Computational Electromagnetics
Introduction to greens Function
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Introduction to greens Function
Lecture – 7.2
A One-dimensional example

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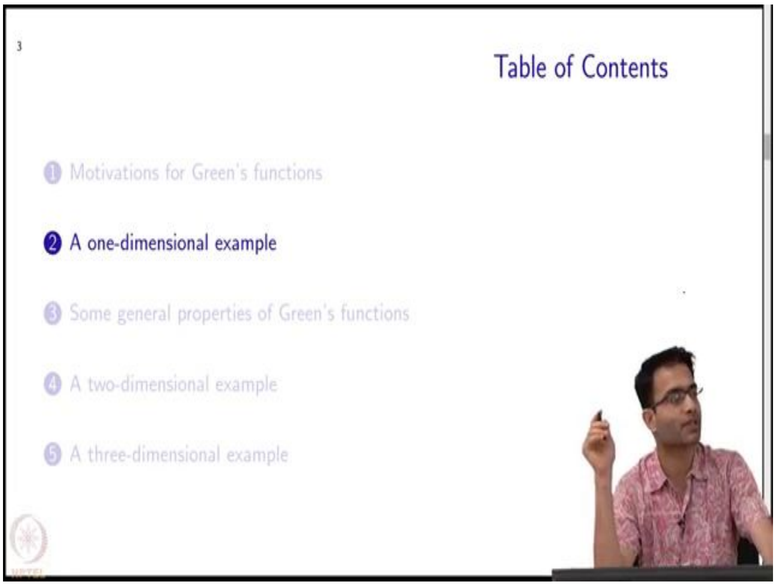

So, let us proceed with the simplest possible example which is a 1-D example.

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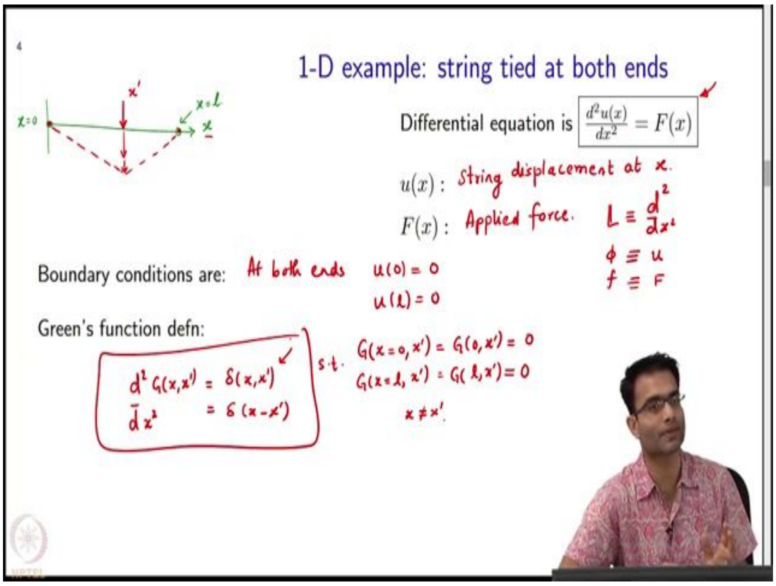

So, what I am going to do is for my 1-D example I am going to take a string and this string is tied at two points x=0 and x=l ok, so that is my string, it is constrained it is tied at two ends and it can oscillate in between anyhow. So, back from you know class 12 you used to solve this kind of equations the tension on the string and calculate the displacement all of those things.

Lots of constraints are there, we will absorb all of those into this differential equation, which tells me that second derivative of u. So what is u? Its the displacement of the string at any position x ok. So, string alright and F is the applied force alright. So, we need not worry how we got this equation right; mechanics has given this equation to us which is the second order differential equation right.

You can already see that if I ignore the well; let me not get ahead of myself let us continue step by step ok, at least the setup is clear? string tied at two ends. So that is my differential equation, so in this case for example, what will my L operator be?

Student: $\frac{d^2}{dx^2}$.

That is all correct. In fact, this is there is no need for me to write partial derivates it is 1-D problem. I can write it as total derivatives and my ϕ is u and f from the previous slide is capital F ok. So, one to one system I have taken. What for this physical problem what do you think are the boundary conditions?

Student: x=0 and x equal to one more 0.

Yeah, it is clamped at two ends, so the boundary conditions are at both ends; the displacement $u(0) = 0$, $u(l) = 0$ ok.

Student: What this, what there are described?

So, these are like you know nails you can say and a string is tied at these two points between this end between.

Student: Force is applied.

Force is applied like this ok, at some point on the on the at any arbitrary point I am applying some force. That force need not be at a point it can be applied throughout and that is given by f(x); f(x) is the is a function of x can be non-zero throughout the string you could be have placed an object on it, so f is distributed right.

So, we do not care what the form of f is ok, as motivated in the previous slide our main objective is find me g, I can tell you everything ok. So now, what is the definition of Green's function now that we will that we can use? I will like this define Green's function very simply like from the last slide.

$$\frac{d^2 G(x,x')}{dx^2} = \delta(x, x')$$

You will find in some places this is also written as $\delta(x - x')$ same thing ok. So, what is the sort of geometry of this physical problem? We are applying, so this is my x axis, at some point x' over here; I have applied a force is at a single point remember the right hand side is non zero at only one point right. It is a delta function. So, at x=x' I have applied some force right.

So, it is like this that the string has now become bent due to the application of this force ok. This is I mean physical interpretation is strictly difficult in this case because I am applying a very very large force at a vanishingly small point ok. So, to draw a diagram like this is slightly misleading because what is it mean to show in infinite I mean, the value of the delta function is infinity at that point.

But its just a representation to keep in your mind that what is happening as one point there is a force being applied to this system and I want to find out the response of the system that is what we will doing over here. So, this response also will follow the same boundary conditions that is; that is one assumption that we will make ok, that helps us to solve the problem and you get a consistent system of equations after that.

So, we can say this such that same boundary conditions what can I say? $G(0,x') = 0$,G(l,x')=0 because the string is clamped over there. And if we want we are going to for now just say $x \neq x'$ just so that we stay clear of this delta function. All right setup is clear next step is solving it ok.

So, we will solve it directly in the spatial domain itself ok, any hints on what should be the first step? When you look at this equation over here what is the difficult part about it is the delta function right, it is going to act in the way that will present some difficulty. So, if I want to solve something what the first thing I could try to do is to consider the case where the delta function is not present and that happens when.

Student: x is.

$x \neq x'$ right.

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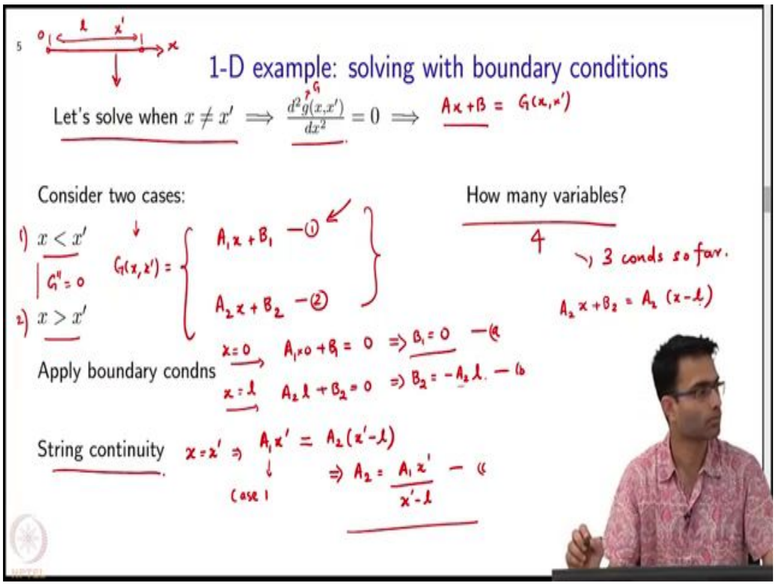

So, what we will do is we will let us solve this when $x \neq x'$. In that case, this equation over here it simply becomes right, so this should be capital G ok. So, second derivative of a function is 0 I am taking the derivative with respect to which coordinates primed or unprimed?

Student: Unprimed.

Unprimed does this have a simple analytical solution, yes what is that?

Student: It is linear.

Linear function right, so I can say that this is simply $G(x,x') = Ax + B$ right. So, this situation $x \neq x'$ can happen in two possible in two possible ways; one is $x < x'$ and the other is $x > x'$ ok, so I will just draw this one. So, again over here this is my x, this is my the length l this is 0 and some point over here is x' that is the physical situation ok.

So, when I consider $x < x'$ what kind of solution can I write? So, $G(x,x') = ?$ two cases will be there. The first case in both cases the solution is $Ax + B$ right, but will these A's and B's be the same on the left and right need not be right, so I will write it as $G(x,x') = A_1 x + B_1,\ x < x'$ and $G(x,x') = A_2 x + B_2,\ x > x'$ ok. I have stayed clear of $x = x'$ for now ok, so no one can argue with this.

In both of these cases, I have a homogenous differential equation which is $G'' = 0$ solution is linear both sides right. So, how many variables does this problem present, how many unknowns are there?

Student: 4.

4 unknowns alright; so, let us use the boundary conditions because that will help us to solve this. Already you can tell one thing that the left hand side over here is a function of x and x', right hand side seems to only be a function of x. So, these constants A and B will be functions of x' that you can sort of anticipate now itself right and that will come from boundary condition.

So, first boundary condition we can apply is at x=0, so I will choose equation 1 right x=0 comes in the 1st case right. So, this is the 1st case, this is the 2nd case. So, what does that tell me, how what does it mean?

Student: (Refer Time: 10:30).

$A_1 \times 0 + B_1 = 0$

Student: B 1 is 0.

I cannot say anything about A_1 right. Next condition happens at $x = l$ right, so here I will say that $A_2 l + B_2 = 0$ right, so I can say that $B_2 = -A_2 l$ ok. The other thing is that we physically expect that the string does not break, so that is equation of continuity right. So, the equation of continuity will happen at the point at which I have applied the stress right other places anyway it is trivially continuous, so this happens at x=x'. So, that implies that. So, what do these two equations tell me? So, B_1 is anyway 0. So, what is the left hand side at x'; what is left hand side at x'?

Student: A_1.

$A_1 x'$ yeah, but we are trying to get a continuity, we use $x \neq x'$ to derive these over here, but whatever solution I get. So, let us say just to the left of x' and just to the right of x' right there are string that is coming it cannot suddenly shoot up it is a string it cannot break. So, these left limit and right limit I am going to set to each other, so that I get a continuous function we have basic definition of continuity. So, that is a left hand side right this is from case 1 and that should be equal to the right limit. So, right limit is going to give me what?

Student: $A_2 x' + B_2$.

A_2 and B_2 I can substitute in terms of this, so this will give me $x' - l$ right, so that tells me that $A_2 = A_1(x' - l)$ right. So, I started with 4 variables and how many relations do I have between these 4 variables so far? 3 right. So, I have a, b and c ok.

So, three conditions so far right is that enough not really right I need one more relation because I to complete give full information about the whole problem. What will that 4th condition be? We impose continuity, that continuity did it directly deal with the differential equation? No right I took the form of the solution I equated it. So, what is the difficult part we are not yet handled?

Student: (Refer Time: 13:40).

The differential equation at x=x'. So, now, we have no choice now we must face this right, so that is what we do next. What is the final trick?

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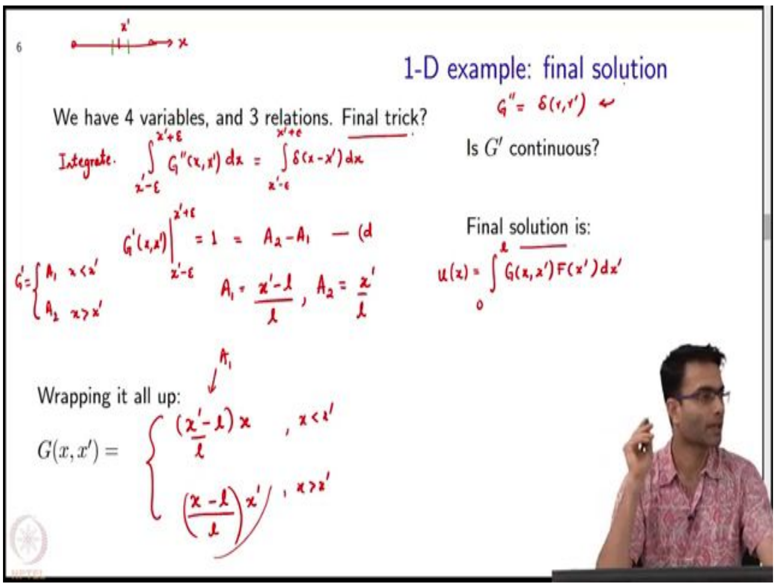

So, I have $G''(x - x') = \delta(x - x')$ right. So, if I integrate the second derivative what do I get? Integrating second derivative will give me first derivative ; obviously. So, if I integrate this intuitively if I integrate this on both sides what will I see? The left hand side will give me first derivative.

Student: (Refer Time: 14:18).

Right hand side will give me

Student: 1.

1 right, so that is something that we can use. So, let us integrate, but what should be the region of integration?

Student: (Refer Time: 14:34).

So, this is my x'. So, if I integrate at any region let us say if I integrate from here to here is there any use? No right because I am not extract going to get any relation this function is continuous over there right. But on the other hand erased everything, on the other hand if I integrate from here to here that is when something interesting will happen right.

So, let us integrate now, so this is my I am going to integrate $\int_{x'-\varepsilon}^{x'+\varepsilon} G''(x, x')dx$. I going to I am integrating over the length of the wire. So, the variable of integration is x itself and this will become an integral of $\int_{x'-\varepsilon}^{x'-\varepsilon} \delta(x - x')dx$ right. So, what does the left hand side give me?

Student: G'.

It will give me G', there integral of the second derivative will give me first derivative right. So, this will give me $G'(x, x')|_{x'-\varepsilon}^{x'+\varepsilon}$ and the right hand side gives me

Student: 1.

1: right because I have included the singularity in the integration. Do I know the value of the left hand side? I do know it right. So, the left hand side is simply going to give me, so it G'.

What is G' in both cases? A_1 and A_2 right, so this is for $x < x'$ and $x > x'$ right. So, this will give me $A_2 - A_1 = 1$ right, so that is my fourth relation that I needed right.

So, I can just simplify this I will tell you the final expression that I get I will get $A_1 = (x' - l)/l$ and $A_2 = x'/l$ ok. So, I have just done the algebra over here using the I had these I had this relation also over here relation c, I put it together with this relation d two equations and two variables I can eliminate a one a two and get it from here in terms of what is given right.

So, the final solution that I get over here I can write it again I will need two cases right for a $x < x'$ and $x > x'$. So, in the first case this will become $G(x, x') = (\frac{x'-l}{l})x, \ x < x'$. $G(x, x') = (\frac{x-l}{l})x', \ x > x'$.

So, it was not so difficult I used the boundary conditions, I used the fact I used the information of where I have placed the impulse and I got the response. So, what will I write the final solution? Final solution is u the displacement right. I am going to $u(x) = \int_0^l G(x, x')F(x')dx'$.

So, I have just applied the whole machinery of impulse responses given it a new name Green's function and shown you in a simple 1-D example how to calculate the Green's function. Now, in case that your operator over here, this in if this operator was something different, then all the calculations will change; but the idea will remain the same right. So, this is your very simple 1-D example right.