Computational Electromagnetics
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Introduction to Integral Equations
Lecture – 5.1
Line Charge Problem

Today's lecture is going to be about Integral Equations and your very first Introduction to an Integral Equation. So, we will keep it very very simple to give you an idea of what exactly is an integral equation and how is it different from any other kind of equation you have seen so far right.

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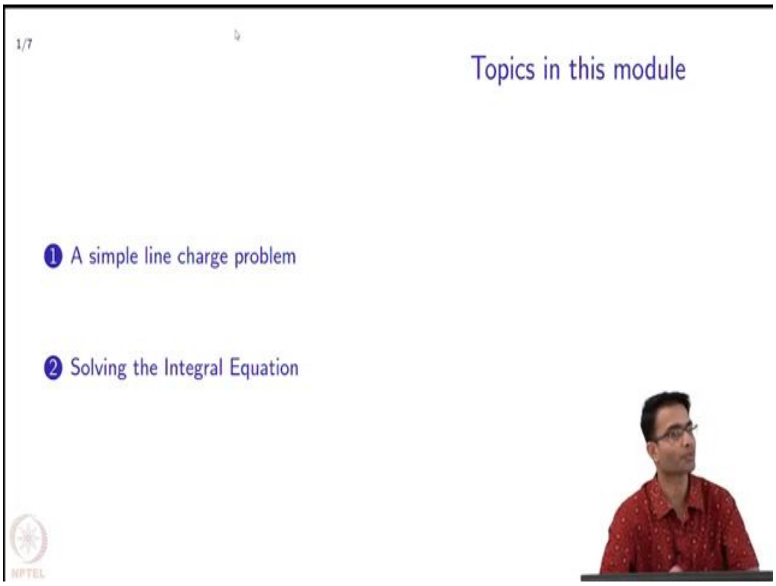

So, in order to look at this, there are what I will do is we will not even go into electrodynamics, we will just take a static case. The equation for which you know a class 9th student can tell you , potential due to a charge that that kind of a thing and then we will solve it .

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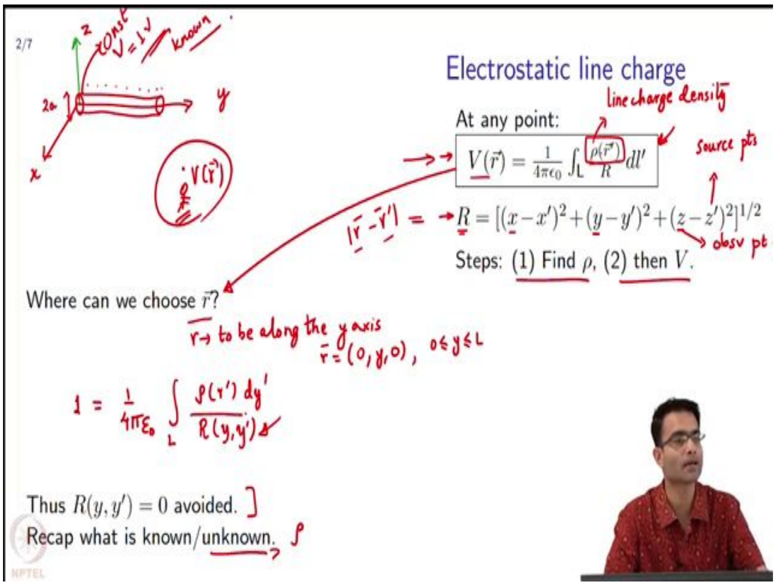

So, let us lo at a Line Charge Problem . So, what do I mean by a line charge is let say that I have x, this is my axis and over here I have put a conductor . It is a conductor and diameter over here is say some 2a, imagine is like a thin wire basically that I have placed along the y axis and it is a statics problem. So, there are no fields that are varying in time and what I have done is I have connected this whole thing to a constant voltage .

So, let us make that simple, let us even just make it 1 volt . So, this wire which is lying along the y axis it has some length L, it has 1 volt you make it even simpler perfect conductor . Now, if I ask you what is let us say, the voltage at some point over here $V(r)$? There are charges everywhere right, now even though this wire has a surface right. It is a I mean it is a two it is a realistic object, it is a wire, it has some surface area, but we can simplify our problem. Because, there is nothing else around, all the charge will be uniformly and symmetrically distributed about the circumference .

So, instead of dealing with the surface we can make life simpler and just deal with a line . If you want to get the correct surface get the correct charge you have to multiply by the circumference or whatever. So, we will deal with a line charge. Now, if I ask you what is the potential at this point over here, how what law would you use? Yes, coming from Gauss's laws right, sum the contribution from every small point over here. So, this equation that I have written here this gives you the voltage at a point r because, I have lots of small small charges along this entire length over here. It will not be a summation it will be a integral right. So, what is rho over here, what would you call it?

Student: Line charge.

Line charge density right so $\rho\int dl$ and I have in the denominator should be what physically?

Student: Distance.

Distance, between the charge and the point right that is what gives me the total potential, for total potential is the integral of all the potential due to everyone of these fellows. And, this capital R that I have written over here is simply the Euclidean distance. So, the x, y and z these are the observer points right and the prime coordinates corresponds to what?

Student: Source.

The source points right; so, those are the also the limits of I mean the variable of integration right. So, those refer to the points along the wire where the charges are right. So, the prime coordinates are the source points and this over here is the observer point . Now, you can sort of appreciate the problem. This observer standing over here wants to find out the potential. In order to find the potential, we have come to this expression over here which gives me the potential as a function of all the other variables. So, what is a problem, what do I know, what do I not know?

Do I know ρ as a function of r'? As a function of the length along the wire do I know the charge density? If I knew it the problem would be done then I just have to evaluate this integral and I will get V. But do I actually know the charge distribution? I do not know right, I gave you a wire all I told you whether it is connected to a constant voltage source. So, the voltage along the wire is 1 volt, but now how will the charges distribute themselves? If I put the charges for example, will the rho be constant as a function of r prime, is that physically meaningful? Imagine you have let us say 10 charges, I put 1 2 3 4 5 6 7 8 9 10 charges and I leave them, will they stay like that, what will where will they go?

Student: According to each other.

They will move according to each other, but intuitively what kind of distribution do you expect?

Student: (Refer Time: 05:45).

They will move away from each other, they are line charges they will move away. Till what point can they go?

Student: Until equilibrium.

Until equilibrium right; so, they will they be clumped up towards the centre or towards the ends?

Student: End.

We would expect towards the ends because, on the end points there is no force from the other side right and we are assuming that the situation is not so dramatic that the charge jumps off the thing right. So, the charges sort of will accumulate there. So, rho actually we do not know, if we knew rho then this problem was simple. I will calculate this integral numerically or analytically depending on the situation and I will get voltage.

So, before we even get into solving something and get lost in the details we should appreciate what is known, what is not known right. So, first of all I need to find rho as a function of space. Once I find rho the problem is simple, I will just plug it into this equation and find the voltage . You will find this general principle applied in all integral equation. There will be some intermediate variable which I do not actually care so much about; I mean this observer over here just wants to know potential. He or she is not interested in what is the line charge density right.

So, but you have to solve for one intermediate variable before you solve the actual voltage. So, that is a strategy right. So, what is unknown over here is rho and what is known? This constant thing over here, constant voltage that is this is what I have given you. Do we have enough to solve this problem? It seems like, I mean think of a wire I connected to a 1 volt source, in principle that should be alright, but looking at this equation it seems a little confusing, how will we find this. So, we need to somehow arrive at a system of equations which captures all the information that is given to me.

So, what is given to me primarily is the fact that voltage is constant 1 volt. Where all is the voltage one volt? Where is it, known to be 1 voltage? In space where all is it known to be 1 volt.

Student: $y = 0$ and $y = L$.

$y = 0$ and $y = L$, why what about in between?

Student: Equipotential.

It is a equipotential; so, everywhere if I pick any point on the wire voltage will be 1 voltage right. So, in this equation over here if I take this equation, this capital R over here is a function of; I mean this is actually written like this right. It is the distance between the source and observer point, there is no choice on the source point, source point is fixed is along the line charge. So, I have 1 degree of freedom over here in specifying where should I choose my r, my observer point.

So, I will not be greedy I actually want the potential at this point over here away from the wire, but let us not be so greedy; let us not fix r to be this point because then, I do not know how will I solve this equation. So, let us what we can do is we can choose this r to be at a point where I know the voltage. So, if I chose this r to be along the y axis. So, how will I characterise any of these points, what will be the x y z coordinates of obverse of a points that goes along the y axis?

Student: 0.

0, y, 0 right for different y and y is going to vary from 0 to L right so, this is my r ; for these points I know the potential right. So, the left hand side of this equation, what does it become? This equation over here, what is the left hand side of it for these points?

Student: 1.

1 I know it, it is 1 volt right. So, I will write 1 is equal to $1 = 1/4\pi\varepsilon_0$ right integral over L and ρ of, let us just let us make this r' right dy' and all others I mean x and y are 0, $x' y'$ also we can ignore it for now so it is a function of these two variables. So, notice another nice thing that happened by choosing r to be this way, this denominator will it ever go to 0? No, right because I have said the sources are on the surface of the conductor and the r that I have chosen is on the axis of the conductor right.

So, this the minimum distance between these two points is how much? a in this case a right. So, this $r\, y\, y'$ equal to 0 never happens and we need that because we are actually dividing by $1/r$ right. So, if come to a come to an equation where nothing bad is happening, there is no singularity nothing is blowing up so.

Student: Sir we are taking the surface (Refer Time: 11:23) of a.

On the surface because, we expect charges will go and flow to the surface they will not.

Student: We are not approximating (Refer Time: 11:29)?

No, we are approximating the charges to be a on the surface, in a realistic situation they will be smeared all over the surface of the conductor right, but they will be symmetrically distributed. So, instead of dealing with a 2-dimensional surface distribution why not I deal with just a line charge. And, once I get that I know that I have just normalised to get the correct surface thing because, it is the same thing that will be repeated at every θ. So, just to make our calculation simple let us pretend that the charges are on one line, just to keep it simple.

Student: (Refer Time: 12:02).

On the surface.

Student: (Refer Time: 12:03).

On the surface . So, these kind of convenient assumptions you will find made throughout the electromagnetics otherwise unnecessarily you will be solving a 2 or 3-dimensional equation; when actually the actual physics is only 1-dimensional . And, and we can make this assumption because this this current I mean this wire is lying in free space. So, the charges will be symmetrically distributed as the function of theta right. So, that is what we will leave in .

So, what you have seen now is your very first integral equation . Now why do we call it an integral equation? If you look at your equation over here once more the unknown, in this case the unknown is ρ. The unknown is sitting inside an integral sign that is why it is called integral equation . So, this is just one example of many types of integral equations. So, we will just make a small detour to types of integral equations right.

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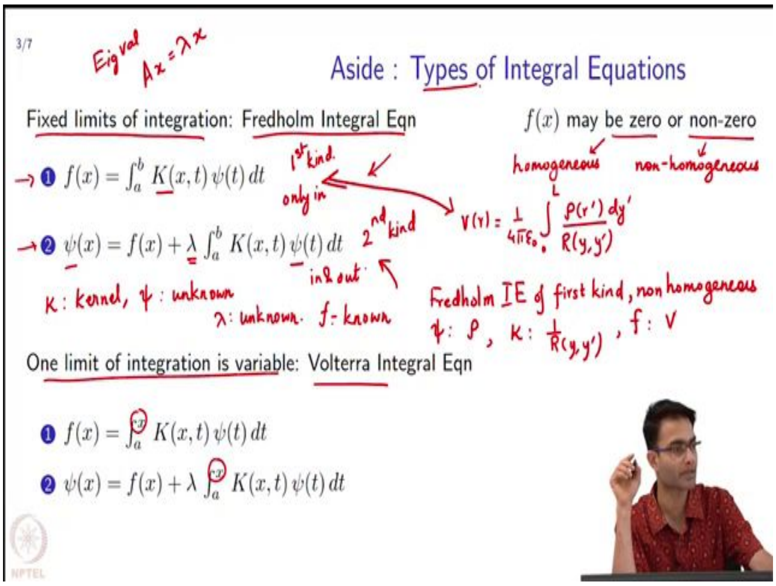

So, we will start from the top. So, what we just saw was a Fredholm integral equation, this is of the 1st kind . So, this K over here is called a kernel , this ψ over here is the unknown . When the limits of integration are fixed right so, that gives us a Fredholm integral equation. Then you also have a Fredholm integral equation of the 2nd kind, what is the difference?

Student: Unknown.

The unknown is both inside the integral sign and outside right, ψ is inside and outside right. So, in and out and this is only in. Later on in the course we will come across both I mean we have already seen the first one and we will later on the course come across the second one also . So, it is not I mean these will be used very soon. Further, I notice the second equation has one more parameter that has come upon which is this guy lambda . So, λ is also unknown, you can think of this as the equivalent of a eigenvalue problem.

When you write a eigenvalue problem, what you write? You write $Ax = \lambda x$ and when you write this equation you do not know x and you do not know λ; you solve for both one by one right. So, we have encountered such equations where there are two unknowns. So, this is a generalisation of that where this λ parameter is also unknown, you can think of it as an eigenvalue. So, these two are Fredholm integral equations and we electrical engineering comes up across these many many times .

There is yet another kind of integral equation which we will not come across in this course, but they also occur throughout engineering and even economics. It is called a Volterra integral equation which is almost identical to a Fredholm integral equation, can you spot the difference?

Student: Limit is variable.

One of the limits of integration is a variable right. So, I have x over here x over here everything else is same . So, these are the two kinds of integral equations that you will come across in the engineering literature right. This f over here had better be known otherwise you cannot solve the problem . This f may be zero or it will be or it can be non-zero. If it is zero standard terminology we will call it a homogeneous equation and else I call it a non-homogeneous . So, when I look at let us say the equation that we had on the previous page right. So, I had

$$V(r) = 1/4\pi\varepsilon \int_0^L \rho(r')/R(y, y')dy'$$

this was our integral equation. So, from these four equations that we have on the board which of these does it lo like?

Student: (Refer Time: 16:48).

Looks like a ; so, it is a complete what is the complete description of this integral equation? So, first of all Fredholm why, why not Volterra? Limits are known and fixed right. So, it is a Fredholm integral equation IE, of which kind which kind?

Student: (Refer Time: 17:13).

First kind right, there is the unknown is appearing only inside the integral sign. In the second one it is also outside and it is a first kind. And is it homogeneous or non-homogeneous?

Student: (Refer Time: 17:32) non-homogeneous.

It is non-homogeneous why? Because what is $f(x)$ over here?

Student: V.

V the potential right; so, it is non-homogeneous. What is my psi in this case? If I want to do a draw one to one analogy between this guy and this guy what is my psi?

Student: Rho.

Rho. What is my kernel K?

Student: 1 by R.

1 by R; what else do I have? f right so, f is V so, that completes the full description of your integral equations. So, we need to specify all three: Fredholm or Volterra, first kind or second kind, homogeneous or non-homogeneous then your integral equation is fully specified . These integrals have been written as an integral in 1-dimension, but as we go further these can be integrals in 2-dimensions, 3-dimensions; the terminology will be the same .