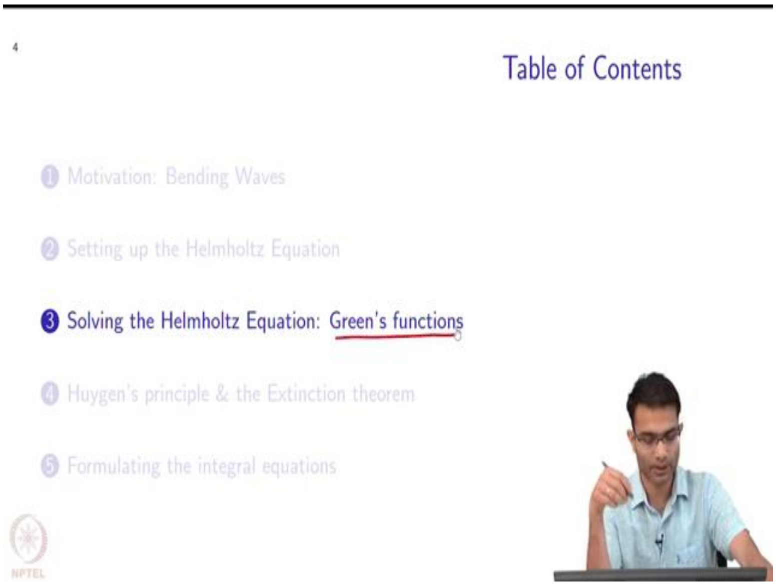
Computational Electromagnetics: Surface Integral Equations
Prof. Uday Khankhoje
Department of Electrical Engineering
Indian Institute of Technology, Madras

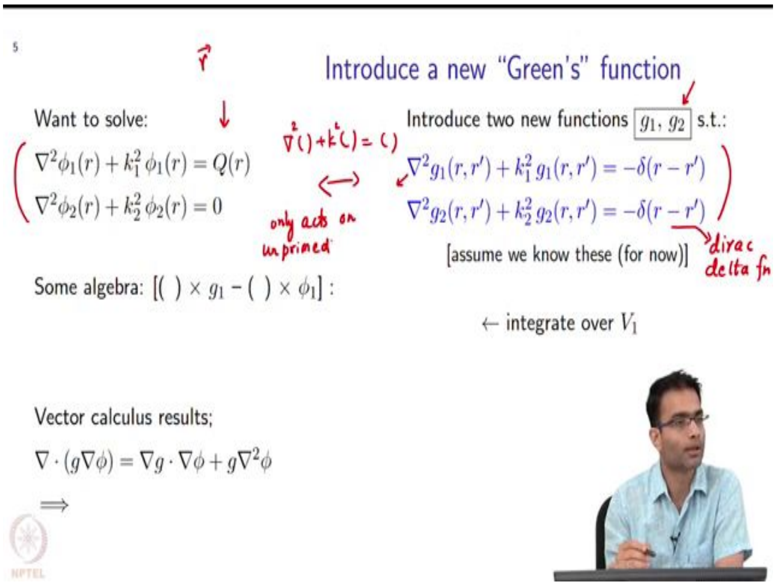
Surface Integral Equations
Lecture – 6.2
Solving Helmholtz Equation

(Refer Slide Time: 00:14)



Now, turns out based on what we already know, we actually cant solve this problem; we need a new mathematical technique. So, let us go in to see what that the new mathematical technique is and that is the technique of greens functions ok. You would have heard the name of green in the context of vector calculus some identities of vector calculus are named after green right. So, he worked a lot in this area, so things go after his name right.

(Refer Slide Time: 00:38)



So, lot of material here, let us go one by one; I have these two equations the entire current source part has been condensed into one term Q ok. This is physical constant multiplied by the current, so, I am going to call it Q. Now, as I said these two equations are not sufficient to solve this problem, so, now I am going to seemingly make life more complicated before making it simple again right.

So, what is the new complication? I have introduced two new functions I am calling them g1 g2. And what are their properties? They satisfy these two equations. So, let us just read out the first equation, so, it is says $\nabla^2 g_1(r,r') + k_1{}^2 g_1(r,r') =- \delta(r-r')$ ok. So, this delta function is your the your usual Dirac delta function right.

So, we all know what are the what is the property of a Dirac delta function, it is 0 everywhere and infinity at the point at r=r', and it is not actually a proper function right. So, but I can talk about it's integral, the integral of delta function as long as I cover this point of singularity is 1 right.

So, what I have done is I have introduced these two new functions, also because we are going to write $\vec{r}$ and $\vec{r'}$ again and again and again, I have drop the vectors sign over r ok. So, even though I have written just r, it is understood that when you see r it is actually $\vec{r}$, but it makes the whole thing very clumsy. So, I have this remove the vector side.

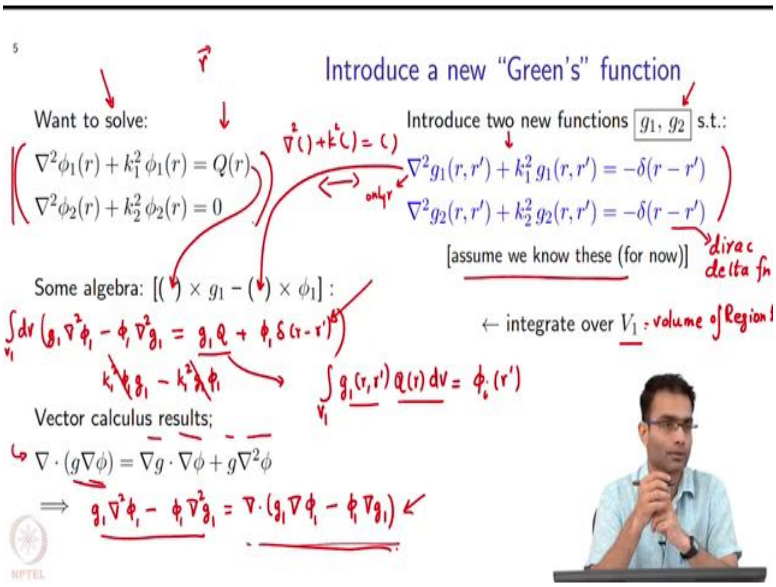
So, it should be clear to you from the context when I am referring to the vector and when I am referring to the value the absolute value of r ok. So, before we get into anything does this equation remind what are the similarities between these two equations are any similarities.

Student: (Refer Time: 02:47) looks the same.

Looks the same right, so, the both of these equations look like this ∇^2 something plus k^2 something is equal to something; that seems to be the general template into which these two equation are fitting right. If I make the right hand side non zero, I get the second equation. The other if ask you what is the difference the first equation is equation in only one variable r.

The second equation now I have introduced one more variable r' ok, which will seem little strange at first, but will see it will make life very powerful. Finally, this del squared operator this is acting only on unprimed co-ordinates ok, so, when it encounters any function of r' it is a constant right. So, this we will say only acts on unprimed that is fine, so that is just a assumption. So, I think I will not clearly write it over here ok, so, now how do we actually proceed with this.

(Refer Slide Time: 04:11)



So, so, well the trick that we will do is, we will take the first equation; supposing I take this first equation multiplied by g_1, take the second equation multiplied by ϕ_1 ok. And subtract these two, by subtracting these two is there any advantages that I get.

Student: The right hand side.

The right hand side, the right hand side nothing much will happen to it. On the left hand side how many terms are there it should be 4 terms, but a result of multiplying this by g_1 and that by ϕ_1 what will happen, what will happened with the k^2 terms.

Student: (Refer Time: 04:50) cancel.

They cancel off right. So, I will have a $g_1 \nabla^2 \phi_1 - \phi_1 \nabla^2 g_1$ right, k^2 term goes away, and now I am left with the right hand side. So, the right hand side was, rather is $Q + \phi_1 \delta(r - r')$ ok. I am being a little sloppy here you know that all of these guys g_1, ϕ_1 all of these guys actually functions of r and r' ok.

Student: (Refer Time: 05:36).

Ok so what will be the k_1, so, I will have a $k_1 \phi_1 g_1$ from the first term and from the second term I am subtracting them. So, I will have a $k_1 \phi_1 g_1$, I am subtracting these two equations right. So, this is canceled, so, I am just left with this all right.

So, still looking at this it is seem even more confusing when we first started out right, but it is a few steps that we have to take. All we manage to do is get rid of this $k_1 \phi_1$ term ok, we still need to do a little bit more vector calculus.

So, here is the identity for what do we call this the product rule in vector calculus; so, I have two functions g and $\nabla \phi$. So, when I want to take the derivative of the product the in vector calculus it becomes divergence right. So, before we get in to anything look at the left hand side is a left hand side a scalar or a vector.

Student: Scalar (Refer Time: 06:50).

Grad.

Student: (Refer Time: 06:54).

Yeah ∇g is a vector.

Student: (Refer Time: 06:56).

g is a scalar, so, the this is overall a vector.

Student: Right.

And when I take a divergence of a vector I get a scalar.

Student: (Refer Time: 07:01).

Right look at this term this is a vector this is vector dot product scalar.

Student: (Refer Time: 07:07).

∇^2 of a scalar is scalar multiplied by g right, so, at least I mean dimensionally it make sense I am not writing scalar equal to vector. So, this kind of basic check you should do you write down this expression ok. So, looking, using this identity, can I rewrite the left hand side of this equation? So, if you look at this one if I try to write down $g_1 \nabla^2 \phi_1 - \phi_1 \nabla^2 g_1$, I can write that as equal to what will I write it.

Student: Del dot del dot g 1 phi.

Del dot g 1 grad.

Student: Phi 1.

Phi 1.

Student: Minus delta phi 1 (Refer Time: 08:02).

$\nabla .(g_1 \nabla \phi_1 - \phi_1 \nabla g_1)$

Correct, because again $\nabla \phi_1 \nabla g_1$ it will come both times it will be cancelled out all right. So, this is one manipulation that we did, so, we will keep this result with us we hold it in memory. Now, so far I mean we started by saying that we are going to talk about integral equations but, so far there is no integral anywhere inside right.

So, to introduce that integral what we need to do is, let us take this equation that I have and let us integrate it over volume V let us, so, let us integrate it over volume V_1 ok. So, this whole thing dV over V_1 that is what we are going to do ok. Do any of the terms do you see them simplifying.

Student: V_1 to (Refer Time: 08:04) region.

Yes V_1 is the region of is the volume of region 1 ok. So, the only possible term that might simplify in this volume integration is which one, what about the very last term.

I am integrating over a delta function that is the only term that seems to be that it might simplify ok. The other terms it is not very clear how they will get simplified ok, so, let us let us see how they will.

Student: (Refer Time: 09:44).

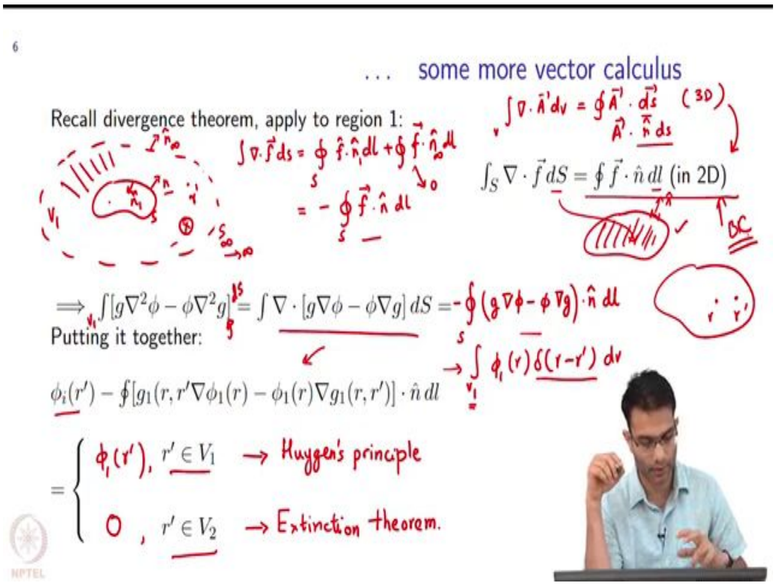
∇^2.

Student: Right.

Student: You can look at that it has sub (Refer Time: 09:48).

We are you are getting to what I am getting to next, you notice that when I integrate this term over here over volume, this becomes an integral of this over volume right.

(Refer Slide Time: 10:02)



So, that is what we are coming to, when I have so what was the divergence theorem in 3D, it said that $\int_V \nabla .\vec{A} dV = \oint_S \vec{A}.d\vec{S}$.

Student: Over (Refer Time: 10:17).

Over ds that was in 3D we are used to thinking of divergence theorem for all the years of schooling, we are used to thinking of it is as 3D thing right there is a volume and there is a surface flux.

But actually there is nothing preventing you from formulating this theorem in higher or lower dimensions ok. The main idea is that if I have a volume in N dimensions on the left hand side, I have a the corresponding surface in N-1 dimension on the right hand side. So, since our problem is actually 2D not 3D, the divergence theorem get simplified in 2D as follows, so, I have to drop one dimension.

So, this volume, so the closed volume, so the volume finite volume right. So, this will become a finite surface right and dV will become dS will become dl and remember this ds is nothing but $\hat{n}dS$ like this right. So, this is you divergence theorem in 2D right, and if you want think of it as a surface it's like this. Some surface the left hand side is being evaluated inside and the right hand side is being evaluated on the boundary it is a contour integral.

Student: $dl\,\hat{n}$ is not (Refer Time: 11:42) $\hat{n}$ is normal.

$\hat{n}$ is a normal, so, it is a outward.

Student: (Refer Time: 11:44) outward surface.

Yeah, so, dl is this.

Is the flux through it is the flux through dl in the normal direction, over here was in that the same interpretation is a flux of A.

Student: (Refer Time: 11:57).

Along $\hat{n}$ in a small area patch that area has now become line length element right.

So, it is a one to one mapping of this divergence theorem to 2D ok, and this is again very simple if we just use the definition of divergence and all you will get this, if you wanted derive it ok. So, we need to now apply this divergence theorem to region 1 over here ok. So, our region 1 ok.

So, let us see, so this is our region 1 I am trying to draw region 1 over here this is my v 1 right region 1, what are the two boundaries of this I have S and S_∞ ok. And I have drawn the normal vector like this actually if I think of these as finite surfaces; if I think of S_∞ as a finite volume. Then when I apply the divergence theorem to this volume over here, how many surface integrals will I have to take care of. In this case the volume was enclosed by one surface and so I had to take care of the flux through that surface right.

So, this was very straight forward, in this case if I applied divergence theorem to this volume I should take care of the flux that is leaving the volume right. That is the conceptual meaning of divergence theorem, divergence of sources inside is equal to outgoing flux. Now, in this case this volume one is bounded by how many surfaces? Two surfaces flux could be leaking at through S outwards or through S_∞ outwards ok.

So, when I write down this divergence theorem, I have to right down the flux through both surfaces right. So, let us call the opposite of this vector over here right that is the outward normal over here. Let us call it n 1 ok, and let us call this n_∞ ok. So, the divergence theorem applied over here will give me the total flux. So, it will give me over s as $\vec{f}.\hat{n}_1$.

Student: $\hat{n}_1$.

$\hat{n}_1$ right this is the outward normal, because I am looking at all the flux that is leaving this volume. So, this is $\hat{n}_1$ outward flux plus $S.\hat{n_\infty}dl$; this straightforward budgeting of the fluxes right.

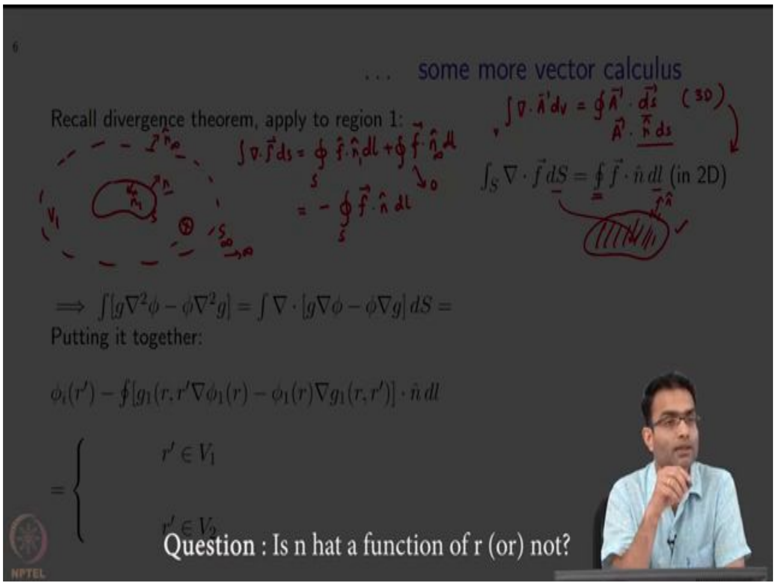
Now, as S_∞ is actually at it is far away and our source is fixed over here. So, what we physically expect to happen on the boundary s infinity.

Student: Field.

Field should go to 0 right, so, this term is going to go to 0 ok. Coming to the second term over here, what I will write it as second terms survive as is ok, $\hat{n}_1$ and $\hat{n}$ what is the relation they just opposites of each other. So, I can write this as ok, and only reason is because I had earlier decided to call the normal to this surface as this normal going outward is in this direction.

But when I am doing the divergence theorem to volume 1, you notice that I have to take the correct out going flux vector right. So, I could solve the whole problem with $\hat{n}_1$ or I could solve it with $\hat{n}$ convention depends on me ok, I am going to choose $\hat{n}$. So, $\hat{n}$ is the inward normal that is why there is minus sign.

(Refer Slide Time: 16:11)



Student: (Refer Time: 16:12). (Refer Time: 16:14) Is $\hat{n}$ a function of r (Refer Time: 16:16).

Yeah as a function of where the boundary is

Student: Yeah so (Refer Time: 16:22) everything (Refer Time: 16:23) have to write an (Refer Time: 16:24).

Yes.

Student: So, at the position of the co ordinate has some value. And other (Refer Time: 16:31).

So $\hat{n}$, so, this remember this is integral is a contour integral, it is evaluated only on the boundary, it is not evaluated in a volume right. So, it comes in the picture only on the boundary where a normally has been defined. So, we will we will make use of this, going back to what the vector calculus identity we had on the previous page was this (Refer Time: 16:52) right and that integrated over volume this is dV is missing over here right.

And in this case is dS, because I have moved down to a 2D problem ok, using what we have derived earlier this became $\nabla.$ this whole thing. And $\nabla.$ this whole thing becomes what here is where we apply our divergence theorem right. And so this will become $-\int_S (g\nabla\phi - \phi\nabla g).\hat{n}dl$ ok, and this is over V_1 that is fine.

So, nothing very complicated we basically took our we started with our two equations over here these two guys, subtracted with these two greens functions I have not told you anything about these greens function. So, we are just assuming that we know them ok, in another module we will talk about how to derive them.

But for now let us assume that they are known, we subtracted them and applied some vector calculus to get this relation after which we applied our divergence theorem to get this relation right.

Now, if I go back to this equation, this also was going to get integrated with respect to volume right. So, this was a $\int_{V_1} g_1(r,r')Q(r)dV$ right that was what the volume integration was. So, if I do all this integration this these are all known things ok, I have not told you how to calculate g, but we will get to it.

So, g_1 is known Q has been given to in the problem, because Q is capturing what the current source, so, this term is given to you ok. So, this is known to me I am going to call this the it like the incident field because it has information about the current source, what comes out from a current source some incident field which is going to get scattered right. So, I am going to call this ϕ_i what should it will be a function of r or r'.

Student: Prime.

r' right, because I am integrating over the place where the current source and all is. Current source is a function of r right. So, I am going to be left with r' ok, and that is thanks to this guy because g_1 is the function of r and r'; if g_1 where a function of r only then it will just be a number this integral right. So, now, we are going to put all of these chunks together this is that term we have to keep a bit of eye on right, because all other terms you simplified.

So, ϕ_i became that current term this minus this term over here has appeared over here ok, what is left was this final term over here right. So, that is $\int_{V_1} \phi_1 \delta(r, r') dV$. So, what will the value of this function be? I have a delta function it is multiplying by another function and I am integrating.

Student: $\phi_1(r')$.

So, $\phi_1(r')$ right that is almost correct, so, look what is happening over here right. So, this is let us say I have one point over here r I have one point over here r', so, if the volume of integration if it.

Student: It will only induce 1 (Refer Time: 21:01).

I will only induce well not one but.

Student: (Refer Time: 21:04) two, that is having one integral (Refer Time: 21:06).

So, this is this dirac delta function is a function in those many dimensions. So, if it is a volume integration, it is a three dimension derived delta function, it is 2D case, so, it is two dimension derived delta function. So, wherever the singularities that is wherever r=r' if it is there then this delta function will extract the value of the function inside the sign right. Except two conditions are there when I say what I am doing is, I am integrating over this region volume 1, that is what I am doing not the entire space right.

So, have I said anything about the location of r' I have said absolutely nothing right when I introduce r' in this equation, I did not say word about where r' is it just came as some variable right. So, when I look at this volume over here I have not told you where r' is r' could either be in V_1 or it could be where.

Student: V_2.

V_2, if r' is in V_1; that means, r' is let us say here, and I am integrating over that volume which contains r'; that means, there will be one point where r is equal to r' and when I integrate the delta function over that volume what will happen to this integral I will get.

$\phi_1(r')$.

Prime exactly, on the other hand if r' were inside here and I am integrating over V_1, what will happen what is the value of the delta function.

Student: (Refer Time: 22:39).

Is 0 everywhere right, so, you can see how Maxwell's equations and vector calculus the sort of going like a like dance all most helping each other along the way. We have to come to this final equation over here right, and in the subsequent slides we will give a very beautiful physical interpretation to it ok. So, we will talk about that subsequently, this the top equations there are actually 2 equations are over here one is when r belongs to r r' belongs V_1 and the second is r' belongs to V_2.

So, when r' belongs to V_1 you do not recognize it yet, but this is actually nothing, but Huygens's principle ok. And we will give a physical interpretation to every term; this thing is something that you have not encountered. So, far or even heard the name of, but this is called the extinction theorem ok.

So, one good question is that why can't we think of solving these equations by themselves right. In the previous problem where we had the line charge we just took them discretize ϕ_1 integrated and got the solution. The trouble over here is that Maxwell's equations have to you also have to impose boundary conditions right, because you have two different volumes right.

So, what is to be respected on the boundaries tangential E field tangential H field has to be conserved. When I look at just these two equations over here these two equations there is no mention of boundary conditions right; this is a point by point relation over here and there is no way for me to impose the boundary condition.

Student: ϕ_1 and ϕ_2 are just what we introduced as 2 (Refer Time: 24:34).

As 2 different variables.

Student: Sir they do not be anything (Refer Time: 24:37).

Well mathematically they are the electric fields.

Student: (Refer Time: 24:40) they do not have the volume differentiate (Refer Time: 24:44).

They do not know which is volume 1 which is volume 2 right, so, that imposing boundary conditions has not is not possible purely with this. In the other problem there was no boundary really right the charges are there over surface and that is it right. Here that is why I am subtracting these equation and then integrating over one particular volume.

So, we integrated only on V_1, then we will integrate only on V_2 and you can see that this equation that I have right. This is actually now operating only on the surface. This right hand side over here is going to be the way I am going to impose the boundary condition.

That is where so that volume has been converted to a surface and boundary condition will get applied on the right hand side over here ok. So, that is why we do this trick of integrating over volume then using divergence theorem ok, good question right. So, with that, we will bring this discussion to an end and subsequently we will look at the physical interpretation of these terms.