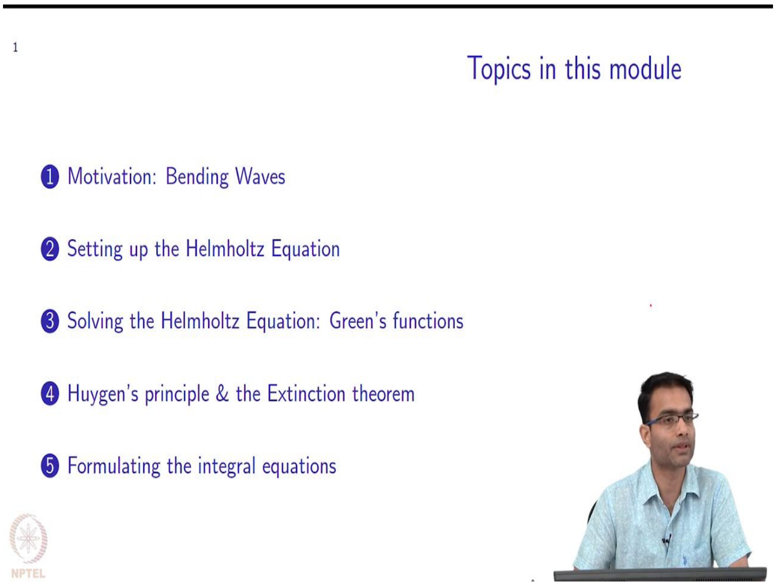
**Computational Electromagnetic**
**Prof. Uday Khankhoje**
**Department of Electrical Engineering**
**Indian Institute of Technology, Madras**

**Surface Integral Equations**
**Lecture – 6.1**
**Helmholtz Equation**

Continuing our discussion that we have been having about integral equations, what we looked at was a simple static case electrostatics problem we found out the charge distribution. Now, we will begin to move into the area of electrodynamics, where there is a time varying field as well because that is the main kind of problems that we are interested in this course.

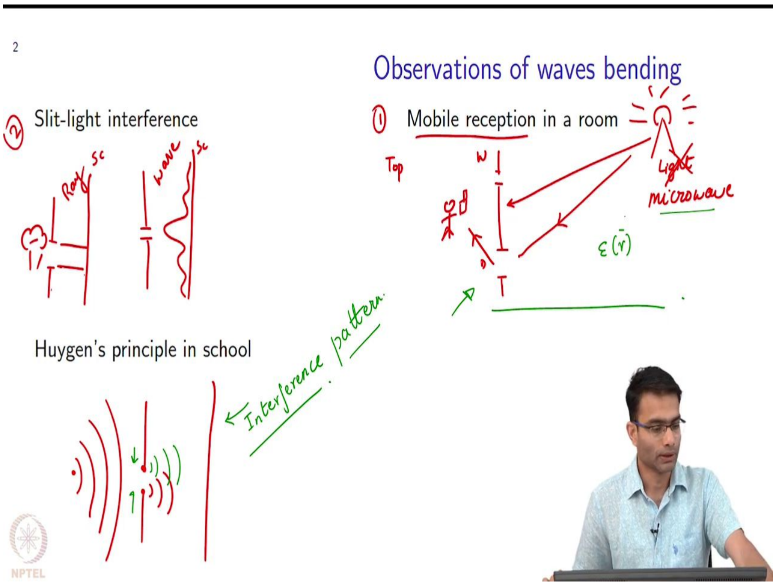
(Refer Slide Time: 00:42)




So, what you looked at was a very simple kind of integral equation, now we'll try to make things little bit more interesting. So, these are the topics that we will cover in this module, we'll start with some physical intuition about how and where wave seem to bend that such the motivation.

And from there we will see how the idea for this idea leads us to what is called the Helmholtz equation and ways of solving the integral equations that come ok. So, let us just start with the idea of waves that are bending right.

(Refer Slide Time: 01:11)




So, let us start with this simple idea of mobile reception in room. So, for example, you all are sitting here in this room, if you pull your phones out hopefully if it is on silent you should find you should find there is that you are getting a signal right if I ask.

So, let us say you are sitting over here right, so this let us call this a top view and this is your let say this is a window and let us say there is a door over here and your mobile tower is somewhere over here right which is sending out signals everywhere. Now, there may or may not be a direct line of sight between you and the mobile tower right.

So, for example, if I draw straight line from here to here it does not go. On the other hand there is a path that seems to go like this and then come to you over here ok. So, if this were light would you be able to see the tower? Of course not right, there is no line of sight you cannot see the tower, but thankfully this is, it is not light it is a microwave. And empirically we find that we are able to get reception on our phone over here, even though there is no line of sight.

So, there is some idea that the wave is somehow bending around obstacles and we are able to receive it this is in day to day life. In physics also we have seen this happen all of us have studied Young's Double Slit experiment and all of these things in high school right. So, I mean the basic idea there is supposing I have light bulb over here and I put a slit and a screen over here. Now, this slit is much bigger than the wave length of light.

So for example, I take a you know 100 watt bulb and I put in front of it barrier with a hole in it and that hole is let us say you know 10 centimeters. So, what is the size of the spot that you will see on the screen, so let us call this is as a screen. If this hole is 10 centimeters wide, how what kind of a spot size will you see on the wall?

Student: Almost.

Almost 10 centimeter right there is no bending happening because it is 10 centimeter this is light will just go in a straight line, so that is what you see, but when you begin to shrink the size of the slit right. So, when you make the size of the slit much smaller right, so this is your screen, then what happens is that you observe that there is a interference pattern on the screen right you will observe something like; something like this is happening which cannot be explained if I assume light to be just rays right.

If it were rays then this is the picture, but this is the wave picture. So, in both of these examples example 1 and example 2, we get a physical idea that waves are seeming to bend ok. So, in school we studied that this is happening because of what was called Huygens principle, did everyone come across this word in school right. So, the idea was that there is wave that travels hits some obstacle in this case the slit and there are secondary waves that are generated right. So, if I just draw this over here.

So, if I take a light source over here which is sending out waves, so these are the waves fronts. When it hits over here this and these two guys they become like secondary sources and they start emitting their own waves. So, this is waves from the first guy and these are waves from the second guy right.

And as a result of this over here on this screen, what happens? These waves interfere right and you observe, interference pattern appears over here ok. So, hopefully this is familiar to all of you the idea of a secondary wave front from which now the rest of the, once I get the secondary wave fronts I can forget about the primary excitation and I get this excellent agreement between theory and experiment to explain the interference ok.

So, as you vary this width that is the other thing you will observe, as you vary this width you will observe that the interference pattern changes and that makes sense because I am changing the interference condition, I am right the relative path difference is changing as I move this slit away. So, what we will look at in this module is how can we study this Huygens principle, how can we bring it into our current understanding with all the sophistication of Maxwell's equations and in short can I study it mathematically ok.

For example to this problem which is which has your microwave, tower and sending your waves to you can I study it mathematically and why would that be of interest? Various things maybe you know telecom provider and I want to understand under what conditions will all my subscribers get good signal strength that can be our requirement right.

So, I want to be able to simulate and predict what are the fields given some obstacles. So, that can be a motivation for solving this. So, in short in this problem what is given to you, what all do you think is given to you in a problem like this microwave problem?

Student: Source position.

The position of the source right. So, I have told you that tower is here. What else should be known?

Student: Obstacles.

Obstacles. So, the geometry of the room should be given to us right, what else?

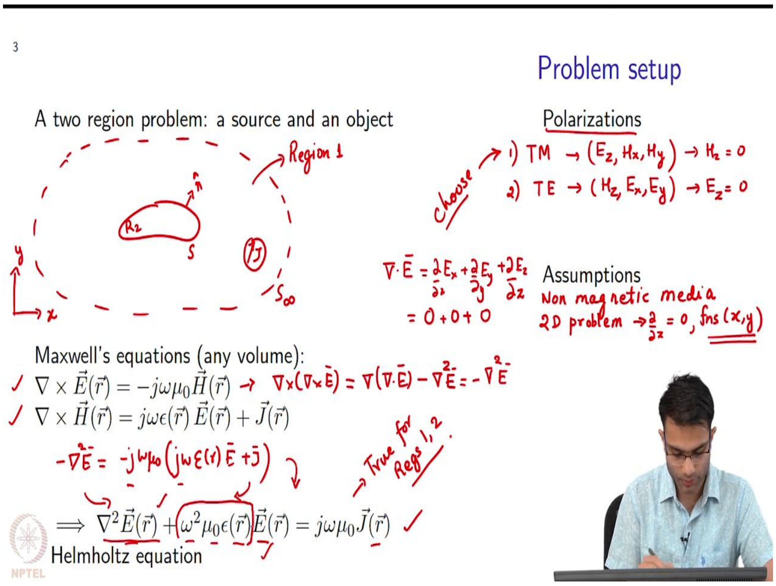
Student: Observer.

Observer right. So, observer could be anywhere, so I will try to solve for the fields everywhere right. So, this is what is given to me, I am given the position of the sources, I am given the geometry and my geometry by now we know what do we mean by geometry into what is geometry encoded in Maxwell questions?

Student: Permittivity and permeability.

Permittivity and permeability right. So, basically $\varepsilon(r)$. So, this is the (Refer Time: 07:34) we should keep in mind as we go towards solving these equation ok. So, whenever you get lost come back to this picture and see what have we solved so far in this picture fine alright. So, now, we will go about setting up these equations mathematically.

(Refer Slide Time: 07:49)




So, we are going to take a two region problem to keep things very simple. So, let us say that there is one obstacle over here and it is inside another medium let us call this thing over here and let us say that there are some sources over here ok, so let us call this is a source. So, this is exactly like our mobile tower problem J is the location of the mobile tower and this object over here inside is some I mean some obstacle could be anything ok.

And so, what I will do is, I will call this as region 1 and this is region 2. This dotted line that I have drawn over here is the outermost boundary which actually it does not really exist, so I am I mean it is far away, so let us just call it $S_\infty$.

Later on we can study how to make more bring more and more obstacles in to it, but once we understand one obstacle we basically would have understood everything else let us call this surface S over here ok. So, this is the problem set up. What differentiates region 1 from region 2 in your opinion?

Student: (Refer Time: 09:13).

$\varepsilon(r)$ right, $\varepsilon(r)$ will be different that is why region one. For example, can be air $\varepsilon_r$ is 1 and this region 2 could be something else wood right, so some different $\varepsilon(r)$. So, now, let us start with as with any problem in computational em we start with Maxwell's equations right, at the end of this course you will be amazed that how every problem we just start with these two or four equations and we get a solution ok.

So, these are the two equations that I have written there is your $\nabla \times E$ and $\nabla \times H$ and the current term come in the $\nabla \times H$ term right. So, this should be familiar to all of you by now. Now, what I would like to do is study this use these equations in this particular problem ok. So, lets say I want to eliminate from here there are two equations in two variables right what are the variables?

Student: (Refer Time: 10:09).

Variables.

Student: E and H.

E and H that is what is unknown that is what I want to determine and what is given to me is epsilon and J right that is what is given to me. So, let us try to eliminate one of the variables from these two equations, so let us make into a equation of one variable, let us remove the magnetic field ok. So, how do you suggest I do that?

Student: Curl of the first equation.

If I take a curl of the first equation right. If I do $\nabla \times (\nabla \times E)$ del cross del cross E and we have an identity from the vector calculus which tell us this is equal to?

Student: (Refer Time: 10:42).

Ok.

Student: So, there is neglecting the (Refer Time: 10:56).

Yes. So, we are assuming that as with almost all real life problems the there are no magnetic materials around ok. So, that is why I have taken $\mu = \mu_0$ ok. So, if you want we can say in assumptions non magnetic media ok. this is not a very fundamental assumption I can relax it, the math becomes a little bit more complicated, but nothing that we cannot handle ok. So, we will deal with non magnetic media for now ok.

Now, if I try to solve this problem in the most general 3d way it is going to be a little bit complicated. So, what I will do is, I will assume that my problem is two dimensional ok. So, the entire problem is in the x y plane. So, what does that mean, what does that tell us about the z coordinate?

Student: $\frac{\partial}{\partial z} = 0$.

or in simpler words there is no physics happening in the z dimension. Everything is invariant in the z dimension. So, this obstacle is infinitely long in the z direction like a tall cylinder right. So, it also means that all my functions all my fields are functions of only x and y right so ok.

Now, when we deal with a 2D problem all of you who have probably studied wave guides and such things in your undergrad, you know that we can talk in terms of a general problem can be studied broken up in to a two orthogonal polarizations, transverse magnetic (TM), transverse electric (TE) right. So, those are two polarizations that we can we can talk about and any general solution or any general problem can be answered as a superposition of TM and TE.

So, may as well break solve TM separately solve TE separately right it just makes a life simpler. So, there is transverse magnetic polarization ok, transverse magnetic. So, how do I characterize the fields in the case of transverse magnetic?

Student: (Refer Time: 13:16) $H_z = 0$.

$H_z = 0$, so then $E_z \neq 0$ right. So, we can say that this is characterized by $E_z$ and what else?

Student: $H_x, H_y$.

$H_x, H_y$ these are the unknowns in the case of TM polarization and analogously for TE polarization the other sets of unknowns are there. So, what would be the unknowns? $H_z$.

Student: $E_x, E_y$.

$E_x, E_y$ right. So, here we are saying $H_z = 0$ here we are saying $E_z = 0$ ok. Now, one thing I want to caution you this idea of what is called transverse magnetic and what is called transverse electric, you will find two different conventions in the literature. The wave guide people have a certain convention for what is TM and people in electromagnetic scattering they have the reverse convention.

So, you might find it confusing when they say TM or TE just see which of the z component is being set to 0 ok. So, that is unambiguous because you might find in some places the other way also alright. So, to further you know simplify the mathematical derivation that happens let us make one more assumption, let us say that let us deal with the TM polarization ok, TE can also be done, but let us start with TM polarization. So, we are going to choose.

So, I mean the basic motivation behind all of these assumption is there are first surface integral equation we want to make it as simple as possible ok, once you get the hang of it you can generalize whichever way you want. Now, as a result of this if I choose the TM polarization what do you think will happen of $\nabla.E$, the divergence of the electric field. So, let us just use the formula, so this will be first term will be?

Student: x.

$E_x, E_y$ right. TM polarizations what happens for the first term?

Student: 0.

0 there is no $E_x$, second term.

Student: 0.

0, third term 0 right because $E_z$.

Student: (Refer Time: 15:59).

Is nonzero, but $E_z$ is not a function of z right we said that all things are function of only x and y right. So, all three terms are 0. So, that is great for us because what happens to this $\nabla \times \nabla \times E$ that I have written over here only one term survives right, the divergence of E goes to 0, so I have a $-\nabla^2 E$ ok.

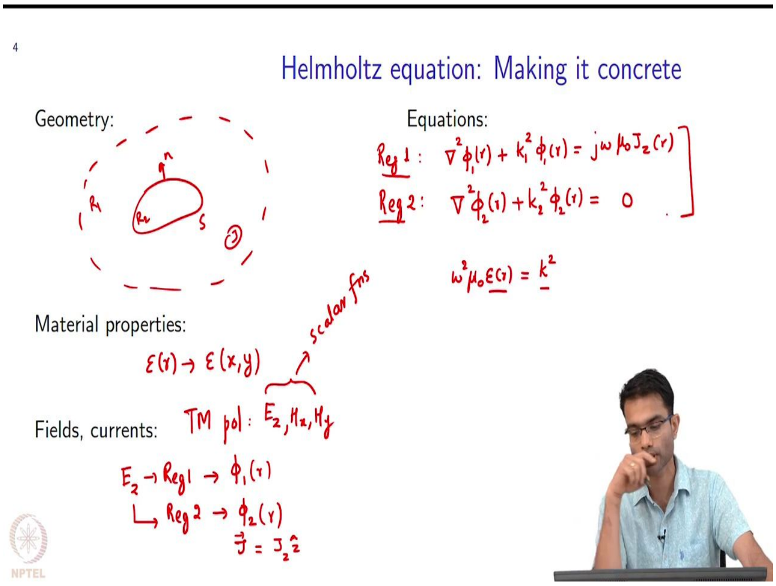
Now, I can substitute this I mean this I took the curl on the left hand side similarly I will take the curl on the right hand side as well. So, what will I get? I will get $-\nabla^2 E$ . This is from the left hand side and the right hand side will become $-j\omega\mu_0(j\omega\varepsilon(r)E + J)$ right.

So, what I can do is just keep the j term on one side, so this is the equation that follows right. So, do all the signs check out, so $-j \times j = 1$ right. So, this $-j \times j = 1$, I bring this $\nabla^2 E$ on the right hand side and I move this J term on the left hand side right. So, that is why, so this is the equation that I get $\nabla^2 \vec{E}(\vec{r}) + \omega^2 \mu_0 \varepsilon(\vec{r}) \vec{E}(\vec{r}) = j\omega\mu_0 \vec{J}(\vec{r})$. So, this is called the Helmholtz equation some people may call it a non homogeneous Helmholtz equation because the right hand side is non zero ok.

Student: (Refer Time: 17:48) question is for region 1.

This is in general because I have not made any assumptions right, whether this is this is true for both region 1 and region 2. What happens in region 1, that epsilon r will be for vacuum and current will be there. In the region 2 what will happen epsilon will be of the object current will be 0. So, this one equation is true for regions 1 and 2, so for so good. Also this surface I am going to characterize by a normal vector over here I will need that alright should we go ahead ok.

(Refer Slide Time: 18:32)




So, let us let us just make it concrete we had a lot of vectors in a in this slide we had a lot of vectors everywhere. Now, because we are dealing with the 2D problem let us try to just simplify notation as much as possible. So, this is our geometry again, this is my region 2 with the normal over here and surface S and this is my region 1 and this is my current source J.

So, material properties we have already seen $\varepsilon(r)$ characterizes the material properties and this $\varepsilon(r)$ is going to be a function of x and y right. Now, the fields and the currents so we are working with TM polarization right, for which $(E_z, H_x, H_y)$ this is what we said.

Now, each of these three quantities I mean they are scalars right $E_z$ is the scalar if the z component of the electric field, so this is the scalar function right. So, all of these are scalar functions ok. So, what I will do is, I am going to use, I am going to denote $E_z$ in region 1 as $\phi_1$ and in region 2 as $\phi_2$.

If you recall the previous equation was a equation only in electric field. So, that is why I am introducing a simpler notation for the electric field in terms of scalar. So, I do not have to keep writing vector; vector everywhere. Similarly, the current if I again go back to this equation the first quantity over here is it a scalar or a vector?

Student: Vector.

It is a vector right $\nabla^2$ is acting on a vector it will produce a vector. In what direction is this vector is in which direction, electric field in which direction in the TM polarization?

Student: Z.

Z direction. So, the left hand side the first term is a vector in the z direction, second term is a vector in which direction? Z direction bunch of scalars multiplying a vector in the z direction, so z. So, it make sense to also have z current directed along the z direction. So, if I take the z component of this entire x equation that is what I will get over here.

So, this J term over here I will write as $J_z\hat{z}$ alright. So, with these simplifications made in place let us write down the Helmholtz equations for both the regions ok. So, region 1 first term was $\nabla^2\phi_1(r)$ and when you had this $\omega^2\mu_0\varepsilon(r)$ right. So, for.. all of you know that we can simplify we can get rid of this $\mu_0\varepsilon(r)$ and write it in terms of speed of light and that speed of light can get converted into a wave number $k = 2\pi/\lambda$ right.

So, once you do all that simplification I can just condense it into a new constant $k_1{}^2$ as the wave number in the region 1 $\nabla^2\phi_1(r) + k_1{}^2\phi_1(r) = j\omega\mu_0 J_z(r)$ just rewritten everything, region 2 is follow the same recipe ok. So, the $\varepsilon$, so in other words this term over here right this term so, $\omega^2\mu_0\varepsilon(r) = k^2$ ok.

k may or may not be a function of space, but if like epsilon is a function of space, then k will also be a function of space ok, just a I am trying to reduce all the extra variables and condense them to one variable, so that is easy to see alright, so these are the two equation that we have got.

Now, we want to solve these two questions right. So, it I mean just looking at these equations, this looks much more complicated than the integral equation that we show in the previous class right. So, how do we solve it?