Computational Electromagnetics
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Numerical Integration
Lecture - 13
Interpolating a Function

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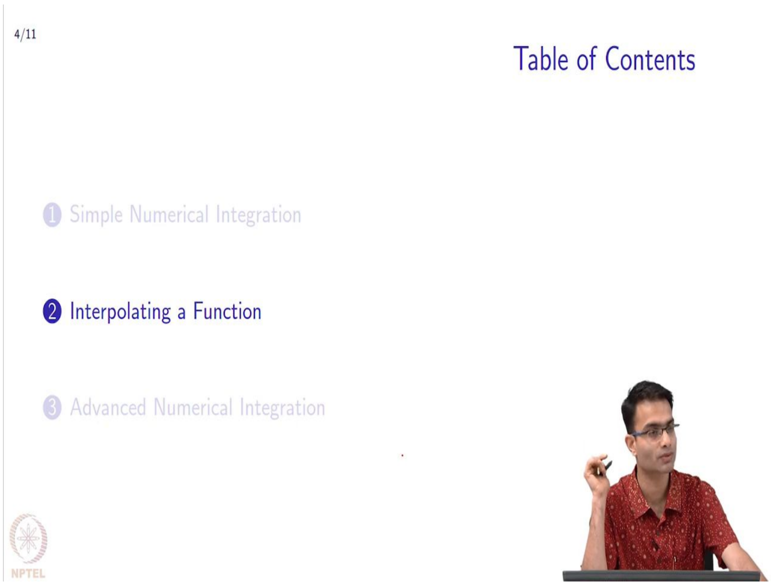

This takes us to the next subsection is, to get to that advanced idea of integration, we need to take a small detour towards how to interpolate a function ok.

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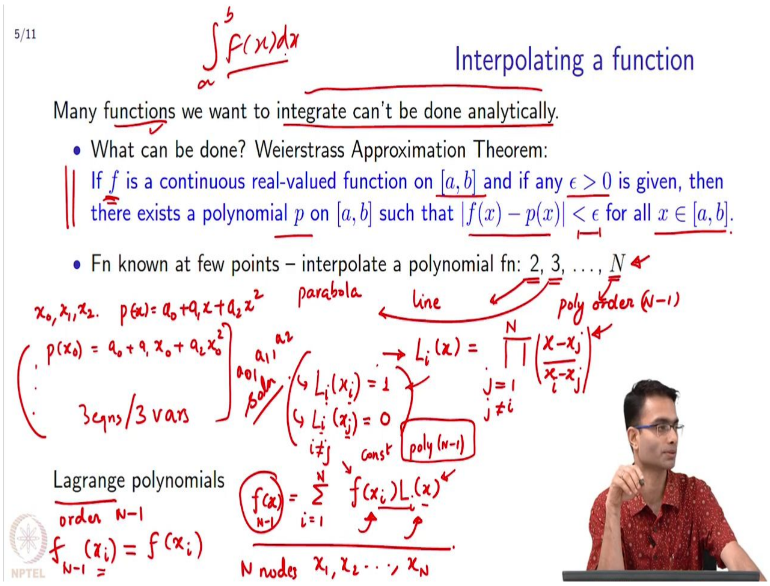

Again this is something that many of you are familiar with; we will just formulize this. Again we are talking about functions that we want to integrate, but it cannot be done analytically that is our approximation.

Here there is a very powerful theorem, called the Weierstrass approximation theorem in blue font over here. And it basically says that if I have a continuous real valued function f that I want to integrate. So, it should be well behaved no funny jumps in the function, and you have given me some interval. Then if you tell me your tolerance, let us say epsilon some small number, then there always exists a polynomial function p such that it approximates f to less than that given tolerance for all x within this integral ok.

So, this is named after Weierstrass. So, you can see the power of this theorem, that is telling me give me any function you know it can be some Hankel function, Bessel function any complicated function. What this is telling you is that, I can find you a polynomial function which approximates this guy very very well. How well? It is up to me right. If you specify epsilon I will find you a p which approximates, this function very well over here.

So, this is again not very surprising you saw that we know that for example, if I take two points over here with two points, I can perfectly put what can I do? What kind of a polynomial can I fit with two points?

Student: Linear.

Linear right so this is a line. three points I can do a parabola and so on. So, four points and you know. So, basically when I go to order N, supposing I say I want to interpolate a polynomial on N points what will be the degree of the polynomial? N minus 1 right for N equal to 2 what is the order of the polynomial? 1. So, always you off by 1 degree right. So, I can fit a poly of order N-1.

So, there is a systematic way of doing this, which we will talk about, but I mean just think about it supposing I gave you three points and I asked you to fit a parabola. So, what would you do? I have given you three points x_0, x_1, x_2 ok. So, I have given you these three points x_0, x_1, x_2 and I tell you fit me a parabola through this point and the functions I mean the through the function values at these points.

So, what would you do?

Student: Sir we have three points (Refer Time: 03:18).

Yeah you have three points. So, you have the functions.

Student: (Refer Time: 03:20).

But these points can be anywhere I means they need not be uniformly spaced, they could be one point here, one point here and one point here.

Student: (Refer Time: 03:35).

Assume the equation of a parabola. So, I can say that. So, say p is $p(x) = a_0 + a_1 x + a_2 x^2$ right. Now I know, I have given you $p(x_0)$ the value of the function has been given to you. So, you get $a_0 + a_1 x_0 + a_2 x_0{}^2$, like this I will get three equations in three variables right and the solution to this will give me a_0, a_1, a_2 right.

So, similarly I can in general reproduce this process for a set of n points also right. So, I will be solving a system of n equations to get these n values. Now, turns out you can you can do it like that there is no problem it turns out there is a more clever way of constructing this polynomial and that goes after the name of a famous scientist called Lagrange. So, this

Lagrange polynomials I will write them down over here.

So, you have $L_i(x)$. So, I will just write down the form and then we will investigate it. So, this big pi stands for product ok. So, before we get into more details just look at this, I have constructed some function over here. So, its product; how many terms are in the product? N-1 terms are there in the product because $j \neq i$ right. So, $j = 1, \ldots, N \quad j \neq i$. So, there is a product of N-1 terms. So, this $L_i(x)$ is a polynomial of what degree?

Student: N - 1.

N - 1 there are N - 1 terms each term has a x minus something. So, over all degrees is going to be N - 1, now this function looks quite interesting right say supposing I ask you what is the value of the function $L_i(x_i)$; what is its value? Each term cancels off and its value is 1 ok. $L_i(x_j)$ =?, $j \neq i$.

What is this value? 0 right; so, if I write if with using this function over here using this so called Lagrange polynomial over here, can I write f like this. So, I can write this as a. So, first of all is this a is this expression that I have written over here on the right hand side is it a polynomial function?

Let us look at it one by one, what is this guy? Is this constant or is it a variable as a function of x it is what is this argument x_i? x_i is a fixed point right there are N points and this is the taking the ith point. So, this term over here is a constant the next term over here $L_i(x)$ what is it? It is a polynomial order N-1. So, this is a poly order N-1 ok.

What is the sum of any number of polynomials of order N-1? N-1 at best right it can be lesser also, but at best will be N-1 ok. So, what I have done is, I have given you a function which is of order N-1. So, it has order N-1 and does it agree with the value of the function at the given nodes. So, I have given you N nodes ok. So, N nodes are this x_1, x_2,..., x_N. So, $f_{N-1}(x_i)$ =? We will use this property over here. So, there are N terms in this summation; of this N terms in the summation which term will survive?

Student: (Refer Time: 08:51).

Only one term will survive because this Lagrange polynomials they are such that there are they are one, only for this i equal to this i, but if this i and j are different then that becomes 0 right. So, in this summation over here there is only one term that will survive when I substitute x_i right and that term will have my value will be 1 from here and what will be left here is, simply $f(x_i)$ ok.

So, keep in mind my original function f was not a polynomial function it was any continuous real valued function. And out of that I have created this guy over here, which is a polynomial function in circle is a polynomial function and this polynomial function it agrees with the value of this arbitrary function at N points right and it is a polynomial of order N-1.

So, what you did in the slightly clumsy way by solving a system of equations over here, you can do in a little bit more systematic way over here. So, this is I mean why the reason why we use this is again it makes the mathematics as we go further on very very simple. It is also very elegant just look at the way this function is created right it such a beautiful thing that it goes to 0 at all other points except at the point that your looking at it.

So, having done that now let us go a little bit further we. So, we wanted to integrate a function right. So, this is the function that we cannot integrate analytically. So, step 1 is we created some approximation of it, a polynomial approximation of it if you give me N points I give you a polynomial of order N-1 because that is the best I can do. So, give me three points I can give you a parabola I cannot give you a cubic right because there is not enough information there ok.

Next, we will try to integrate it right. So, why do you think we first took instead of directly approximating the function directly integrating the function, why did we make this change to a polynomial? Our objective was to do this.

Student: (Refer Time: 11:8).

Exactly right I do not know how to integrate this because its non integrable, but I am approximating it by a polynomial and polynomials I know we can integrate by hand right. So, step 1 was to find a good approximation.

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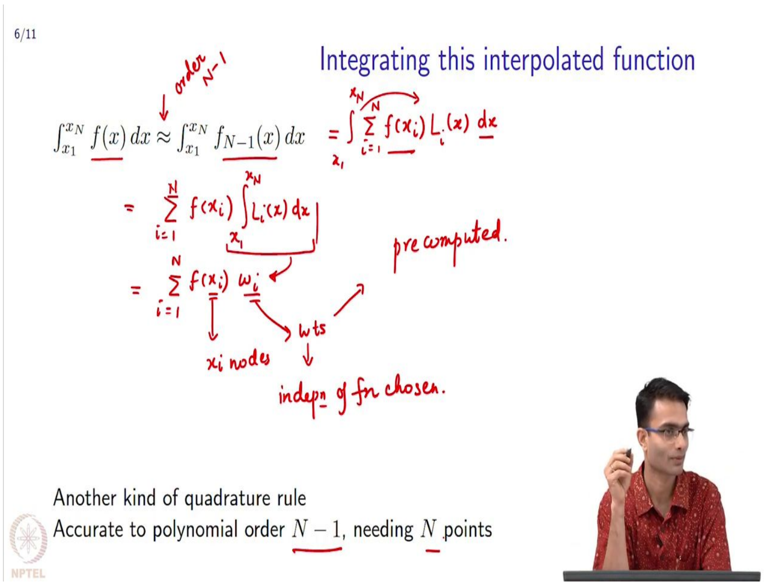

So, now let us just do the obvious let us integrate this expression that I have. So, this was my original function; note that this is an approximation. It is an approximation of order N-1 that is the function that we had from before and we will just put this N over here.

So, f this term over here, I am going to replace it by what I had already used before. So, that is going to be a $\int_{x_1}^{x_N} \sum_{i=1}^{N} f(x_i) L_i(x) dx$ ok. So, I have just rewritten what I had done previously, remember this is the integration sign the variable of integration is x.

So, what do you think that I should do next? Can I simplify this expression? Can the integral sign move anywhere ok? This guy what is this guy? This guy is a constant. So, I can move this integral sign further inside over here right. So, I can write this as $\sum_{i=1}^{N} f(x_i) \int_{x_1}^{x_N} L_i(x) dx$ right ok.

This thing over here this term that I have written over here, does it depend on the choice of function f not at all right there is no f over here right. So, that is see, the power of that. So, now, I can rewrite this expression as $\sum_{i=1}^{N} f(x_i) w_i$ where this w_i is some other w_i right this expression over here.

So, would you call what we have done also a quadrature rule? It is a quadrature rule because it is giving me an approximation of the integral it is telling me what all I need to know?

Student: Function value.

Function value at a few discrete points. So, these are the x_i nodes and weights that do not depend on the function right that is the important part of a quadrature rule, that the weights are independent of function chosen. So, another advantage is that this w_i what you can do is, you can pre compute and keep it you can store it as a table you do not have to solve it when you are actually when you are given a integral, it is not that that time you should start to compute w i you can pre compute and keep it. In fact, there are many libraries mathematical libraries which will tell you the values of w for various I mean various intervals or whatever right. So, these can be pre computed.

So, when you get your function that you want to integrate what you have to do? All you need to know are the nodes calculate the function values of those nodes multiply by pre computed weights right. So, this we should do pre computed fine ok. So, this has given us a systematic way where we have calculated the approximated the integral of a function to order N-1 and for that how many points we need? N points.

So, this is as far as interpolation goes, now there is, you will be surprised to know that if I told you the result beforehand now we want to make this even better you know. So, even better means what? Supposing I tell you that the number of points at which I want to evaluate my function is fixed N points you have no choice over it; can you somehow give me a answer that is accurate to a higher order of polynomial? No right order N-1 is the best I can do.