly different functions ok.

(Refer Slide Time: 20:18)

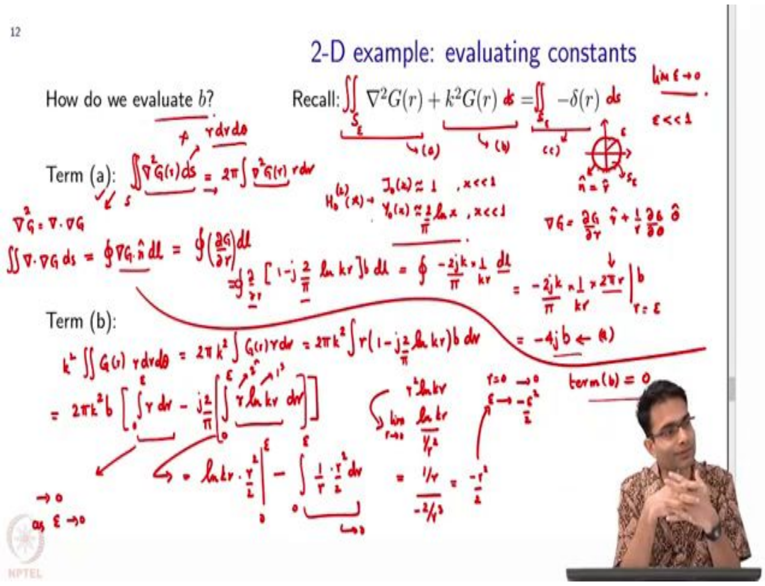


Even in the case of the 2-D so look at the behaviour far away right. So, far away from the origin you can see I mean the its there is a 1/r dependence over here ok. In the 2D case what was the dependence if you go let us go back over here all the way right, as I went far away from the origin these are the far field expressions what is the dependence like?

Student: 1 by.

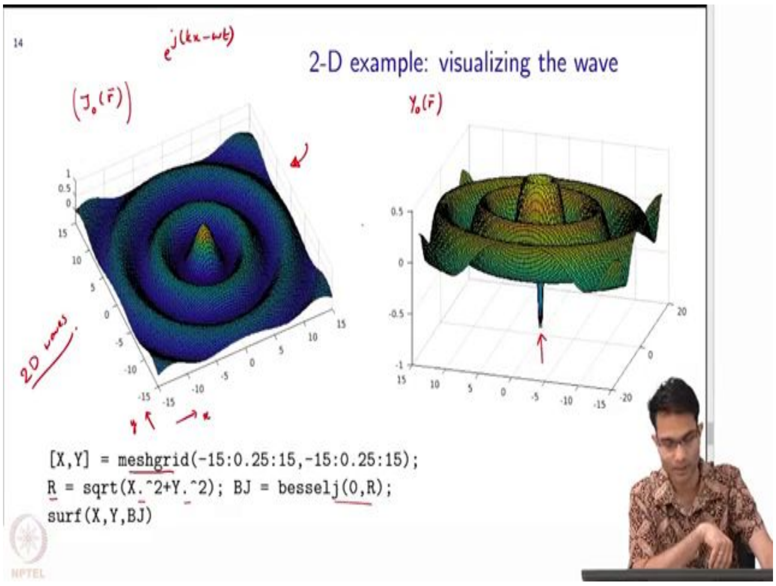
$1/\sqrt{r}$ right, so you can see that this whether its 1/r or $1/\sqrt{r}$ and even when you go to 1-D case what is it there is no such term in the denominator right. So, in some sense is if I look at 2-D and 3-D because they are truly the physical waves, it is in my opinion the where energy has to be balanced in multiple dimensions two versus three that is what is giving us different form.

(Refer Slide Time: 20:53)



In 1-D yeah, 1-D is a strain skills I mean it is a quite unphysical way which is goes on forever alright. It is very difficult for you to actually think of a physical 1-D problem what is that mean? There is always a second dimension in world somehow.

(Refer Slide Time: 21:11)



But this; these graphs really they convey the institution of it, if you plot the 3-D Green's function it looks similar to this right, it will have the maxima I mean it will have some large value at the origin and then begin to decay of. So, physically that is what we expect ok. I think what we will do is, we will call it a quit over here instead starting the next module on method of moments so we will stop here.