Computational Electromagnetics
Prof. Uday Khankhoje
Department of Electrical Engineering
Indian Institute of Technology, Madras

Review of Vector Calculus
Lecture – 1.3
Common theorems in vector calculus

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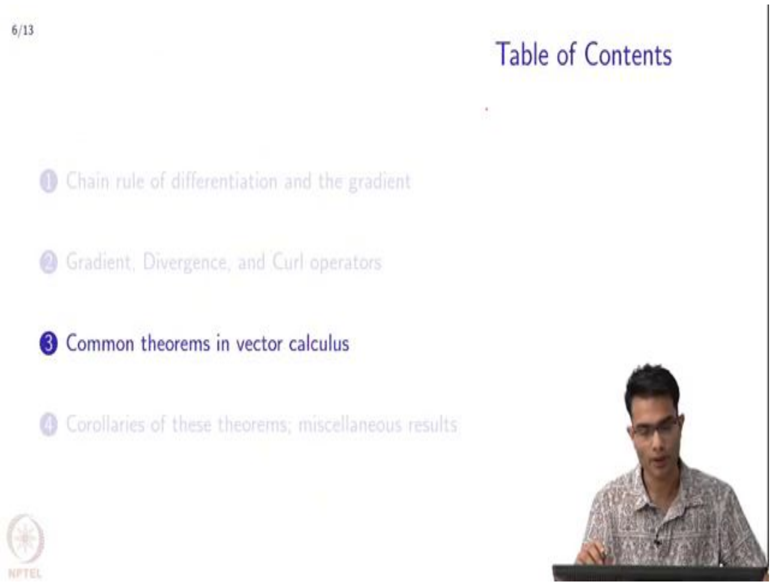

So, moving on, we will talk about some Common Theorems in Vector Calculus ok.

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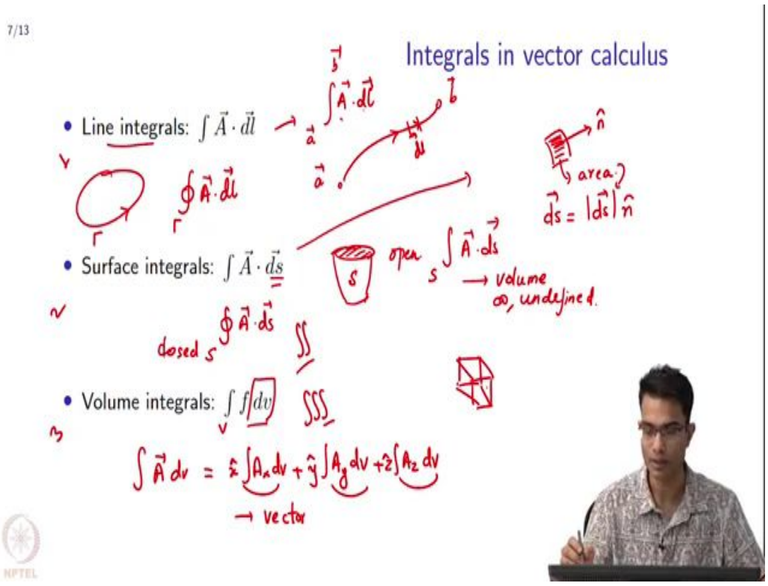

So, before we come to theorems we again have to get familiar with the different kinds of integrals that are there ok. We have so far looked at partial derivatives, we have looked at looked at dot product, we looked at cross product. So, those are differential elements the next thing important component is the integrals. So, as you can imagine there are three kinds of integrals that we will look at.

So, there are line integrals, surface integrals and volume integrals. So, the line integral over here is as before I can there are two kinds. I can go from point a to point b like we are we had done that before right. So, that is one point over here, one point over here ok. So, depending on the nature of a this integral may or may not depend on the path because we have seen that if a can be written as grad f, then this integral does not depend on the path ok.

So, this is one kind of integral, the other kind of line integral is also something which we have seen which is a integral around a closed path right. And the notation for that is we can put usually a path that is denoted by some symbol. So, for example, this is a path and I denote it by Γ. So, these are the two kinds of integrals that we will come across; pretty simple. Then comes surface integrals. Now how many kinds of surfaces do you expect very simply?

Student: (Refer Time: 01:50).

Open and closed surfaces right. So, for example, if I take you know a cup over here or the surface over here. So, this is the surface of the cup represents a.

Student: Open.

Open surface right. So, let us call this S, so this integral can be of this form ok. Similarly if I now close this lid over here right so, it is a cup with a lid on top of it. Now it is a closed surface. And then that same integral is represented with the same symbol over here and I write it as. So, an open surface, I cannot talk about how much volume it encloses right.

So, I cannot say so the volume is either you can say infinite or you can say undefined but a closed surface encloses a certain finite amount of volume. So, that is important to remember and finally the other kind of integral that we come across as a volume integral. There is not much different over here it is just integration over closed volume V.

Sometimes you might encounter an expression like this, it is an integral of a vector over the volume, we should not get afraid all. We have to do is open up the components. So, this is going to be integration along each component dV remains the same $\hat{z}\int_V A_z dV$ ok. So, this quantity finally, is going to be a scalar or a vector? It is going to be a vector right.

And the components of the vector are given by these amounts over here; those are the components of this vector ok. One small thing that you will have to get used to is regardless of the dimensionality of the problem. For example, this is a 1D, this is 2D, and this is 3D we are only going to put one integral sign ok.

In other courses, you may have seen that if it is a two dimensional integration; you will be putting a double integral and you will be putting a triple integral, but we find that from the context it is clear whether it is a single integral double or triple integral because of the differential element ok. So, if you see a $dxdydz$ you know it is a volume integral. So, we will not be putting these multiple symbols over here, it will be clear from the context alright.

So, just one thing to keep in mind, here you can in the one dimensional integral over here this dl, you can see is a small line element with some length and direction. In the case of the surface, how does the surface element look? It will be having something like this. It is a small patch with a normal vector right.

So, it has some direction and it has some area ok. So, this $d\vec{S}$ over here that I have written or you can write it like this $|d\vec{S}|\hat{n}$ which is the area and the vector over here ok. So, that is implicit and understood when we write ds vector. For volume integrals, we need not worry there is no direction, it is just you know the take a differential volume like this that is the volume inside $dxdydz$.

So, far I have been using the language of x y z, but it is not necessary I can use the spherical coordinate's (r, θ, ϕ) or use cylindrical coordinates (ρ, z, ϕ). It depends on the problem at hand we use the convention depending on what makes it easy to solve that problem. So, these are the integrals that we will come across.

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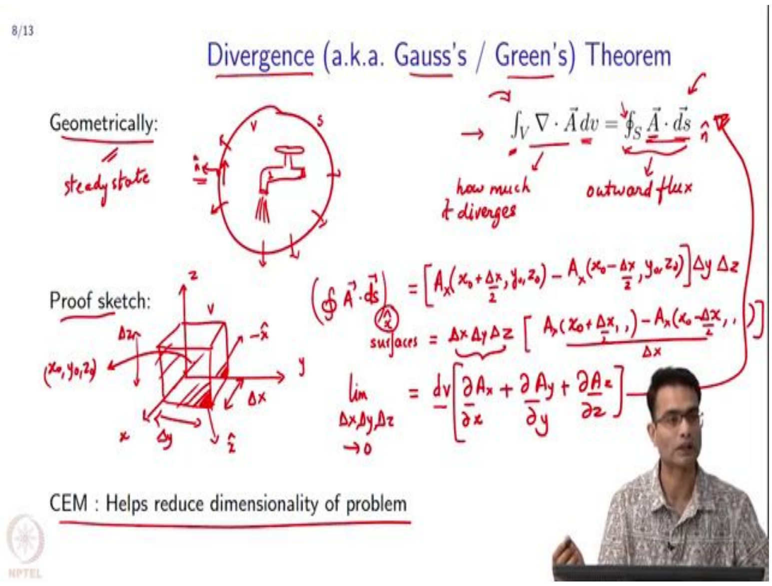

Let us move to our first main theorem in vector calculus which is most commonly known as a divergence theorem. Different books will refer to them as different by different names, we have also studied the same theorem by Gauss theorem or in electrostatic called Gauss law. It is also called Greens theorem ok. Both gauss and Green are names of famous very old scientists ok. So, to explain this, let us first write a intuitive sort of given intuitive geometrical interpretation of what this divergence theorem is and the simplest way to think about it is let us say that I have a tap and which is giving out water over here. And let us enclose it in some imaginary volume over here; so it is a closed so, it is a closed surface that is what we are doing.

And I want to find out how much I want to do a balance of the water inside this so this diagram that I have drawn. So, right now there is only one tap over here and let us imagine I am considering this in steady state ok. Steady state; that means, the tap has been on for a long time whatever water had to fill up in this volume has filled up right. So, if I wanted to measure; let us say the water outflow from this surface.

So, this is a surface S include enclosing a volume V. So, water is going to be going out from the surface; obviously, because the tap is here it is a magical tap; it has most connection it just there it just appeared. So, there is water that is going out from here.

So, what could you do? You could just stand on the surface over here and just measure how much is the flux right; flux is how much is going out. Now if water were for example, in this surface over here if water was flowing like this along the surface would it; obviously, not flow out. So, there is a flux that is depends on the orientation of the surface.

So, there is a normal vector over here and the vector how much it is going out from here right. So, if you look at the expression of the theorem that I have written over here, this expression over here is outward flux ok. Remember the ds had the normal vector inside it over here as it made sense and I had to take the dot product of the vector field with that normal vector, that gave me how much was going out ok.

So, this is this makes sense for counting the water going out. I can do a more direct calculation. If I know the tap if I know how many taps are in this, right now I have shown only one tap, but they could be more taps right. So, I have taps, what was the expression or quantity which I had described which measured how much was how much outgoing a vector field was? Was a divergence right? It told me how much something was going out. So, I take this vector field a which is representing water flow. If I find out this divergence at various points inside this, I will get a sum total of how much is outgoing right, but that is at each point in the volume.

So, that is the expression over here. This is the how much it diverges and I have to do this integration over the volume because there could be lots and lots of taps right. So, these are two ways of counting the same problem and that is the geometrical intuition of how do we calculate this, how do we need to do the budgeting of the water. You could either do it over a volume V or you could do it over S and remember that this surface has to be closed because I am talking about a volume right.

So, hence this closed surface over here. So, this is your divergence theorem ok. This was the geometrical intuition of it will come to the proof sketch shortly; one comment I want to make over here. How many dimensions is the left hand side being? How many in the integral is in how many dimensions? In three dimensions; so volume integral. Whereas on the right hand side is an integral in how many dimensions?

Student: Two dimensions.

Two dimensions right. So, from a computational point of view this is the point I want to make that it helps to reduce the dimensionality of a problem. Let us say that there is a problem that I am using I am solving by the left hand side. Then to numerically solve it, I have to break up the volume into small small pieces and do some calculation over each part of the volume.

Whereas, the right hand side is telling me that the same calculation can be done only on the surface I need not go in right. So, we know that the volume is a larger quantity, then surface right. So, this helps us this is a trick which we use to reduce the cost of computation ok. This is again a very computational idea of the divergence theorem.

Now the proof sketch for this is fairly simple. So, I will not outline it fully, but we will just have a brief idea of it. Let us start with the let us start with the right hand side ok. So, the right hand side is telling me it is the flux of a from some surface. So, let us take a small box. So, this is my x, y and z is pointing upwards over here this is my small box ok.

And let us say that the center point over here is (x_0, y_0, z_0) and in each dimension I have the width of this box Δx, Δy, and vertically over here is Δz ok. Now for this small for this very small volume V, I let me try to calculate the right hand side so the right hand side now this for the surface. For example let us take the surfaces this one and this one ok. So, these are surfaces for which the, what is the outward normal for this surface.

Student: (Refer Time: 11:55).

Right. So, for this surface it is $\hat{x}$ and for this surface the normal is pointing in the backward direction right. So, it is going to be -x so this is -x ok. So, let us just do the calculation of the surface integral over here. Let us just take the x surfaces $\hat{x}$ surfaces. So, let us assume that this volume is very small such that this function does not vary very much inside it.

So, if I take the let us take this side first over here. So, what can I call it? So, it will be now we can assume that the value of the function is more or less constant. So, I can call this A of which component of A is going to come? It is going to, I am taking a dot product with the normal right. So, normal is along $\hat{x}$. So which component of a will survive?

Student: x.

Only x ok. So, I will write this as A_x ok. What are on this surface over here what are the values of the coordinates right. So, I will have $A_x(x_0 + \Delta x/2, y_0, z_0)$ So, I am what I am doing is, I am trying to find out the value of the function at the center of this space. Center of the space is simply going to be y_0, z_0 and minus the contribution from this side ok. From this side, the contribution again which components survives x is what survives that is right. What is the value what is the argument over here?

Student: x_0.

x_0 exactly. So, it is the backside. So, it is away from the center by a distance of $-\Delta x/2$, the other coordinates remain the same ok. So, this is the vector component coming from A and I need the only thing missing is now what the surface area. So, what is the surface area of this?

Student: dy.

Right not dy, but Δy and Δz right ok. Now does this remind you of something this expression over here? A very tempting thing to do is to multiply and divide by Δx right. So, if I do that right so, I will get $\Delta x \Delta y \Delta z$ and this thing over here $x_0 + \Delta x/2$. I am going to leave the other components the same. Because they do not change right this is going to be $x_0 - \Delta x/2$; do not change and the whole thing divided by Δx and if I take the limit as these guys tend to 0. This expression will become so this over here. What does this become?

Student: ΔV .

dV right this becomes the differential volume right. And this expression over here notice which is the component that is changing in this only the x component right. So, it is $\frac{\partial A_x}{\partial x}$ right. So, this is considering the surfaces with normal $\hat{x}$. Similarly this will they will be two surfaces with normal's $\hat{y}$ two surfaces with $\hat{z}$ right.

So, I have got what will happen is common will be dV, then I will have a partial with this other terms will give me this and finally, the right. So, that is how we get it and this looks exactly like our expression over here right. So, we have a satisfactory a geometrical idea over here which talks about volume flowing out and we can prove it in a very simple way and hopefully this has given you enough confidence on how to solve this.

So, this is the divergence theorem and as I mentioned that it helps us to reduce the dimensionality of the problem. Let us move to the next object which we have considered which will be involving the curl right. So, that is the curl theorem also known as the Stoke's theorem right.

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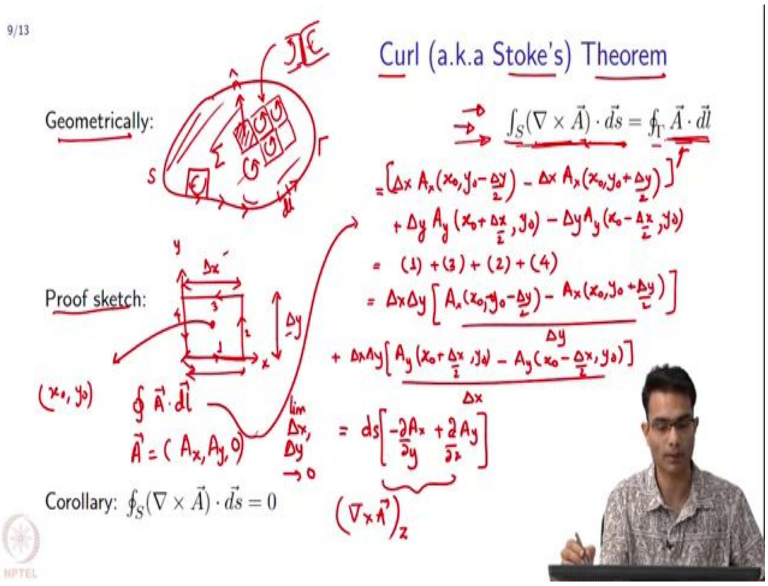

So Stoke's theorem is going to relate the curl to something over here. So, this is the x this is the statement of the theorem as before will take a geometrical picture of it and then look at how to prove it ok. So, the geometric intuition of the curl itself was how much it swirls right. So now, that is the curl of A over here. So, it is telling me how much a vector field swirls, but that is not all. It is there is a dot product with a surface vector differential surface element. So, it is the swirl of a vector field and after that the flux of that and then an integral ok. So, note that this is a what kind of an integral is this a closed or a open surface.

Student: Open surface.

It is an open surface right; it is not the circle is not around it. So, I can think of a surface like this. And in this so this is my surface s over here, I will write it over here. So, let us take one small patch dS over here this dS has some normal vector over here right. Now in this surface patch over here, what is happening is I am calculating the outgoing flux of this swirl and it is to be done over the whole surface.

So, I will have to break up the surface like this into various patches and sum it up. Now remember we are talking about the swirl of a vector right. So, this in this patch how much it is swirling, in this patch how much it is swirling and so on. Now you can see if I take let us take a zoom in of this interface over here right.

So, I have two patches that are side by side right. This patch over here the swirl is going in this direction. In this patch, the swirl is going in this direction. So, if I look at this common edge over here, these swirls are in some sense cancelling off. So, what will I be left with if I sum it up over this you know if I sum it up over all of these patches, what are the parts that will not cancel?

Student: (Refer Time: 18:29).

The outermost edges right, so whatever is happening over here that is not going to cancel right. So, geometrically what we can say is you want to calculate the outgoing flux of the swirl very good. Instead of doing it over every element over here, why not just walk along the surface over here and collect the vector field that is along the surface that will give me the swirl that is my $\vec{A}.\vec{dl}$.

So, $\oint_{\Gamma} \vec{A}.\vec{dl}$ is what I can get from here ok. So, geometric picture is very clear there are swirls over here; they all cancel at the internal boundaries the only place, they do not cancel. So, look at this let us zoom in over here. Let us look at this swirl over here.

So, over here it is going in this direction; no one to cancel it similarly here, similarly here. So, may as well just directly take $\vec{A}.\vec{dl}$ over here sum it up over the curved surface and that is our geometric picture of the Stoke's theorem right. If you did not know this you know you might look at this expression and feel very scared what. So, many of ugly looking quantities, but the geometric picture is very clear. Similarly now for this expression, what can we do?

Let us try to sketch out the proof like we did previously ok. So, again we will take the right hand side and to make matters simple, we will just restrict ourselves to patch in the x y plane. So, this is my x y, this is my patch. Center point again, I am going to take x_0, y_0 and this patch has some Δy and some Δx is the width of this patch. And this is how I am going to sum up this do the evaluate the right hand side over here.

So, this is what I am going to do $\vec{A}\vec{dl}$ ok. Now my vector field that the way I have drawn it. Let us just assume that A is of the is in the xy plane so $(A_x, A_y, 0)$ that is my vector field.

Now I want to so let us start from here and go in this direction. So, let us assume that this Δx and Δy are very small quantities.

So, I can approximate this part of the integral simply by the value of the function at this point multiplied by the length because we are assuming A_x and A_y do not change. So, much it is a very small element anyway we are going to take the limit Δx, Δy going to 0 right. So, when I do this I am going to write it as, so let us write it over here sorry this is what I am going to write. The first term what will I get $A_x.dl$. So, for this path what is dl along. So, let us call it path 1, 2, 3 and 4. Along path 1, what is my $\vec{A}.\vec{dl}$?

Student: Δx.

So, Δx is the element right. So, I am going to write Δx is the length element that is along which component of A will contribute to it?

Student: A_x.

A_x right. So, it is going to be A_x what are the coordinates I should put down for this function at what am I evaluating it. So, $(x_0, y_0 - \Delta y/2)$ remember the center is (x_0, y_0). So, this point is $x_0 - \Delta y/2$ right that is the distance over here, so this is this thing. Now so next. So, this is from path 1, let us write down the contribution from like sense of writing from path 2. Let me write for path 3, let us just change the order it does not matter. So, from path 3, what will be the contribution which way is dl?

Student: (Refer Time: 22:32).

Minus right it is along $-\hat{x}$ direction. So, I am going to write a minus the length element is again going to be Δx. And which component contributes A_x or A_y?

Student: A_x.

A_x again is going to contribute. And what do I have over here?

Student: (Refer Time: 22:50).

$(x_0, y_0 + \Delta y/2)$ ok. So, this was for path 1 and path 3 ok. Now let us go back to the remaining two paths which are path 2 and path 4. So, for path 2, what is the differential element Δy? What component of A is along this path A_x or A_y?

Student: A_y.

A_y is going to be my component A_y. What are the coordinates?

Student: x_0.

$x_0 + \Delta x/2$ and.

Student: y_0.

y_0. And when I go along path 4, my dl is pointing the $-\hat{y}$ direction right. So, this is going to be $-\Delta y A_y(x_0 - \Delta x/2, y_0)$ ok. So, in other words what I did was I took path 1 plus path 3 plus path 2 plus path 4 that is the order in which I did it ok.

So, now, let us look at these terms 1 and 3 together. What is common is Δx and in paths in 2 and 4 Δy. So, like we did previously, our trick will be to multiply divide by missing one. So, in the in the first two terms what I can do is multiply and divide by.

Student: Δy.

Δy right. So, this will become $\Delta x \Delta y$ common right and what I have is $\Delta x \Delta y (\frac{A_x(x_0, y_0 - \Delta y/2) - A_x(x_0, y_0 + \Delta y/2)}{\Delta y})$ A x x naught y naught minus delta y by 2 minus A x x naught y naught plus delta y by 2 whole thing divided by delta. What did we divide by y and the other term that I will get is delta.

So, $\Delta x \Delta y (\frac{A_y(x_0 + \Delta x/2, y_0) - A_y(x_0 - \Delta x/2, y_0)}{\Delta x})$. So, as before we will take the limit that $\Delta x \Delta y \to 0$, so this is a limit. So, then this expression $\Delta x \Delta y$ will become.

Student: dS.

dS right dS what happens to the first term over here. This is the, it is a partial it looks like a partial derivative of which component.

Student: A_x .

A_x with respect to.

Student: y.

y, but with a negative sign right. So, this becomes $-\frac{\partial A_x}{\partial y}$ and the second term will is the partial derivative of the y component. So, with a plus sign alright. And that is if you work out the curl expression over here this is which component of curl right. So, there is x and y; x and y what is missing is z. So, this is actually the z component over here right.

So, this is the starting with a geometric idea which gave us the statement of the theorem very intuitively. The proof is again not very difficult we just looked at a surface and the line that is around the surface and we did a simple integration right. So, this gave us a z component. If you had a general surface oriented along with normal's in each component, you will get all three components. So again very straightforward using the simple intuition about integrals and flux; so this tells us about the Stoke's theorem.

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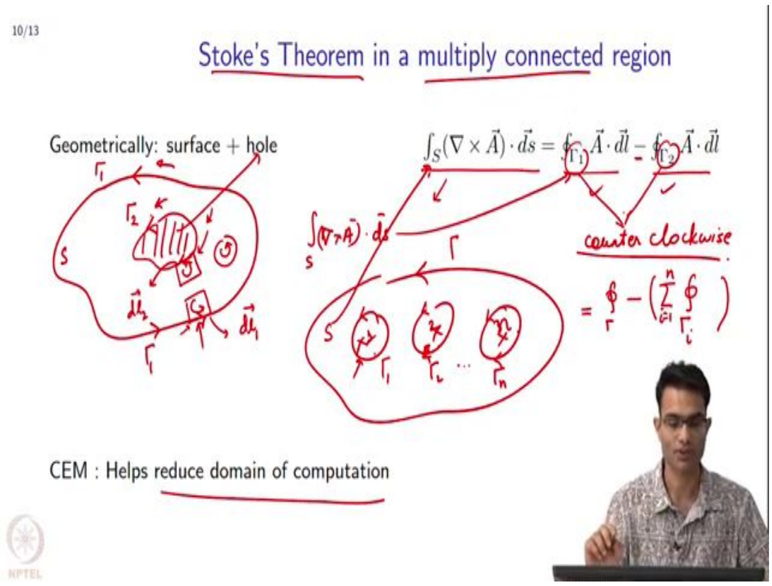

Now, moving on there is one small variation of the Stoke's theorem which we will talk about. So, Stoke's theorem as before but in what is called a multiply connected region ok. So, previously we had our surface over here was hole right there I mean homogeneous there was there were no holes in it right. And the integral was around the bounding surface gamma. Now in a multiply connected region, what we will have is let us say a surface like this, but let us say there is a hole in it that hole does not exist.

For example, maybe it is there are there are no electromagnetic fields there or let us say it is a room and inside a room there is a Faraday cage, inside the Faraday cage we know that the field is 0. So, why try to calculate? It so we exclude that area or volume from the computation by means of a hole right, so this is the hole ok.

So, now, this surface S over here is bounded by two lines or contours right. So, one is this one, so we can call this Γ_1 and here is another line which contours which bounds in Γ_2 ok. And we want to know what is the Stoke's theorem saying for such a volume for such a surface over here right. Now as you can see the statement of the theorem shows you the difference between two line integrals. Earlier we had only one line integral.

Now we have two line integrals Γ_1 and Γ_2. And what is the intuitive explanation for this? It is again the same as the geometric picture that we had that there are many many swirls over here right. So, if I take this over here this swirl over here did not cancel from anywhere else. Similarly when I take this surface element over here, I am going to have the part along here is not going to cancel from anywhere right.

So, if I call this let us call this dl_1 and let us call this part over here dl_2 ok. So, dl_1 is pointing in this direction and dl_2 is pointing in the opposite direction ok. So, when I do this the budgeting over here $\nabla \times \vec{A}$ over the whole thing what will happen is that the first component will come from dl_1 right and the direction is the same as the direction over here that I have shown off Γ_1 right.

So, it is in Γ_1 is in this direction. So, this term is fine why is there a minus sign over here? You can notice that for this surface element the part that did not cancel out was actually flowing in this direction, but if I take a convention that I am going to always give a orientation for my contours in let us say a counterclockwise way right. So, Γ_1 here is counterclockwise, Γ_2 is shown in a counterclockwise way, then this contribution here is appearing in a clockwise sense. So, that is why I have to put a minus sign over here ok. So, this is the second term.

So, again this convention may not be universal some books may have different orientation or sense of orientation. So, when I write it in this way it is implicit that Γ_1 and Γ_2 follow the same convention which is counter clockwise ok. And then you can extend this idea to multiply connected regions ok.

So, let us say I have a volume like this multiple things that have been canceled off over here. So, I have overall let us say gamma and I will call this $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. So, what will happen is that I just have to subtract off the line integrals along each of these things right. So, if I have say n such holes over here this hole 1, hole 2, hole 2 I mean sorry this is hole n. I am going to have how many summations how many integrals on the right hand side? So, one from Γ and then minus $\sum_{i=1}^{n} \Gamma_i$. So, I have a total of n plus 1 integrals on the right hand side again.

Student: (Refer Time: 31:14).

The directions of that is a good point right. So, the direction is in should be in the counterclockwise way right. So, it should not be this it should be this thing right thanks for the correction so all are counterclockwise ok. So, as I sort of motivated this to begin with this is helping us reduce the domain of computation right.

If I know that there is something which is of not interest which is not of interest to me why should I calculate it? So, I can remove these elements from the integration small price I have to pay is a few more line integrals have to be evaluated ok. And now the surface is excluding this that is the meaning of S over here ok. So, this is Stoke's theorem in a multiply connected region which we will use in the course and so.

Thank you.