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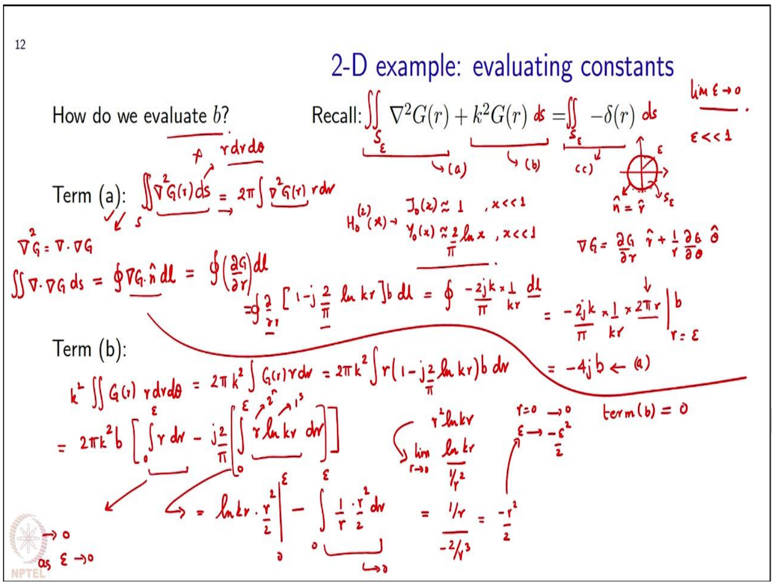
# Introduction to greens Function
## Lecture – 7.6
## 2-D example: Evaluating Constants - Part 1

(Refer Slide Time: 00:14)




So, to evaluate this constant b, I am going to take my definition of my Green's differential equation over here and do an integral ok. And that integral is going to be what kind of an integral should it be, 1D, 2D,3D?

Student: 2D.

2D integral right, so this is going to be an integral over S $ds$, what kind of S should it be? Where is my singularity?

Student: Origin.

Origin so I should include it right. So, the easiest symmetrical way of doing it is to have a circle of radius epsilon ok epsilon is some small number and integrate about this. So, I will call this surface to be S infinity sorry, not S infinity, $S_\varepsilon$ that is what I am going to do ok. So, evaluating these integrals requires some little bit of care ok. So, the first term over here when that integrates I am going to call it term a, the second term I am going to call term b and the third term over here I am going to call term c ok.

So, we will individually go over a b c and do a budgeting to get b that is the approach that we are going to take ok. So, let us look at the first term for a. So, it is going; it is saying what now I will just write one integral symbol let it be will just be consistent we will write $\int \nabla^2 G(r) ds$, what is $ds$ in polar coordinates?

Student: r d r.

$r\ dr\ d\theta$ right. So, this is $r\ dr\ d\theta$ ok; is there any $\theta$ dependence?

Student: No.

No right. So, that integral over $d\theta$ we can immediately write as $2\pi$ right. So, I am going to write this as $2\pi$ and $\int \nabla^2 G(r)\ rdr$; that is what I have to evaluate ok and this is (Refer Time: 02:36). So, that could be one way of doing it ok, but the trouble is I still have this $\nabla^2 G(r)$ sitting with me ok. Another way of doing it could be if I look at this term over here, can I apply; can I think of first of all can I write this $\nabla^2 G$ as a divergence of a vector right? This follows from the rules of vector calculus that $\nabla^2 G$ can be written as the $\nabla \cdot \nabla\ G$, you can just write out the operators take the dot product will get exactly this ok.

So what I am doing now? So when I am doing integral of $\nabla \cdot \nabla\ G\ ds$ what theorem of vector calculus comes to mind? It is a divergence under a closed volume or in 2D closed surface. So which theorem?

Student: (Refer Time: 03:40).

Divergence theorem right; so divergence theorem will convert this del dot something into the outward flux right. So, this will become outward flux of what vector, $\nabla G \cdot \hat{n}\ dl$ right. So, one root was what we were trying over here, splitting it into $r\ and\ \theta$, another root is this root. This root seems to be little bit more intuitive because I have reduced that integral straight away what is $\hat{n}$ over here?

Student: r cap.

$\hat{r}$ exactly. So, $\hat{n}$ is for example, always pointing readily outwards, so that is my $\hat{n}$ is equal to my $\hat{r}$ . What is the definition of $\nabla G$ in polar coordinates, this one is not that difficult $\nabla G$ in polar coordinates, the first term is simply.

Student: Dou G by.

$\partial G/\partial r$ along $\hat{r}$ and.

Student: 1 by r.

1 by r.

Student: $\partial G/\partial r$.

$\partial G/\partial \theta$ along $\hat{\theta}$ ok. So, you know $\hat{n}$ is $\hat{r}$ , $\hat{\theta}$ has what relation to $\hat{r}$ ?

Student: Perpendicular.

Perpendicular right, $\hat{r}$ and $\hat{\theta}$ at perpendicular to each other right. So, only one term is going to survive from this integral and that is going to be.

Student: Del G by del r.

$\partial G/\partial r$ right. So, this is going to become integral $\partial G/\partial r$ right and what else do we have over here that is about it right this will become a $dl$ right. Now I am going to choose $\varepsilon << 1$ like $\varepsilon$ is very small, it is just enough to include the singularity right. So, I am going to keep $\varepsilon$ very small ok.

So, here is another approximation that I will tell you I mean that I will use. So, $H_0^{(2)}$ right, that is the function that I have right. There exists an asymptotic form for this when x is very small and that is based on the fact that $J_0(x)$ is approximately 1, when x is much much less than 1 and $Y_0(x)$ is approximately $2/\pi\ log(x)$.

Again this is just information I am giving you, with this is very very well documented in various mathematical hand books asymptotic forms of special functions. So, what do I do next? I will use this form when you can see that the second this $Y_0(x)$ you can see that it has the correct singularity kind of a shape right, as x tends to 0 what happens to log of x?

Student: (Refer Time: 06:47).

It goes to minus infinity and if so it is capturing the singularity well and $J_0(x)$ as a finite quantity 1 ok. So, from these two terms what survives is going to so, let us write it down over here. So, $\partial G/\partial r$ that is what I have to do ok. So, $\partial/\partial r\ (1 + 2/\pi log(kr))\ dl$ ,that is what; I have to do I have just substituted the form of G. So, the first term goes away second term what will I get? Oh, you are right yeah, there should be; it should be a $-j$ here right, because of the way that I have defined $H^{(2)}$ is $J - jY$ right, that is right.

So the first term goes away, when I take the $d/dr$ second term will give me what, what is the derivative of $log(kr)$ with respect to r.

Student: (Refer Time: 07:55).

k right. So, I am going to get a $-2jk/\pi\ 1/kr$ integrated over $dl$ ok. What will this integral over I am now doing this integral only on the surface right, on this surface the value of r is constant right on the line rather. So, what will this integral evaluate to? Integral $dl$ will just simply give me $2\pi$.

Student: r.

r and r is equal to $\varepsilon$ right so we will just write this as $-2jk/\pi$ into $1/kr$ into $2\pi r$ evaluated at r is equal to $\varepsilon$. What I am left with k k goes away $\pi\ \pi$ goes away r goes away, r is small but finite. So, I can cancel r in the numerator and denominator right. So, what I am left with.

Student: $-4j$.

$-4j$ is that clear. So, remember I mean they the meaning of this is $\partial G/\partial r$ evaluated at all points over here along the circle boundary right, there is no variation. So, this integral over $dl$ simply becomes $2\pi r$ ok, r is not varying; if r were varying then it would be different thing r is constant over here. So, it is more like an integral over d theta, so I get a $-4j$. So, term a evaluates to $-4j$ then we come to term b ok. Now term b is where we can use our this $r\, dr\, d\theta$ approximation.

Student: Sir, there is a.

Yeah.

Student: b also right.

There will be a b yeah you are right.

Student: (Refer Time: 09:51).

Yeah G has a b everywhere that is right. So, the b will come in over here.

Student: $-4jb$.

Yeah exactly very good, so there is a b over here and there is a b over here yeah very good. So, let us come to the b term so, I have a $k^2 \int G(r)\, r\, dr\, d\theta$ right and again I can write this as $2\pi k^2 \int G(r)\, r\, dr$ ok.

Student: Sir.

Yeah.

Student: (Refer Time: 10:28).

No that is here integral over $dl$ give me $2\pi r$.

Student: (Refer Time: 10:34).

This is what we started evaluating. So, there is no additional $2\pi$ right there 2 over $\pi$ has being taken account into account over here ok. So, again what do I do? I have a special form for G I will just substitute that right. So, $2\pi k^2$ integral r and $1 - j2/\pi \, log(kr)$, this time I should not forget the b and $dr$ yeah that is my G, $b \, H^{(2)}$ is $J - jY$ is this and $Y$ is this over here.

Student: (Refer Time: 11:28).

No we did not use we did not follow this approach.

Student: We (Refer Time: 11:38).

We followed this approach where we use the divergence theorem to convert it. So, that is the $2\pi$ that you are referring to yeah we did not use that approach over here, we used that approach for this term b ok. So, this is the approach that we used over here so let us write it over here. Now when I look at this expression let us just gather all the constants outside. So, $2\pi k^2 b$ I can take out and then finally, I have an integral of $rdr$ right and then, I have an integral. So, 0 to $\varepsilon$ and then I have a term which is $j2/\pi \, r \, log(kr) \, dr$ those are the two terms right.

Now, without actually doing all the math, you can see what will happen to the first term over here? Integral $rdr$ will give me $r^2/2$ evaluated $\varepsilon$ and 0; and I am going to take the limit $\varepsilon$ tending to 0 eventually right. So, we are going to take limit $\varepsilon$ tends to 0 that is going to happen anyway ok. So, this first term over here is tends to 0 as $\varepsilon$ tends to 0 fine. Then the other term that I have to worry about is integral of $r \, log(kr)$ ok.

So, what can we do about integral of $r \, log(kr)$? We can just integrate it we know how to integrate this right we can take, log as the first term integration by parts right. So this if I just let us just look at this term over here ok, otherwise I will have to keep writing the constants each time. So, this square bracket term I am writing over here. So, first term multiplied by integral of the second function right. So, I am taking this as the first function and this as the second function in integration by parts. So, $log(kr)$ multiplied by integral of second function which is $r^2/2$ evaluated between 0 and $\varepsilon$ minus integral 0 to $\varepsilon$ derivative of the first function right which is going to give me a what is going to give me?

Student: 1 by (Refer Time: 14:15).

1 by.

Student: $1/r$ .

$1/r$ right, the k k will cancel out multiplied by $r^2/2\ dr$ right ok. Now what happens over here is there are so, is equal to this let us let us look at the second term over here, second term is going to cancel of 1 r I will be left with a $r\ dr/2$ integrate 0 to $\varepsilon$ what is going to happen? This guy is going to tend to 0 finally, what I am left with, $r^2 log(kr)$ right. So, $r^2 log(kr)$ that is the term that will have to evaluated at 0 and $\varepsilon$; how do I evaluate this function? Is there some does are you reminded of some way of evaluating this limit as r tends to 0.

Student: (Refer Time: 15:19).

L'Hopital's rule right. So, first I have to get it into a 0 by 0 form right. So, I can do that as so $\lim_{r \to 0}$ this becomes $log(kr)/(1/r^2)$ right. So, either 0 by 0. I mean infinity by infinity form whichever way right. So, derivative of each term so first term will become $1/r$; second term will I mean the term in the bottom will become.

Student: r cube.

$-2/r^3$ is equal to $-r^2/2$. So, what is its $\lim_{r \to 0}$?

Student: 0.

0 and at $\varepsilon$ it will be $-\varepsilon^2/2$. So, what is its contribution? Overall 0 right; so, all of this hard work is going to give term b right, term b is equal to 0 ok. So, this is I mean we have gone through a lot of very sort of I mean this is basically class $12^{th}$ math right L'Hopital's rule and integration by parts and all of that right. So, it is not very

complicated, but just notice what happened, G is your Green's function, we have physically seen that it has a singularity at 0 because at a Bessel $Y$ function was going to 0, even though the function is singular what is the integral of that function?

Student: 0.

Its finite in this case its 0 right, it is just like a delta function the delta function is undefined at the singularity, but its integral is finite similar thing has happened over here ok. So, next we will look at the remaining terms and put them all together.