Computational Electromagnetics
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Numerical Integration
Lecture – 4.1
Simple Numerical Integration

Alright so, this module is going to introduce you to a concept which we will use extensively throughout the course and that is dealing with what is called Numerical Integration. So, we will start with the review of Simple Numerical Integration ok.

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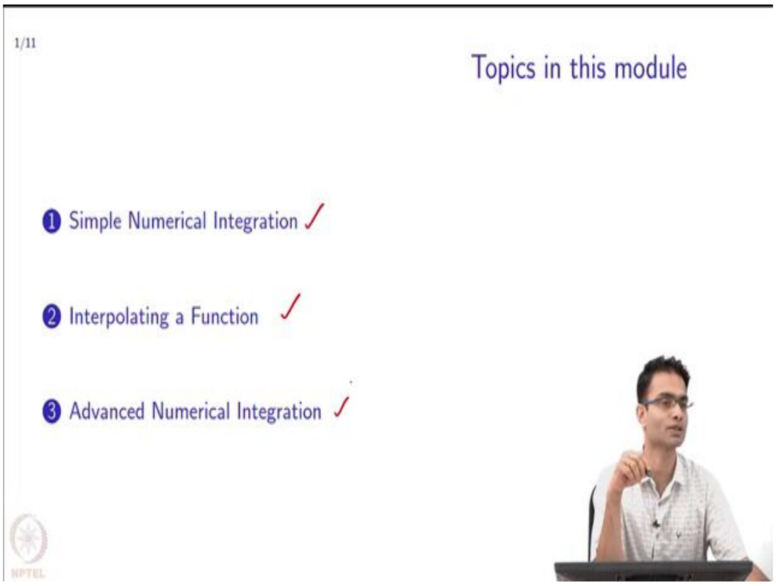

Something which is very familiar to you, you would have seen it in high school, undergrad alright. We will proceed to the idea of interpolation of a function and use the idea combine the first two ideas into more advanced ideas for numerical integration ok. So, that is going to be the flow in this module, we start with simple numerical integration right.

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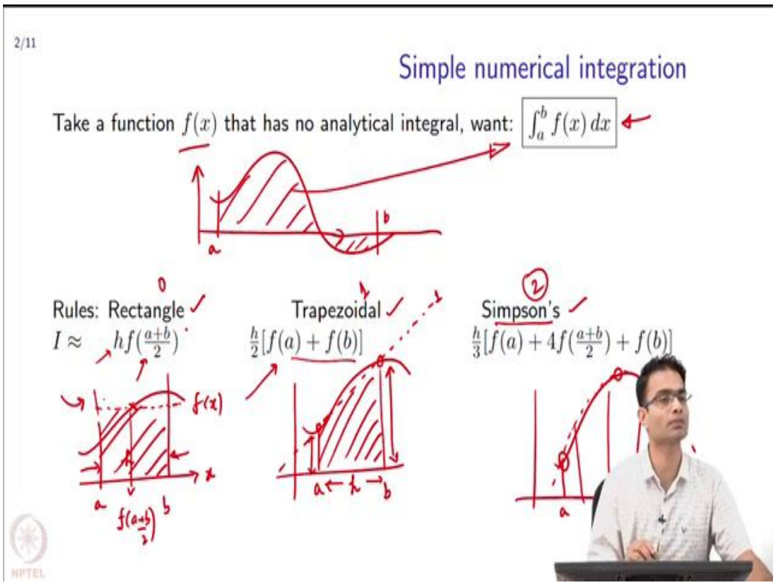

So, let us take a function f which has no analytical solution for its integral ok. We all know that there are these kinds of functions are there which cannot be integrated exactly right. So, and what we want is the integral of this function ok. These kind of integrals they appear throughout electromagnetics because, the kinds of functions that appear in electromagnetics you will have Bessel functions of various kinds, they do not have any close form analytical expression. So, you are faced with this problem of how do I integrate ok.

So, everyone knows intuitively the idea of Riemann integration that you studied earlier in high school probably, its integration is area under the curve right. So, if I draw it like this, if this is say a and this is say b over here then the area under this curve is what is the integral of $\int_a^b f(x)dx$ ok. Now, you would have also encountered simple rules of integration in your schooling which are about how to do this integration approximately right. So, the first rule of course is the what is called as the; is called the rectangle rules, its the midpoint rule that if I have some function over here.

And, this is my interval a comma b, this is $f(x)$ then what is it say? If this gap between a and b I denote as h, I replace this by one single value over here right. So, that is the midpoint right $f((a+b)/2)$ and approximate this integral by what? By the area of this rectangle right; so, it is so, that is my the length $\times$ width right. Hopefully everyone has seen this before, this is as simple as it gets. So, you can this is also called a staircase approximation of the function right.

So, as you can imagine this will not be very very accurate, to make it very accurate what do I need to do? Make h very very small right that is one way of doing it, the second rule which you would have seen would we call the trapezoidal rule. Take the same function over here; try to draw it again in the same way. So, this is let us say the 2 points a and b what does it do is say is instead of putting a rectangle through the midpoint, I let me put trapezium that covers this right. So, it is going to take a trapezium like this and estimate the area as this quantity over here ok.

And you can see $f(a)$ is going to give me this length, $f(b)$ is going to give me this length and this is the formula for the area of a trapezoid right; $\frac{h}{2}(f(a)+f(b))$ ok. Obviously, we expect this to be more accurate than the midpoint rule ok, because instead of a flatting I am fitting a linear function over here ok. If you wanted to get a even more accurate answer for the same integral, the next step logically you can see this was order 0, this was order 1; order 2 means I what is a kind of curve that I will fit?

Here I fit this dotted line was a polynomial of what order? This dotted line this dash line over here, this is a polynomial of order 1 straight line is polynomial of order 1. In this case this is order 0, this is order 1. So, stands to reason a next step to take is let us fit a parabola through this function right. So, let us say this is my a, this my b ok. So, I can take some three points like this over here, and what this rule will try to do is ok, let me somehow fit a parabola through these three points. Because, I need three points to uniquely specify a parabola right.

And, then compute the area any polynomial function the advantage is that what about its integral, I can integrate a polynomial in close form right. So, doing that gives me all of these rules right. So, back in the day when it was innovative people's names went behind it. So, some person by the name of Simpson discovered this rule, I mean if you were born 300 years ago, it would be named after you right; it is nothing very great about it. But, this Simpson's rule tells you gives you one expression which is accurate to order second order. So, as I go to higher and higher order I will be able to better approximate the function and once I approximate the function integrating a polynomial is easy. So, that is the idea behind simple numerical integration. Let us move a little bit ahead with this.

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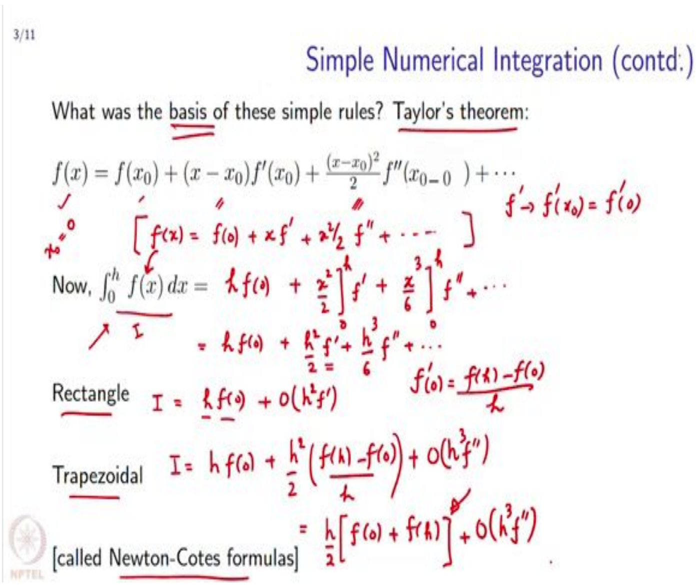

Let us try to find out what is the underlying idea behind these simple rules that we have seen. So, what is the basis of these simple rules? Not surprisingly the Taylor's theorem comes of use over here, we have used this extensively in numerical techniques and what not right. So, this tells you that a function f can be approximated about a point x_0 in terms of increasing orders. So, the first term is constant $f(x_0)$, then I have a linear term and then I have quadratic term and so on right and it is a infinite summation.

So, let us take; let us just try to make the math simple ok. So, let us just write $x_0 = 0$ ok. So, what is $f(x)$ then? So, $f(0) + xf'(0) + x^2 f''(0) + \ldots$ ok. I can put the origin wherever I want. So, I have just $x_0 = 0$ just to make the math easy. So, now I have a sort of analytical form for f which is approximation. It will be an approximation depending on how many terms of this series I keep ok.

Now, I want to integrate this. So, if I integrate this what will happen, I am going to put this inside over here. So, in the first term what happens to the first term? It is a constant right integrating from 0 to h. So, integrating from 0 to h will simply give me a?

Student: h.

h right. So, this will give me a h, $f(0)$ is a constant that is the first term. Then the second term is $\int_0^h x dx$ right. So, I am going to get a $x^2/2|^h{}_0 f'(0)$ ok, when I say f' what I am saying? I am saying $f'(x_0) = f'(0)$. So, that a constant right and the next term is going to give me $\int_0^h \frac{x^2}{2} dx$ integrated once will give me?

Student: x cube by 6.

$x^3/6|^h{}_0 f''(0)$. So, this is again very simple to do, all these higher order terms ok. So, now you can see where these the simple rules that we have seen where do they come from. So, the rectangle rule that we saw earlier what did it say? It kept it, it replaced the function over here, it replaced the function by a constant value right. So, in this term if I keep only the constant term what will I get? Rectangle rule will tell me that this integral over here, let us call this I; we will say $I = hf(0)$ and plus the error; what is the order of the error?

Student: Squared.

Squared right $o(h^2 f')$ this is the order of the error right; so, that is the order. So, you have got the value of a function at one point multiplied by the interval that is your rectangle rule ok. So now, when I go to trapezoidal rule what am I going to do? I want to keep linear right, but dealing with linear terms now I have to do somehow do something about this f'. So, how do you think I should replace?

Student: (Refer Time: 10:00).

Replace f' by?

Student: (Refer Time: 10:04).

Right, finite difference.

Student: (Refer Time: 10:07).

Right so, I can say $f'(0)$, I can write it as roughly what? $(f(h)-f(0))/h$ ok. So, if I keep the first two terms I will get $hf(0)+\frac{h^2}{2}(f(h)-f(0))/h$ and the error is going to be order of h^3f'' ok. So, this thing will simplify one h will get cancelled, what am I going to get? I will get so, h I can take common, what is the coefficient of $f(0)$?

Student: Half.

Half and what is the coefficient of $f(h)$?

Student: (Refer Time: 10:56).

Half right. So, I can take the 2 outside I am going to get $f(0)+f(h)+o(h^3f'')$. So, does not this look exactly like your trapezoidal rule right. So, you notice that this is the underlying principle is Taylor's theorem, you can keep going to higher order term you will get more and more accurate expressions. So, these formulae after Simpson they stopped being named after anyone because, it became too boring.

But, these general set of principles they are called Newton-Cotes formula ok, this is something to know does not really matter over here. So, what is what I wanted to convey in this particular module is the basis behind numerical integration, simple numerical integration ok; as we go further we will look at little bit more complicated ideas ok. So, we were looking at numerical integration right and we found out the basis of the simple integration rules that we came across in high school like the rectangle rule, trapezoidal rule etcetera it was basically Taylor's theorem. We want to extend this idea and sort of formulize it a little bit more. So, let us just do that.

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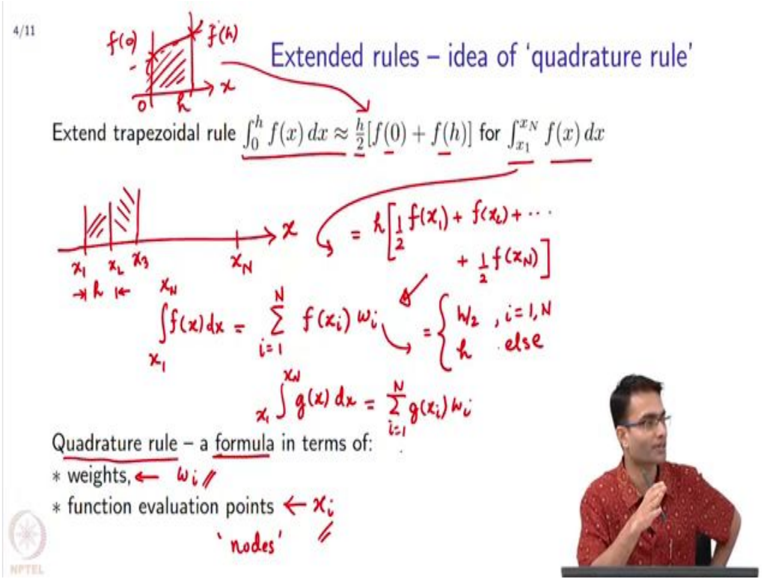

So, what we saw was the trapezoidal rule was for an integral from 0 to h; everyone is familiar by now $h/2$ and the length from at 0 and the length at h right. So, this was my function over here. This is 0 and h right and this is my $f(0)$ and $f(h)$. And, what this approximation is doing, it is putting a straight line like this and telling you the area over here right that is what is computing over here. But, this was for one interval only from 0 to h; now I want to basically just extend this rule over an entire interval ok.

So, let us say I have n points x_1 through x_n right. So, this is my x axis over here and I have $x_1, x_2 \ldots, x_n$ and let us assume that the spacing is uniform although it need not be ok. So, if I want to just apply this trapezoidal rule to find out the whole integral from x_1 to x_n, I just have to add the contribution for each of these segments correct. So, if I want this over here so, I can say this is equal to what do you think it will be? So, for the first segment over here.

Student: $h/2$ (Refer Time: 13:41).

Right so, $h/2$ will come and I will get?

Student: $f(x_1)$.

$f(x_1)$, then I will come to x_2, I will $f(x_2)$ with its coefficient will be?

Student: $h/2$.

So, its coefficient will be $h/2$ from this interval; similarly there is a x_3 over here. When I take the contribution from this interval x_2 will come again.

Student: (Refer Time: 14:01).

So, its contribution will be h another $h/2$ right. So, this whole thing I can write as; so, I can take an h common, I will get. So, $f(x_1)/2 + f(x_2)/2 + \ldots + f(x_n)/2$.

Student: x_n.

x_n right, pretty straight forward. Now, we could have stopped right here; if this is all we wanted to do we could have stopped right here. But, let us try to write it in a more standard way and what is that standard way? Let us write this like this a summation ok, $\sum_{i=1}^{n} w_i f(x_i)$. This formula fits into this mold right and you can see if the w_i's are pretty straight forward; $w_i = h/2, \quad i = 1, \ldots, n$ and for others $w_i = h$ right. So, $w_i = h/2, \ i = 1, \ldots, n$ and $w_i = h$ else ok.

So, we did not do anything new, we just re-wrote what we already had in a little bit more what I am going to call the standard notation. So, this is the approximation of $\int_{x_1}^{x_n} f(x)dx$ ok. Now, this way of writing it might look a little strange, but this is what we call a quadrature rule. Now what are the ingredients of a quadrature rule? It is the case where I do not know the analytical integral of an equation ok, that is the first thing. I do not know how to integrate my function analytically that is why I am doing all of this circus.

So, instead a formula tells me an approximation of the integral, what is this formula? So, quadrature rule means a formula ok. So, the word quadrature rules sounds fancy, but what is it? It is a formula. What are the two ingredients of the formula? Weights so, w_i and the points at which this function should be evaluated x_i ok.

So, once I give you once I tell you that here is a quadrature rule, you should ask me what are the weights, what are the function evaluation points? These evaluation points are also sometimes called nodes ok. And, once that is specified; now supposing you had this function $f(x)$, now supposing I tell you now I want to integrate some other function $g(x)$. So, will I have to change the quadrature rule?

Student: (Refer Time: 17:02).

No, I have given you I have invented after a lot of hard work I have invented a quadrature rule ok and, I use it to integrate $f(x)$. Now you come along tomorrow and say no I want integrate $g(x)$, where g is some different function. Do I need to change the quadrature rule?

Student: (Refer Time: 17:21).

Answer is yes, answer is no; any other answers? Objective is to integrate the function, do I need to change the quadrature rule? So, to answer that you should ask me what is a quadrature rule we just discussed, what is the quadrature rule?

Student: Depending upon the function (Refer Time: 17:37) points.

Ok.

Student: Because when we using 0, x_2 because, we decided these points by the more of the function, because something looks like a trapezoidal then we have to take that section (Refer Time: 17:49).

No, not really what we did is we took the line over here the x axis, we chopped it out at uniform this points $x_1, x_2, x_3, \ldots, x_n$ ok. And, the price I am paying is that I am going to say in each segment x_1 to x_2, x_2 to x_3 I am going to approximate by a trapezoidal whether or not it is a good approximation or not. So, if I come up with a new function g I do not have to change the quadrature rule, all I have.. I am given x_i and w_i these are fixed. So, how will the integral of the new function g come? It will simply be, if I want integral now $\int_{x_1}^{x_n} g(x)dx$ I just change the value of the function right.

So, $\sum_{i=1}^{n} w_i g(x_i)$, I already knew x_i I knew w_i so, I have to just find out the value of the function. So, you see how much of a power this is, I do not need to know the analytical integral of any function; I just need function values. Knowing the value of the function at a few discrete points I am able to approximate the integral, that is why this so, called quadrature rule is so, useful you make it once, you can use it for any number of functions under certain conditions. And, we will talk about those conditions as we go along ok, that is why this quadrature rule is very important useful rather ok.

Now, so you notice one thing that what we were doing so far is for example, this was trapezoidal rule. We could as well have made a quadrature rule out of Simpson's rule. Simpson's rule what did it do? It fit a parabola in between each segment right and then that approximation will of course be better right. So now, you can think how can we make this more accurate ok. We want more accuracy for the same number of function evaluations. Now, it seems that if I want more and more accuracy then I should evaluate my function at more points right; for the if I want to fit a parabola I need more points then if I want to fit a line right.

Now so, the question is you can be greedy and you can ask can I get better accuracy while still retaining the same number of points. So, that is what we are going to investigate.