Computational Electromagnetics
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Surface Integral Equations

Surface Integral Equations
Lecture – 6.4
Formulating the integral equations

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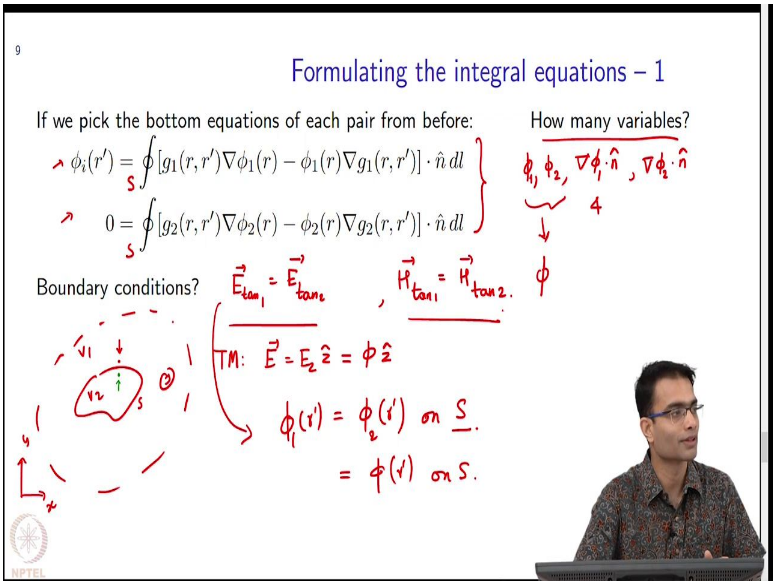

That is what I have done over here. So, the first term had the incident field, the second term had no incident field and we are interested in solving this problem now. What was remember the overall objective? I want the field everywhere. To find the field everywhere,

Student: (Refer Time: 00:29).

I need to know these terms over here. So, $\nabla\phi_1$ and ϕ_1 right, I was assumed that g_1 and ∇g_1 through some effort I have calculated. For now we will leave it as it is. So, I need to solve these equations. Let us say, I solve these equations; once I solve these equations, how will I find the field everywhere? Does this tell me the field everywhere? Do this equations tell me the field everywhere?

Student: Yes.

Where is $\phi_1(r)$ and $\phi_2(r)$ at an arbitrary location? Is it there or it is only on the boundary?

Student: Boundary.

Its only on the boundary. So, supposing even if suppose, we solve this equation how will this help us? You see the catch let us say, I solve it through some you know very good math we solve it then what? Is my objective achieved? My objective was to find the field everywhere. Go back to the volume integral, that we had started with, the current distribution on the wire. What did we do there first? We first found the charge density rho and then we substituted back into something to find the field the potential.

So, your hint is which of the theorems will be used to write down these 2 equations, we will discuss 2 theorems, one was Huygen's principle, the second was extinction theorem these 2 equations are which ones.

Student: extinction.

Extinction theorem. So, what have we not used?

Student: Huygen.

Huygens principle we have not used it, we have only wrote the extinction theorem.

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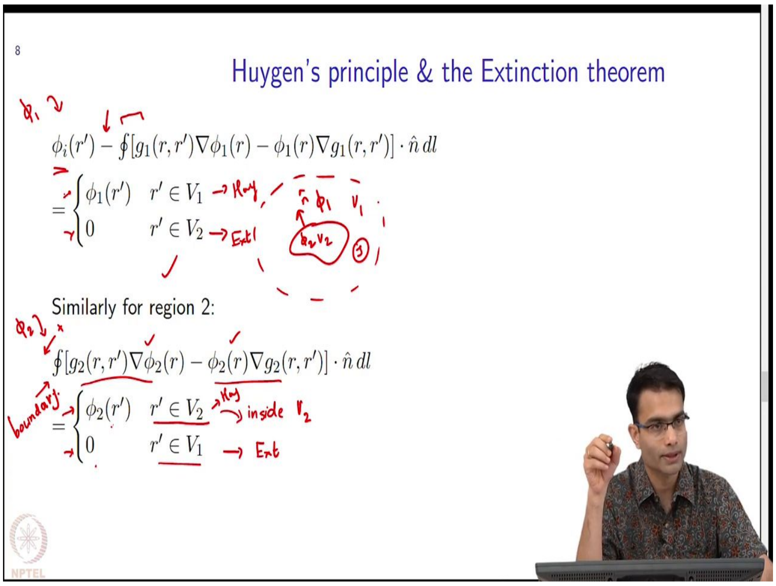

So, let us go back to the expression over here, I use the bottom 2 equations I use this and this. So, now can you tell me supposing I solve for and find ϕ_1 and $\nabla\phi_1$ how will I find the field everywhere?

Student: Huygens principle.

Use Huygens principle right because, ones I evaluate this guy and this guy I can use this theorem which will tell me ϕ_2 everywhere similarly, ϕ_1 everywhere right. So, always in an integral equations there is a 2 step process, first evaluate an intermediate variable, in that case it was charge density on the wire in this case, it is $\phi_1 \nabla\phi_1$ $\phi_2 \nabla\phi_2$ on the boundary, then I can substituted back into Huygens principle and get the field everywhere.

So, you should always have this high level picture in my mind that this is the strategy that we are following otherwise, it's easy to get lost in math all right clear. So, let us go back here. So, this was the extension theorem as I already mentioned. Now, let me ask you how many variables are there in these 2 equations? What all do I not know?

Student: Phi 1, phi 1.

I do not know ϕ_1 anything else.

Student: ϕ_2.

ϕ_2, anything else?

Student: Grad phi.

I do not know $\nabla\phi_1 \cdot \hat{n}$ and $\nabla\phi_2 \cdot \hat{n}$. So, you must be thinking that, if I know ϕ_1 I can calculate $\nabla\phi_1$, but that is actually not going to happen, because let us assume that there was no $\nabla\phi$ in these equations and only phi. We have seen that if I have an unknown variable inside an integral equation we have seen one way of solving it and what was that one way? Discretizing and solving that give you phi all right. But it gave it to you in analytical form or numerical form?

Student: Numerical.

It is gave you bunch of numbers from that, how will you accurately get grad phi? You will not get right. So, grad phi 1 dot n hat is actually your second variable it looks like, I can derived one from the other that would be true if I were getting ϕ_1 is the function, then I can take its gradient, but I am not getting it as the function. So, there seem to be 4 variables and how many equations?

Student: 2.

2 equations. So, some budgeting seems to be off right, I cannot solve for 4 variables from 2 equations, but what I keep, what I have been saying. So, far is boundary conditions I have not yet use the boundary conditions right. So, what are the boundary conditions that we have studied in the electromagnetic? Simple boundary conditions.

Student: E tangential.

Tangential E fields are conserved, tangential H fields are conserved as long as there are no pure surface currents. In this problem, there are no pure surface currents, because I did not say that it was a pure metallic object (Refer Time: 05:26) you know some dielectric object right like, you know like a piece of wood or whatever right. So, I know that physically there are no surface currents. So, $E_{tan1} = E_{tan2}$ let us write it and the second equation, second boundary condition is these are the two boundary conditions.

Student: Sir.

Student: Can we considered this scattered field is from the object volume V_2.

Student: Means, it is not penetrating itself.

No, it is penetrating of course, its penetrating its dielectric object, it has some permittivity. So, the incident field will go inside the object, some of it gets scattered by the surface, some of it will go inside and will reemerge. So, it is fully complicated situation. So, it is not, if I had a perfect metallic say object, then the field will only bounce off that is a special case of this. If I make the conductivity part of this epsilon tending to infinity I will get that relation that will become an impenetrable object but right now, I am taking it to be a penetrable object.

Student: It is a dielectric.

It is a dielectric, it has a real part or imaginary part. So, let us take the first of these equations. So, I should draw this again over here, this is V_2 and this is my current source over here. So, what was in the assumptions that I made? What polarization am I working with?

Student: TM.

TM polarization. What is the electric field in this case?

Student: E_z.

Only E_z right. So, E is actually only $E_z \hat{z}$ and the further notation that we made was we called it by the symbol phi that was the variable that I was solving everywhere right. So, remember this being a 2D problem, this object extends infinitely in the z dimension. So, if I take 1 point over here and 1 point over here and I move this point over here, and I move this point I am trying to move as close as possible to the surface.

As I move as far as possible, I mean as close as possible to the service, I will be able to ask what the boundary condition is. So, for E_{tan1}, what would be the boundary condition? On the boundary S what can you say about $E_{tan1} = E_{tan2}$? Which direction is electric field only along z, is this you know constant this z axis vector is tangent to this surface? Yes right, because this object extents infinitely out of the plane of this paper or board.

So, the z axis any point any line parallel to the z axis is tangential to the surface. So, what does this equation turn into?

Student: $E_z = \phi_1$.

$\phi_1(r') = \phi_2(r')$ on S and S is the place where I am doing this integral. So, now how many variables can I say? So, these two have been condensed into I can call it equal to some phi r prime, but only on S not anywhere else right. So, from here, I have combined I have merged into one variable.

So, it seems like, now I have three variables two equations this part clear. So, I mean you would have guessed by now, the next equation is going to give is going to help me further reduce the number of variables.

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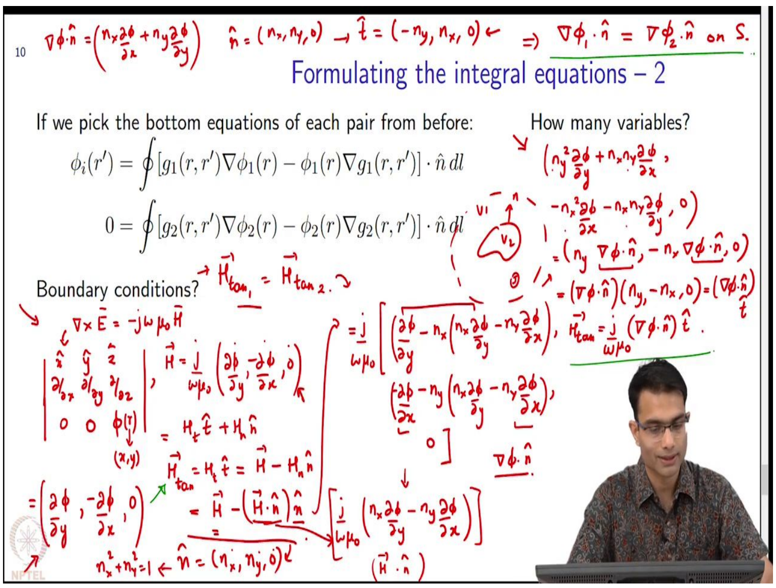
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So, let us look at it now the second boundary condition is going to be $H_{tan1} = H_{tan2}$. Now this is going to be a little bit tricky, how do I get the magnetic field now? Before I get to the tangential component of the magnetic field, I should just I should first get an expression for the magnetic field and then take its normal or tangential component right. So, how do I get the tangential field I mean, how do I get the magnetic field?

Student: $\nabla \times \vec{E}$.

$\nabla \times \vec{E}$ right. So, we have $\nabla \times \vec{E} = -dB/dt = -j\omega\mu\vec{H}$, that is my Maxwell's equation right. So, if I know $\vec{E}$ can I get $\vec{H}$ right, so I can express $\vec{H}$ in terms of ϕ right. So, let us just write this over let us write this over here. I have $\hat{x}, \hat{y}, \hat{z}$ right. What are the components of $\vec{E}$?

$E_x = 0,\ E_y = 0,\ E_z = \phi$. And ϕ is going to be a function of r , which is the function of x, y and not z right. So, which component will be which component will survive over here? Let us look at the $\hat{x}$ component. So, $\hat{x}$ component will be? $\partial\phi/\partial y$. The y component will be $-\partial\phi/\partial x$ and the z component 0. Not surprising, I started with a vector in the z direction curl of it give me something in the $x\,y$ plane right, so this is my equation. So, from here, I can say that what will $\vec{H}$ be? H will simply be, so let me get this out of here. So,

$$\vec{H} = j/\omega\mu_0(\partial\phi/\partial y, -\partial\phi/\partial x, 0)$$

So, I have got the magnetic field, at step 1. Next what do I need? I want to enforce tangential components. What do I know about the surface, what has been given to me on the surface?

Student: Normal vector.

Normal vector. So, assume that I know the normal vector. So, I can write down $\vec{H}$ as, so sort of $\vec{H}_{tan} + \vec{H}_{normal}$. I can break up any vector in the plane in terms of a normal component and a tangential component, I can do that. So, that tells me that the $\vec{H}_{tan}$ which will be along the tangential direction, I can write that as $\vec{H}_{total} - \vec{H}_{normal}$.

How do I write out the normal component of the field? Let us take the dot product of $\vec{H}$ along $\hat{n}$ right, so this can be written as $\vec{H} - (\vec{H} \cdot \hat{n})\hat{n}$. So, this term over here is what I have to enforce on the boundary right. So, it looks a little complicated, but all I have done is have extracted the tangential field in terms of the magnetic field and the normal to the surface.

So, let us try to calculate this now. So, let us see, so what should we do? We can just substitute this over here. So, what will this be? So, let us further write down the normal vector as $(n_x, n_y, 0)$. So, we can keep this $j/\omega\mu_0$ outside common, the first component over here. So, what will that be? So, I will let us take this term over here, this is simply going to give me $\partial\phi/\partial y$, $(\vec{H} \cdot \hat{n})\hat{n}$.

So, let us write down $\vec{H} \cdot \hat{n}$ first. So $\vec{H} \cdot \hat{n}$ is going to give me $j\omega/\mu_0$, what else? What is $\vec{H} \cdot \hat{n}$? It is the dot product of $\vec{H}$ which is here and $\hat{n}$ which is here. So, the x component gets multiplied by the x, the y component gets multiplied by the y and the 0 is 0. So, this is a scalar right, this is my $\vec{H} \cdot \hat{n}$.

$$\vec{H} \cdot \hat{n} = n_x \, \partial\phi/\partial y - n_y \, \partial\phi/\partial x$$

And then I am taking this scalar and projecting it along the $\hat{n}$ direction right. Everyone with me so far. So, the same component here will get multiplied along n_x and along n_y right. So, when I write down the n_x part over here, the x component of the total H_{tan} over here, what do I get? So, this is coming from $\vec{H}$.

$(\partial\phi/\partial x - n_x(n_x \, \partial\phi/\partial y - n_y \, \partial\phi/\partial x))$, this is the x component. y component is going to give me $(- \partial\phi/\partial x - n_y(n_x \, \partial\phi/\partial y - n_y \, \partial\phi/\partial x))$. So, on expected lines I have got some vector with a x component or y component and no z component right, that is what we expect. So, I have always been writing this has $\hat{n}$; that means, its unit vector. So, what does that tell me? In any relation, $n_x^2 + n_y^2 = 1$.

So, look at these; look at these two terms over here, can may be combined somehow it's $1 - n_x^2$ something, so that becomes n_y^2. And what about the other term? $+ n_x n_y \partial\phi/\partial x$.

What happens to the second term? Again this and this can be written as $n_x{}^2\partial\phi/\partial x$ will come and what will I get?

$$\vec{H}_{tan} = \vec{H} - (\vec{H}\cdot\hat{n})\hat{n} = (n_y{}^2\partial\phi/\partial y + n_x n_y\partial\phi/\partial x,\ n_x{}^2\partial\phi/\partial x - n_x n_y\partial\phi/\partial y,\ 0)$$

Student: Sir minus and (Refer Time: 17:52).

This will be minus correct and the $n_x n_y$ term will be? Again still does not look very very straightforward over here, can I take something common from, can I look at this the first term has a n_y in both the terms, the second term has a n_x in both the terms right. If I ask you what is $\nabla\phi\cdot\hat{n}$, what will you write? Let us write it over here, $\nabla\phi\cdot\hat{n}$ what you will write it as $n_x\partial\phi/\partial x + n_y\partial\phi/\partial y$. Do I observe that term anywhere over here in this expression? It is there right. Its

$$\vec{H}_{tan} = \vec{H} - (\vec{H}\cdot\hat{n})\hat{n} = (n_y\nabla\phi\cdot\hat{n} - n_x\nabla\phi\cdot\hat{n}, 0)$$

Final manipulation, what does this look like? There is something common, which is $\nabla\phi\cdot\hat{n}$. Does this look like the dot product between I mean, think of this as a not a dot product. There is a tan. So, if $\hat{n}$ is $(n_x, n_y, 0)$, what can I say about the tangential field?

Student: Minus.

$(-n_y, n_x, 0)$. That is a tangential field because look at the dot product of this, what is the dot product of this?

Student: 0.

0 right, so this is a tangential field. Now, having seen this guy over here and having seen this expression over here, can we further simplify this? Can we write this as $(\nabla\phi\cdot\hat{n})(-n_y, n_x, 0)$, which is nothing, but $(\nabla\phi\cdot\hat{n})\,\hat{t}$.

Student: H tangential.

We have finding H tangential field. So, what I have got is

$$\vec{H}_{tan} = j/\omega\mu_0(\nabla\phi\cdot\hat{n})\hat{t}$$

Granted we have to do some amount of work to get here, but patiently we have only work this is just you know working with vectors cross product dot product so on. Now finally, now we can say, so given that this is the tangential field and it should be conserved at the boundary, what does this tell us?

So, it tells us that $\nabla \phi_1 \cdot \hat{n} = \nabla \phi_2 \cdot \hat{n}$ on S right. So, all we did was we wanted the magnetic field, we started with Maxwell's equations, which allowed us to express $\vec{H}$ in terms of $\vec{E}$.

So, I got an expression for $\vec{H}$, when I got the expression for $\vec{H}$ I found out the I try to get the expression that give me $\vec{H}_{tan}$. I had to do a fair amount of algebra to write it in terms of grad phi and the tangential field which is what I got over here. And I say that it should be equal and that gave me this boundary condition.

So, the first time you see this kind of our derivation its seem a little intimidating, but you get the hang of it, the rest of the stuff is just repeating this in you know being clever about the boundary of the problem and just doing this manipulation. And notice that, we kept the surface to be completely general as long as you tell me $\hat{n}$. I can do this right, nothing very special, I did not need like a rectangular boundary or a circular boundary can be any boundary right.

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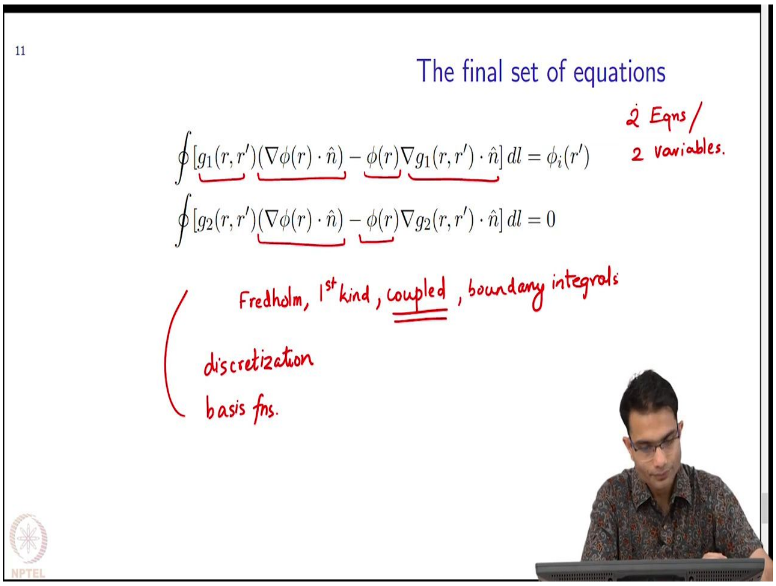

So, these are the final sets of equations, that we have are unknowns are over here, $\nabla\phi\cdot\hat{n}$ and ϕ similarly here. So, these are 2 equations, 2 variables. So, what kind of an integral equation do you think this is? We have studied different types of integral equations, this is a slight variation. So, does it look like a Fredholm or Volterra? Fredholm, because the limits of integration are unknown. So, its what kind does it look like, first or second kind? The unknowns are the inside and outside or inside the integral sine?

Student: Inside.

Only inside. So, we can say 1st kind , but what is different about it is that, I have 2 variables. So, these are actually coupled system of equations, it is not 1 equation and 1 variables 2 equations in 2 variables, these are coupled set of Fredholm integral equations of the 1st kind right, many many adjectives to describe two simple equations . How will we solve this? Actually solving these equations is now there is no great mystery over here, we already know it what is the process?

We repeat, we have done this in the previous problem. So, what would you do first? Any equation you want to solve it, what would you do discretize it right, so we would do a discretisation. Then what is the other important choice? Basis functions right, I have to choose the basis functions and solve the system of equations. The parts which we have not studied, so far are what are these guys and what are these guys? So, that will be another module on Green's functions which will come to.

So, how these equations for different from previous ones was because of the coupled nature and also these are boundary integrals. Previously we had just I mean, volume we have chopped up into segments. So, these are the final sets of equations, solving this gives us grad phi dot n hat phi 1 substitute that back into Huygens principle and I can get the field anywhere, I want that is the basic idea is this clear So, how do we go about will cover how will solve this grad detail.

Student: (Refer Time: 25:30).

We will not need to decouple them, we can solve them as we will form a linear system of equations, and solve it as $Ax = b$, that is how will solve it. That is nothing very fancy involved in solving these equation like you do not have to do some changes of variables, so some other domain where they become decoupled or at all that is not needed; just discretizing and choosing basis functions.

There is a lot of work that will go into evaluating these terms, I will give you just 1 hint, what the challenge will be. Along this boundary, so what is your variable of integration over here primed or unprimed?

Student: Unprimed.

Unprimed right. They will be points where $r = r'$ all right and when $r = r'$ go back to the definition of g_1 what happens? Is the delta function sitting at $r - r'$. So, this is singularity there. So, we have to do the evaluation of these integrals in the presence of a singularity. So, some amount of careful work is needed over there, then the rest is simple, but will get it you will solve it in great detail.