Computational Electromagnetics
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Applications of Computational Electromagnetic
Lecture - 14.19
Finite Element-Boundary Integral - Part 1

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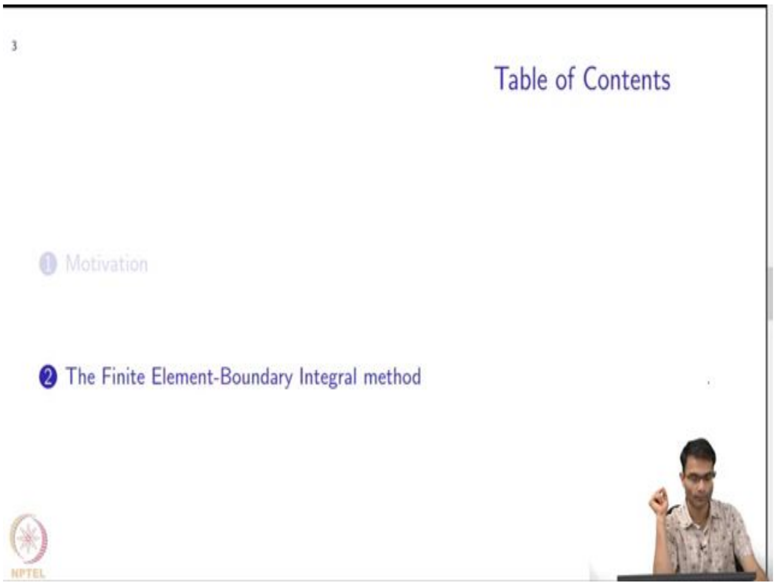

So, let us get into this FE Finite Element Boundary Integral method ok.

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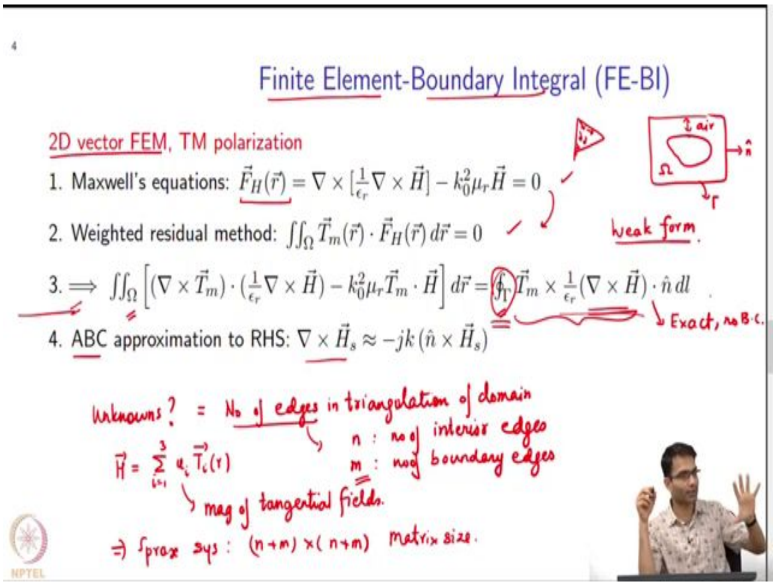

So, this is a little bit of revision of our 2D vector FEM not the your regular note base FEM, but 2D vector FEM is what was covered in a previous module and as you know that there are basically four steps in your FEM approach. So, let us there is a lot of equations here let us just go one by one.

What is the first equation telling you or rather where is it coming from?

Student: Maxwell's equation.

This is just coming from Maxwell's equations right. So, you take you have two equations, you move your $1/\varepsilon_r$ on one side and again take a curl. So, you can replace curl of E by H and all those constants become $k_0{}^2\mu_r\vec{H}$. I have not put the dependence on space, but everything is a function of space, ε_r is a function of space, H is a function of space, μ_r could also be a function of space although we will not consider it and H is of course, the function of space.

So, this entire left hand side I am I have called it this functional $\vec{F}_H(\vec{r})$ which in the ideal world I would like to be satisfied exactly at every point in the computational domain that is my wish list. Of course, that is not possible. So, I say I will do a weighted residual method, I take some testing function and the this the average of this residual I mean the average of this testing function with this functional, this average should be 0 instead of saying and instead of making you know the first equation is simply obtained if I put T_m to be a delta function.

So, I say that is too rigorous that is not possible. So, let us make it into a function with finite support right and we have seen what kind of function did we choose for the T_m in our earlier discussion?

Student: (Refer Time: 02:17).

They were these vector fields that look like this right. So, arrows, arrows, arrows everywhere right with the property that they were they had constant tangential component.

Student: On one edge.

On one edge and the tangential component on the other edge is where.

Student: 0.

0 right. So, those were our very nice basis function which was derived from the Lagrange polynomials ok. So, this we remember so, this for step 1 step 2 then we just plug this guy into this guy and use a little bit of vector calculus again this is just revision. Final this equation that we get it looks big, but it actually very easy we call this is the weak form of the FEM ok.

So, this is the point where no approximations have been made, no boundary condition has been applied yeah. So, you have the left hand side is purely over the boundary. So, this is your computational domain, capital omega and the boundary is gamma right. So, that is where your capital omega and gamma come into play ok. So, this is exact no boundary conditions.

Then we said this is the point of departure right this is where we will make a departure, but just to refresh your memory what did we do to apply the absorbing boundary condition we did this right we said that this $\nabla \times \vec{H}_s$ for a plane wave we have derived it, it was exactly equal to $-jk(\hat{k} \times \vec{H}_s)$. So, it is true for a plane wave going along any direction on the boundary we said we are going to enforce it along the $\hat{n}$ direction right. So, $\hat{n}$ is the outward normal.

So, when $\hat{k} = \hat{n}$ this relation is true for any plane where, but in general the waves hitting the boundary are not going to be along $\hat{n}$ they will be along any direction. So, that is why it was in exact hence the approximation and we are also said that this condition applies only to the scattered field not the total field that is why I am writing this as H_s. So, in the right hand side of equation 3 what I would do would be to write $H = H_{in} + H_s$ I will replace this right. So, that gives me my non-zero right hand side and rest of all we have already seen in the FEM module.

So, let us see now. So, this is why when I have my object over here I need to have some air over here right. So, how many unknowns are in this problem? What are the number of unknowns in this problem? As many triangles I mean as many elements? No.

Student: As many edges.

As many edges right is going to be equal to number of edges in triangulation of the domain, how I mean if you remember how did you express the unknown fields right you wrote this as $\vec{H} = \sum_{i=1}^{3} u_i \vec{T}_i(r)$ and T_i was my known vector field u's were the unknowns which I wanted to determine, and by choosing the these T s to be those first order Whitney elements these had unit magnitude unit tangential component along their respective edges. So, I multiply that by a unknown u_i which tells me what is the magnitude of the tangential field right. So, this is magnitude of tangential fields ok.

Student: Ok. So, far we (Refer Time: 06:44).

So far, well I will not call this volume integral because there is no integral equation over here, but this is the volume discretization it is a volume discretization, until I mean step 1 to 4 are standard FEM standard and standard FEM with ABC ok. So, what we will do is we will not go to step 4 we will stop at step 3 and we will see how we can marry it with boundary integral ok.

Now, if the number of edges ok so, I basically the number of unknowns is the number of edges. So, I am going to break this up into two different components, one is the number of interior edges and the number of boundary edges ok. So, I am going to say n is the number of interior edges and m is the number of boundary edges ok. So, my sparse system what is the size of my equations system of equations? My Ax=b what is the size of A? When I form assemble a system of equations using all of these what is my system the size of my system of equations?

Student: (Refer Time: 08:00).

(n+m) × (n+m) matrix size ok. So, this is as far as your revision of FEM was concerned. Now the second part is I mean look at the title is finite element and boundary integral. So, next we will see what is the just revise the boundary integral part ok.

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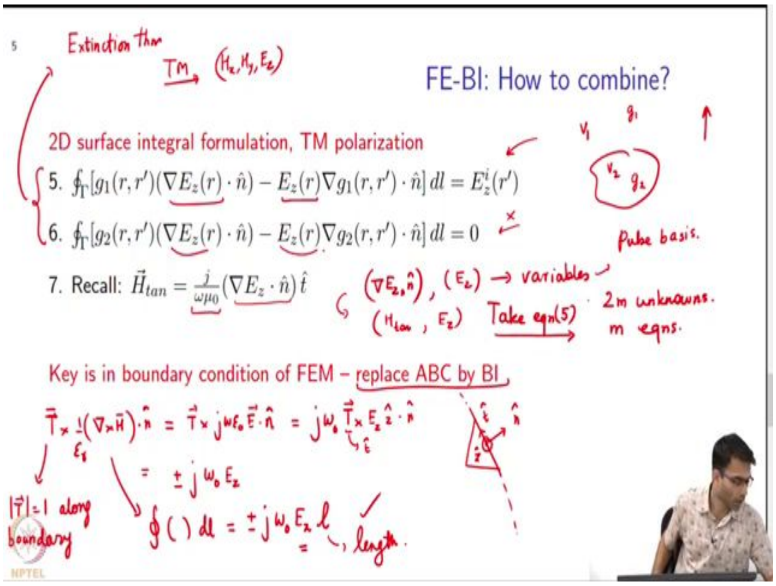

This is the stuff which you have already seen, but just a revision. These two equations that you see over here they are your familiar extinction theorem right. So, what was the problem that we had this was my V_2 and this was my V_1 right which had some g_1 Green's function and g_2 Green's function ok. The current the exciting source was in volume 1 which is why there is a incident field term over here, but not over here ok.

So, this hopefully it is just a revision for you. So, what were the variables over here again this is TM polarization. So, TM is H_x, H_y, E_z right that is my TM ok. So, E_z is a scalar right. So, your E_z is appearing here and here similarly here and here ok. So, now, the challenge that we have discussed using these boundary integral equations is that g_1 is fine because its free space g_2 is the problematic guy right. So, what do we do with problematic guys? We will check them exactly right. So, we do not use equation 6 let us for the moment ignore equation 6 why because it is not practical to use it.

So, now let us focus on equation 5, and well regardless of equation 5 or 6 if you notice one other calculation that we had done in the module on surface integral was that this $\nabla E_z.\hat{n}$. It was proportional to the tangential H field with some constants it was constants over here right correct.

So, this problem has been written as the variables are these scalars right sorry this is just E_z these are the variables. I can as well replace this as $\vec{H}_{tan}$ and E_z how many when I when I solved the when I for example, take equation 5 what will be the first approach that I will do to solve this equation? How do I what basis will I choose for my variables E_z and $\vec{H}_{tan}$ what did we do when we solved it? Pulse basis. So, pulse basis is what we used?

Student: (Refer Time: 11:34).

Exactly yeah that is a good that is exactly the observation will use here we if you use a pulse basis then my E_z and $\vec{H}_{tan}$ we are both expanding them on a pulse basis in each discretization of the boundary it is constant and with some coefficient which I will determine and zero everywhere else that is the basis right. If you look at the FEM basis functions they also if I think look at just a boundary edges it is also like that right it has a constant U_i on the boundary and its 0 everywhere else. So, that similarity is going to be what we will exploit ok.

So, now the key part is to get rid of this absorbing boundary condition. So, in other words this step 3 is where we draw the line here is where we draw the line. So, what term is it that we have to deal with? This is the term that we have to deal with ok. Notice that this equation is consisting only of H there is no E over there ok, but if I want to link it with the boundary integral equations somehow I should also get E over there. So, what is your guesses as to what to do over here can I get a electric field from in the right hand side somehow?

Curl of H reminds you a what?

Student: (Refer Time: 12:53).

Electric field itself right. So, what I can do is this $\vec{T} \times \frac{1}{\varepsilon_r}(\nabla \times \vec{H}).\hat{n} = \vec{T} \times j\omega\varepsilon_0\vec{E}.\hat{n}$ right. Is that approximate or exact? Exact right, so far no issues.

So, that just becomes I can take this $j\omega\varepsilon_0\vec{T} \times E_z\hat{z}.\hat{n}$ ok. So, let us take a sample case over here this is my $\hat{n}$ my object boundary right and I am just looking at one triangle on it may it is too skewed, may be you can just make it a little better this is the triangle on the boundary and let us say that my t hat is my tangential vector it goes along the boundary ok.

And which way is my $\hat{z}$? Coming out of the board now $\hat{t}$ and remember this term is only appearing on the boundary I do not have to think about it on the inside. So, on the boundary what is the value of T? It is a very simple question on the boundary what is the value of capital T?

Student: (Refer Time: 14:48).

Yeah, it was a trick question.

Student: 1.

1 that by definition that is how I constructed it, these capital T its one along the boundary 1 or minus 1 depending on the orientation of the boundary and choice of t right. So, what do you think will happen over here? Can this can this expression gets simplified? So, I have a vector of magnitude one along the $\hat{t}$ direction. So, $\hat{t} \times \hat{z}$ right hand term rule gives you $\hat{n}$ right $\hat{n}.\hat{n}$.

Student: 1.

1 right. So, this is actually going to give you $\pm j\omega_0 E_z$ that is it. So, this is actually I mean we are almost done with the conceptual part of this, what I have done is my original equation which was equation 3 here had H on both sides then I just use Maxwell's equations over here to recognize that curl of H is related to electric field, replace get that electric field in and realize that this whole thing is going to evaluate to just $\pm j\omega_0 E_z$ why where will the minus sign come from?

Student: (Refer Time: 16:31).

Right if T was pointing along $-\hat{t}$ its possible depending what is it depend on? When will it what determines the direction of t?

Student: (Refer Time: 16:42).

The node convention right the numbering of the nodes which is done by the CAD software. So, sometimes it may be this way sometimes it may be that way we just have to be consistent. So, that is where I get my term from. Now if I take this expression over here its not just this that I want I want to integrate it right. So, if I want to do integral of this guy over dl what will this become? I am assuming that my E_z is going to expanded on a pulse basis; that means, E_z is constant along that edge.

So, what will survive in this integral?

Student: (Refer Time: 17:20).

l the length of the edge right. So, this is going to be $\pm j\omega_0 E_z l$ ok. So, if I use this is I mean this is the key of what I am saying replacing the absorbing boundary condition by a boundary integral now what I have done is, my this FEM equation over here equation 3 the left hand side is the entire machinery of FEM left hand side is what is going to guarantee a sparse matrix for me.

And the right hand side term I am now replacing it in terms of E_z ok. Now you can ask I could have done this earlier also right. So, far I have not yet invoked boundary integral methods, all I did was to recognize Maxwell's equation saying that this term over here can be replaced by E. So, why did not I do this earlier?

Student: (Refer Time: 18:22).

No, I mean I how can replace it meaning Maxwell's equations makes this to be identically true regardless of any approximation, what is the problem here by doing this?

Student: (Refer Time: 18:37).

I am adding another unknown right then the and there is nothing known there is no incident field information coming into it. So, I will just the set of true equation which are not helpful ok. So, but what will what is going to give us the additional information is going to be the boundary integral part over here ok. So, I am just this is sort of step 1 first we just change the boundary condition term and we notice that I get a E_z over here ok.

Before we further if I take equation 5 over here how many equations in how many variables will I get? If I take only equation 5 I remember I have how many edges on the boundary? m edges on the boundary how many unknowns are there? m+m there now each boundary edge has a unknown E_z and a unknown $\vec{H}_{tan}$ right.

So, you have 2 m unknowns at how many points on the boundary are you testing are you enforcing equation 5? At the center of each edge that is what we are done.

Student: (Refer Time: 19:59).

Yeah. So, n is the number of interior edges. So, you have your triangulation of the scheme and the H could either be in the interior or on the boundary. So, n is interior m is boundary. So, that is why my matrix size was $(n+m)\times(n+m)$.

Student: (Refer Time: 20:15).

Yeah exactly so, yeah ok so, 2 m unknowns over here, but if I take only equation 5 how many equations will I get? m equations because there are m edges at the center of each I am enforcing this side. So, I have m equations which is why this is not alone if I use equation 5 alone I cannot solve this, if I add a another m equation from 6 then I can solve a $2m\times 2m$ system of equation, but I am going to stop short over here ok. So, now, we are in a good position to put this together.