Computational Electromagnetics
Introduction to greens Function
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Introduction to greens Function
Lecture – 7.4
2-D Wave Example : Finding solution

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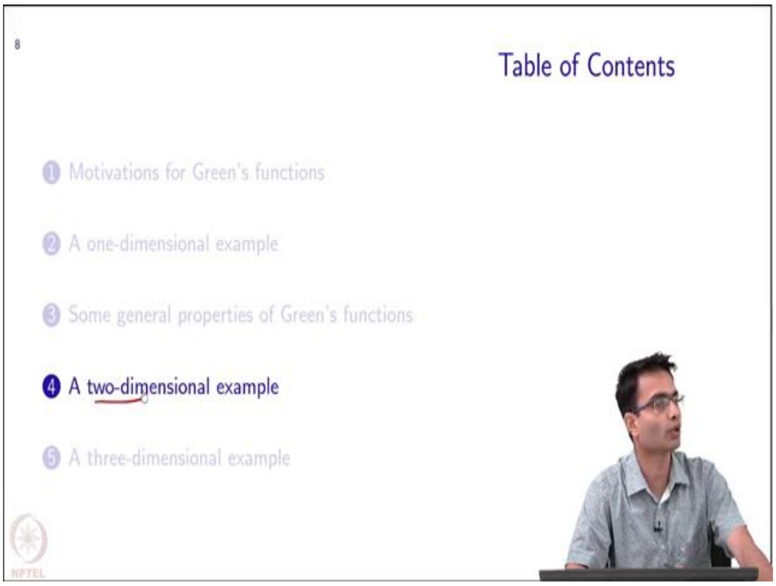

So, what we will do next is we will go to a 2 D case and we will spend a lot of time on 2 D because the scattering problem which we had started with was a 2 D problem. So, we will be using this Green's function right and so, let us go to the 2 D Green's function right.

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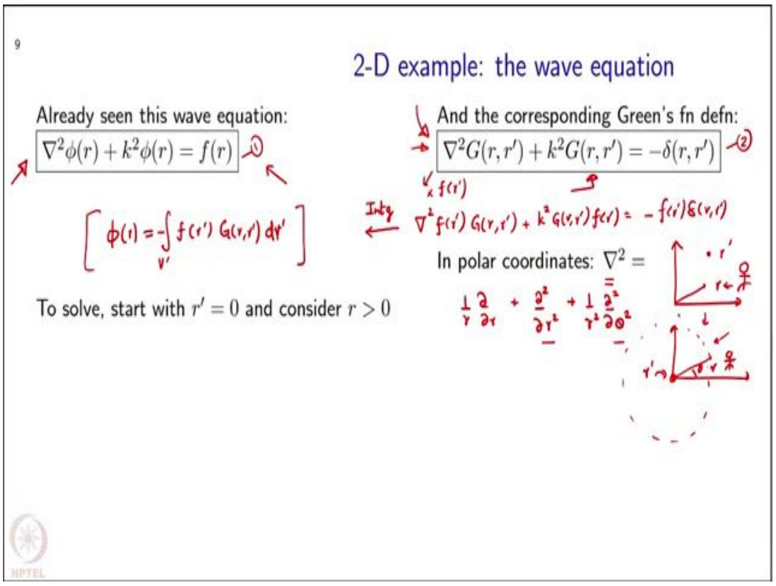

Now let us look at the wave equation that you are all already familiar with because you have seen it once or twice before, this is the wave equation that we had written right. How did we arrive at this? We had taken Maxwell's equations right and made some simplifying assumptions. What were they? For example, I had considered TM polarization, 2 D problem, all of those assumptions I had made and I had written it in terms of the variable ϕ . ϕ was my E_z from the previous module.

So, this equation we are familiar with. We were also familiar with the definition of the Green's function; we had used this kind of a relation. So, again, so this is mostly revision for you all. If I wanted to use the theory of Green's function, I would calculate this G once and then use it for whatever f is given to me that is the idea of Green's function.

So, the first step is to calculate or rather solve this equation over here. Again repeating from previous few slides, what would be the recipe of these equations say 1 and 2, what do I start with? Remember objective is ϕ in terms of G and f that is what I want for that I want G right. So, of equation 1 and 2 what do I begin with?

Student: 2.

2 multiplied by what?

Student: (Refer Time: 01:55).

$f(r')$ right, so I take this guy

Student: (Refer Time: 01:58).

Multiplied by $f(r')$ and then integrate; integrate both sides right with respect to prime coordinates right. So, because I am multiplying by prime coordinates, I can this is just once again I will write it. So, $f(r')$ goes in because the operator only acts on the unprimed coordinates right. And then, the next step was integrate and when I integrate it, it looks just like equation 1 right and from that I concluded that my $\phi(r) = \int f(r')G(r,r')dr'$. It is called dv' or dr' whatever. That minus sign is come because here the definition has a minus sign over here right when I integrate on both sides, I will find this minus sign just convention.

Student: Actually you took plus (Refer Time: 03:15).

Initially we you know because here.

Student: Ending problem you have took.

Yeah, yeah it is just a matter of convention. There is no great mystery to it. Once you choose once you fix your definition from equation 2, all the minus signs take care of themselves. So, my primarily in this example, I am keeping the minus sign because the reference material which is there for reading they also assume a minus sign which is in Chew's book. So, when you go back to reading those notes, you should not get off by minus sign each place, but just know that it is just a matter of convention.

Student: Sir, where in the green just G gets (Refer Time: 03:45).

Yeah that is all absolutely nothing else, so this is the solution. Now, let us before we get into solving this, what do you think is the most sort of intuitive coordinate systems to solve this problem?

Student: Polar.

Polar coordinates right. So, I mean in polar coordinates, what will be right your ∇^2 operator does anyone remember right? No yeah, no one usually remember it. So we will just write it down, so it is $1/r$. This is the operator ∇^2 in polar coordinates. Yeah, you just look up the Wikipedia article whatever and remember write it down each time, but I mean this still looks complicated.

So, what is another simplifying assumption we could do? So, let us try to interpret this equation a little bit physically. I have a left hand side which is some operator and right hand side is an impulse placed at which point? $r = r'$ and I am construct I want to find out what is the response when I place that impulse at r'. How can I make my life a little bit easier?

Student: (Refer Time: 05:15).

I do not want to care about theta that is the right idea right out of these two derivatives with respect to r and theta if you can somehow remove the theta dependence it would make my calculation easier. And what would be the way to do it? If you are standing at the impulse, do you find an angular symmetry about it right? If the impulse is at one point and you are standing over there everything around you looks the same with respect to theta right.

But in your in our coordinate system right now this is what we have let us say this is our impulse is here at r' and this is some general point r. So, with respect to this observer, who is standing at r right; this is an observer standing over here. The impulse is over here. So, is there any angular symmetry for this observer? Not at all. When will there be angular symmetry?

Student: r equal to.

If I move r' to the origin right, so supposing I do; do this small transformation over here; I move r' over here and this guy r remains over here. Now wherever this observer is with respect to theta; if I keep the same value of r and this observer walks around like this, there is no orientation with respect to theta anymore right. But has the physics of the problem changed? Not at all I have just choosing my location of my coordinate system the origin of my coordinate system is what is being chosen a little bit cleverly. Once I get the solution to this problem, all I have to do is do a change shift of origin and I get my general solution.

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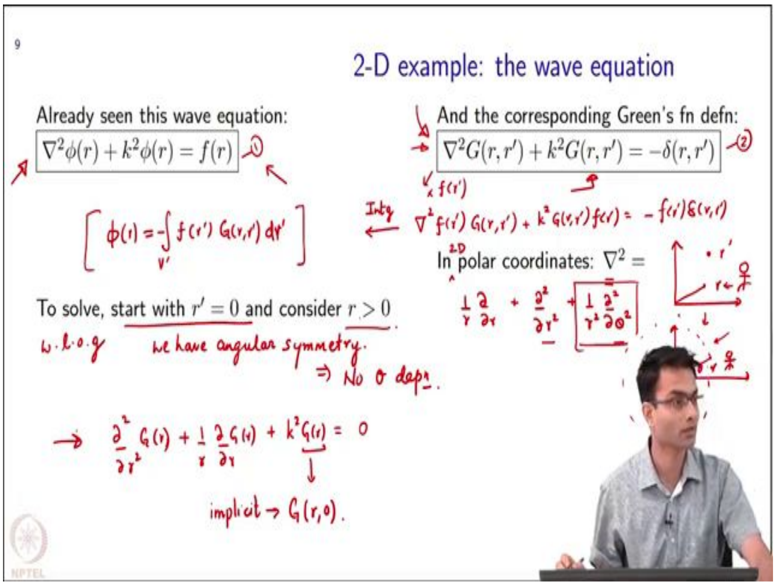

So, we will without any loss of generality, so without loss of generality, we can start with $r' = 0$ and consider all points which are away from the origin. So, in this case, we have angular symmetry which means that the what is the physical significance of angular symmetry is that whatever function or output I get will not have any theta dependence right; so no theta dependence.

And as great for me because this guy the third term in this Laplacian can be set to 0; it is not an approximation it is exact right. So, let us just put it all together, what do we have then? Our equation becomes $\partial^2/\partial r^2 G(r)$. Now I will just write $G(r)$. Why I am writing $G(r)$?

Student: (Refer Time: 08:16).

I have said $r' = 0$, so you will get irritating to write it each time, so just

$$\frac{\partial^2}{\partial r^2} G(r) + 1/r \frac{\partial}{\partial r} G(r) + k^2 G(r) = -\delta(r)$$

Again I do not write $\delta(r, r')$ and here it is understood that this is $G(r, 0)$.

Student: Sir, in (Refer Time: 08:42) coordinates we have r, θ, ϕ right.

This is 2D polar coordinates; you are thinking of 3D cylindrical 3D spherical coordinates where there is a r, θ, ϕ but this is just a 2 D problem, so there is no phi yeah. So, this is less write it over here 2 D polar coordinates, but when we move to the 3D example we will take spherical polar coordinates for using the symmetry; just fine right.

So, what does this look like? This looks like a second order differential equation right. There is a 1 by r in the denominator right. So, let us let see what we can do with this. actually I forgot one think I am saying $r > 0$. So, what should the right hand side actually be?

Student: 0.

0 right; so I should not write this over here write it 0. We will deal with the singularity the $r = 0$ point in the very end as we have done in the 1 D example fine. So, now, let us now this is the second order homogeneous differential equation that we want to solve. Let us try to get it into a little bit more of a familiar form. So, to do that I am going to take this equation and just multiply everywhere by r^2; I do not like to have the $1/r$ in the denominator. So, I will just multiply by r^2 right.

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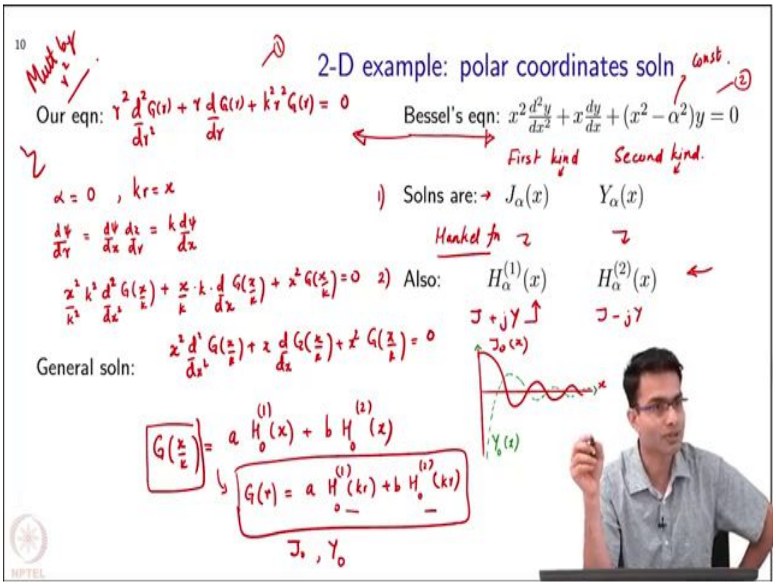

So, I am going to get a

$$r^2 \frac{d^2}{dr^2} G(r) + r \frac{d}{dr} G(r) + k^2 r^2 G(r) = 0$$

So I just multiplied by r squared. Now, those of you who have done course on special functions whatever in your math courses might recall what is called the Bessel differential equation.

This Bessel differential equation no one will actually remember, but yeah I mean if you look it up this is what it is. It looks like

$$x^2 \frac{d^2}{dx^2} y + x \frac{d}{dx} y + (x^2 - \alpha^2) y = 0$$

where α is a constant. So, do you see any similarity between this equation and our equation?

Student: (Refer Time: 11:00).

It looks almost the same right. I have a second derivative multiplied by squared, first derivative multiplied by linear term and $x^2 y$ or a $r^2 G$ term right the something is slightly off and that is.

Student: k.

k right, so we will have to deal with that k we will get to that, but let us just look at the high level conceptual idea here. This being a second order differential equation has two independent solutions and those solutions are named after Bessel right. So, these two solutions are the two independent solutions are this $J_\alpha(x)$ and $Y_\alpha(x)$. So these are called Bessel functions of the first kind and of the second kind.

So, this guy is the first kind and this guy is the second kind over here. So, this is just information I have not derived anything, we got very lucky that our Green's function relation came very similar to a preexisting differential equation. So, all the hard work was done by the gentlemen Bessel for us, we just have to carefully adopt his solution.

So, the Bessel's functions are written in two sort of popular ways; first is this way which is J and Y, the second is linear combinations of J and Y. They will also be solutions right linear combinations of J and Y will also be solutions, so that is the second type. So, for example, if I take $J + jY$, I get this guy that is the definition. So, what is this guy called? It is called a Hankel function. Again of the first kind and of the second kind, this guy is defined as $J + jY$.

So, linear combinations of the original solutions are still solutions to the differential equation right. So, you can either have Bessel's functions or Henkel's functions which are solutions to this differential equation. How I am getting a little ahead of myself, but how would you know which solution to take? You have a differential equation, you want to find G remember that is our objective because once I find G, I can convolve with f and get my ϕ.

Student: You can write a $J + jY$.

Yeah. So, I can write a $J + jY$.

Student: (Refer Time: 13:52).

I can also write a $H_1 + b\,H_2$ both will work.

Student: Yeah.

Which one should I choose? When I know it is going to be real, that could be one thing. What is the in solving electromagnetic problems what are the two things we have heard of? Maxwell's equations second thing is.

Student: Boundary condition.

Boundary conditions, so far we have not talked about boundary conditions. So, boundary conditions are actually what will help us to choose which is the most convenient form, you can choose either representation. But one form may be more convenient than the other like; we saw the Green's function in 1 D. Now, the closed form solution or a series solution, depending on the problem I will choose one or the other; so will leave that choice for now. So, looking at our differential equation over here, this is not yet exactly in the form of the Bessel's differential equation because as was pointed out the factor of.

Student: k^2 .

k right. So, first of all in our problem what is the value of α?

Student: 0 0.

α is 0 there is no constant r term right $\alpha = 0$. So, to convert this equation what would I, what is the how do I convert our equation 1 into equation 2?

Student: Put kr as the new variable.

Put kr as a new variable right; so supposing I have put kr is equal to x. So, if I take $d\psi/dr$ that is what I want to worry about $d\psi/dr$ is any function, I can write it by chain rule as $d\psi/dx\ dx/dr$ right. And this becomes what? dx/dr.

Student: k.

Right so that becomes the k right, so this becomes $k\ d\psi/dx$. This looks weird dx yeah. So, with that substitution what will this equation become? First term. So, r will get substituted as x by.

Student: k.

k right, so I have a x^2/k^2 second term becomes.

Student: x^2/k^2.

Will it become k or k^2? k^2 right because the second derivative when I apply this chain rule, once again it will become k^2. So, this will become $k^2 d^2G/dx^2$. Let just write it I mean let us it write it as r I can write it as x/k, so same thing, if you want I will write it as x/k. Next term becomes $\frac{x}{k}k\frac{d}{dx}G(\frac{x}{k}) + x^2 G(\frac{x}{k})$ is equal to 0.

So, conveniently for us all the all the ks cancel off and I get exactly, the Bessel's differential equation which is as follows. So, the general form can be written I know this is exactly the Bessel's differential equation, so the solution is I can write down $G(\frac{x}{k})$ is going to be equal to. So, let us let us assume the Hankel form. I could have assume the Bessel form, but I can as well as assume the Hankel form.

So, $a\,H_0^{(1)}(x) + b\,H_0^{(2)}(x)$ right, this is the object inside the differential equation, so it's solution is linear combination of the two solutions nothing very great over here. But remember x is our intermediate variable I am not really interested in x. So, I should resubstitute and get the solution in terms of r, so that will become left hand side becomes $G(r) = a\,H_0^{(1)}(kr) + b\,H_0^{(2)}(kr)$.

Student: r.

r is equal to a times H becomes.

Student: k r.

So, that is how these two functions I mean that is how the Green's function can be written. So, it seems that without too much effort because, the effort was done by Bessel, we have got r we have got some form for the Bessel's for the Green's function, just one small piece of information. How do these functions J_0 and Y_0 what do they look like? J_0 and Y_0 because that is what H_1 and H_2 are made of. What do they look like? So, if I plot it over here; let us plot this as x, so J_0 looks something like this.

So, this is my $J_0(x)$ and on the other hand, my Y_0 actually has a singularity at the origin and both decay for large r. So, we should have a physical I mean a mathematical intuition of what this function looks like right. So, the mathematical difficulty with this Green's function G is going to happen at which point? r is equal to 0 because that is the point where Y is going to blow up. So, this is unlike your Green's function from the 1 D example which did not blow up, this guys blowing up. So, working I mean dealing with this is going to be a little bit interesting.