Computational Electromagnetic:
Review of Maxwell's Equations
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Review of Maxwell's Equations
Lecture -3.4
Equivalence Theorem

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Now, this equivalence theorems are theorems where usually they are not encountered in undergraduate course, but they are very useful in computational problems. So, let us look at what equivalence theorem say.

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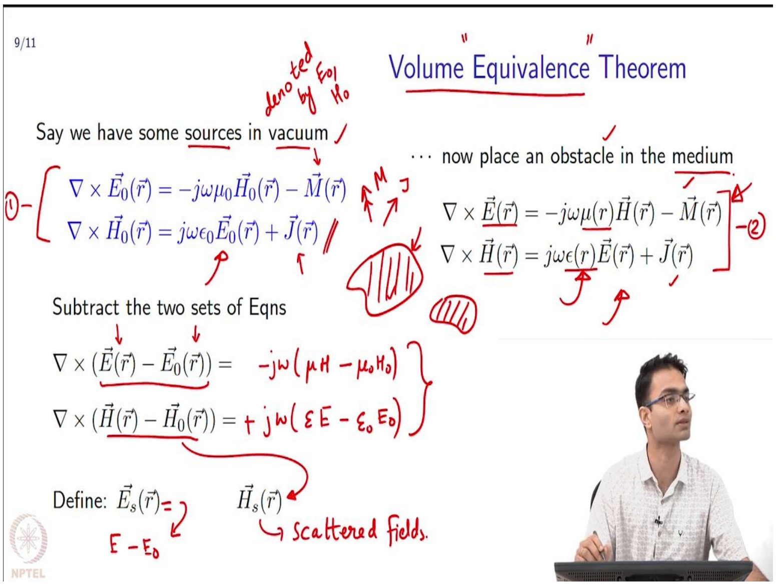

So, its lot of sets let us go step by step ok. So, first we look at what is called the volume equivalence theorem. So, let us start with the left part of the slide over here. So, let us say we have some sources, and these sources are given by M and J really simple. So, I have some current source over here M and some current source over here J. They are nicely radiating and looking at this blue equation over here, you can tell that there is no object because which part of the equation tells you that there is no object? Where do the medium properties going to? Epsilon and mu and what are the value of ε and μ here? $\mu_0\varepsilon_0$. That means there is actually no object. So, it is vacuum right.

So, this is setting these currents are setting radiating in empty space. So, the fields in empty space or vacuum they are denoted by the 0 under script right. So, denoted by $E_0 H_0$. So, the naught on the subscript tells you that there is nothing; it is just a current sources ok. These equations, we already had brief glimpse of how to solve them it was simple you take curl of first equation, substitute the second equation; you get a you will get a kind of wave equation.

But only difference will be that wave equation will have these a current sources. Now let us make life a little bit more interesting because real life will objects ok. So, now, let us say I play obstacle over here ok. What how should I characterize an obstacle or any object? By the medium properties.

Student: μ_1 .

ε and μ which will be a function of space. For example, here there is nothing and here there is the object right. So, all of that information is captured in ε_r and μ_r . So, that is why denoted as a function of space. So, these equations that are written now are in the presence of a medium. So, apart from epsilon and mu what is the other change that happened? Do you notice any other change between these two sets of equations?

Student: E has changed (Refer Time: 02:53).

E has changed; obviously, once I introduce an that object, that object is going to scatter the fields. So, the fields will no longer be $E_0 H_0$ and now the fields become E and H. Will the currents change? The current sources will remain the same; I may have kept my transmitter still standing out there right. So, M and J are the same as before. So, this is the situation that we have; now remember that this is what is called equivalence theorem. So, equivalence means I am going to convert one problem to another kind of problem which may be easier to solve that is the objective ok. So, we will see how that objective is achieved over here.

So, far it is clear I started with vacuum, then I went to an obstacle. Now I do not want to deal with the external current sources M and J because, they may be very complicated to specify. So, the natural thing that I could do is I can take these two sets of equations and subtract them. So, if I subtract them the advantages what vanishes?

Student: M and J.

M and J go away right. So, if I subtract these two equations what happens is I will get a $-j\omega$. So, $\mu_0 H_0$ from the first term sorry; this will be μH because I subtracted 2 from 1 and minus $\mu_0 H_0$, the M term is vanished. I can do the same thing; I can do the same thing for the second set of equations involving curl of H. So, I am going to correct a $j\omega\varepsilon E - j\omega\varepsilon_0 E_0$. Really simple manipulation so far ok. The beauty of these two equations is that you only have, you do not have any current sources, you just have the fields and the material properties which I know.

Now, this is a good time to introduce one concept there is quantity called as the scattered field you all heard this word scattered field. So, what is the scattered field defined as? So, the difference between the electric field in the presence of object and absence of object right; so, this term over here is defined as the scattered electric field ok. So, the word make sense scattered field means the fields scattered by the object, I do not want to include the incident field that was already there. So, I subtract off E naught ok.

So, that is what is the scattered electric field, similarly I can take the difference in the magnetic field with and without object and difference I will call as the Hs, s for scattered right we call this scattered.

Student: Sir, suppose we have two objects (Refer Time: 06:01).

There can be as many objects in this problem it does not matter all I have to do is include it into epsilon r as the function of space epsilon r can be any complicated function does not matter ok. So, I can have one another object over here no problem, I just have to include that in the definition of epsilon ok so, this is very general. Even the number of sources can be as many it does not matter they all cancel off. I that fine?. So, so far we have not actually solved anything we have just eliminated sources. So, what further can we do this?

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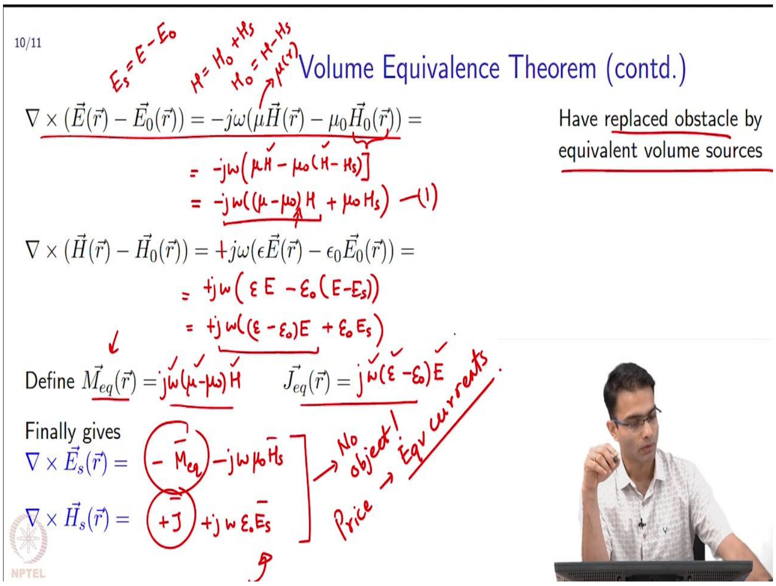

So, let us have a look at that. So, I have rewritten the equation from the previous slide over here right. So, the difference in the electric field incident and total is this is the scattered field as I already mentioned and this is what we already had ok. So, let us try to do a little bit more algebra on this to bring it into convenient form ok. So, E_s, I have written as $E - E_0$ right. So, what I can do is this term over here H_0 how what I can write H_0 as. Can I write H_0 in this form right? The total field that is not clear we can say that the total field H is the sum of $H_0 + H_s$. So, it says that H_0 can be written as $H - H_s$ sure. So, this term over here can be simplified as $-j\omega(\mu H - \mu_0(H - H_s))$ right.

Student: (Refer Time: 07:50) me and epsilon are not function of r?

They are I have just. So, μ and. So, μ over here I am being lazy over here μ is still the function of r μ_0 is of course, a constant. So, in shorthand notation I am just writing it as a new, but they are all functions of space ok. There is no need to be giving special treatment to mu or epsilon they all functions of space ok. And what I can do now is I can combine the H terms right. So, I can get $(\mu - \mu_0)H + \mu_0 H_s$ ok. Very good I can repeat the same algebra for the second equation also right. So, I will write this as $-j\omega\varepsilon E_i$ leave it as it is and $-\varepsilon_0(E_0 - E_s)$ in this case right. Again I can combine the $\varepsilon - \varepsilon_0$ terms into a E and I will get a.

Now, to make our life a little bit more simpler, let us define yet another term over here just to simplify the math. So, this term over here that you see right it is a longish term. So, let us what should we what we can do is, we can write define something called a equivalent current and what is equivalent current? Simply $j\omega(\mu - \mu_0)H$ ok. This term over here I have put into M equivalent ok. I have not solved any problem I am just re writing these terms. Similarly the j equivalent term over here will be written as; so, this term over here yeah.

Student: (Refer Time: 10:08).

We will see alright; So now, when we combine these equation. So, what will the first equation? So, let us call this equation 1 over here. So, equation one will become. So, the first term over here will become minus M equivalent minus j omega mu naught H s right the second term becomes what do I have over here minus J and I have a minus j omega epsilon naught s. So, let us see I have a feeling I missing minus there is a extra minus sign over here right if I look at these equations yeah right. So, there was there was this was a plus sign over here right. So, there should not be any minus sign. So, we will keep this plus over here yeah that was a mistake of a minus sign no problem.

Now, when you look at these equations, what do they what do they look like?

Student: (Refer Time: 11:33).

They look like original Maxwell's equations in which case.

Student: (Refer Time: 11:37).

When there is objective or there no object.

Student: There is no object.

No object right. So, this is, but what have we have to pay as a price?

Student: (Refer Time: 11:53).

We have to now introduce some now new equivalent currents right, but the price here. So, this volume equivalence principle what now we can actually have a statement what it is. What have we done we started with an object over here and we started with these sets of equations. As a result of some algebra I have reduced down to these sets of equations which are about sources radiating in free space right $\mu_0\varepsilon_0$ is there free space. So, what I have done is I have replaced this entire obstacle by a set of equivalent volume sources right. So, these M and J these are sources ok.

So, as we will see later on in the course, these kinds of equations the once in vacuum are simpler to solve then these kinds of equations where epsilon is the function of space. So, I have not actually solved the problem, I have just converted it into a form that is easier to look at. So, once again I want to say look at the definition of the equivalent current that I have. The definition of the equivalent current contains something which I do not know. I know let us say j omega mu minus mu naught I know, but I do not know H I do not know the total field.

Similarly for j equivalent I know omega epsilon. But this one the electric field I do not know. So, I have made it I have not actually solved the problem I have just transferred the problem into M equivalent and J equivalent ok. So, the reason for this will become sort of clear as we go along in this course, but this is a popular way of replacing an object by current sources magnetic and electric ok.

Remember this is a equivalent problem this does not physically exist. There are not physically current sources sitting in space. I am mathematically thinking of them as this is the same the solution is the same, where I use the this an equation all I did was add subtract and things like that right. So, this is the volume equivalence theorem right ok. So, this was the volume equivalence theorems.

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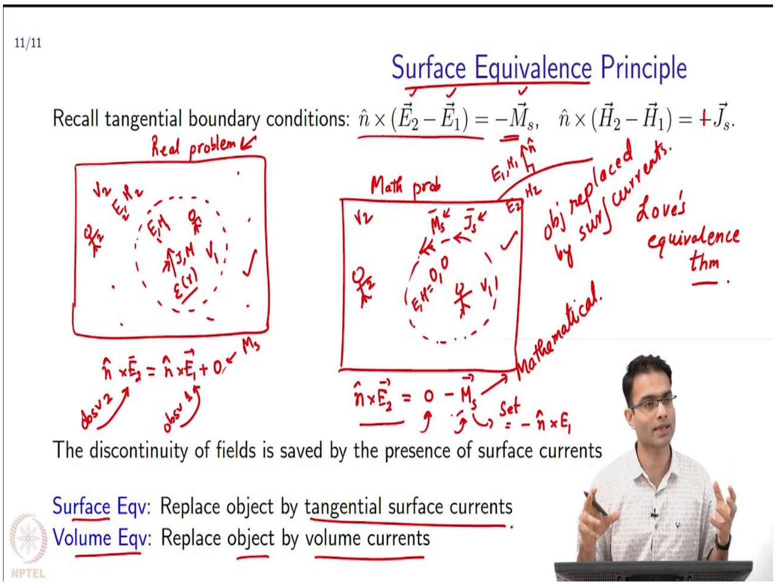

The second equivalence theorem is called the surface equivalence principle ok. So, surface equivalence principles as you can the word suggest I have to now think of the interface between two objects ok. So, there is a sum normal over here let us say $E_1 H_1$ and $E_2 H_2$.

Student: Say in that H 2 minus H 1 is positive (Refer Time: 14:56).

Again yeah the mistake here in the minus sign this should be plus J_s ok. So, let us let us start with start with the real problem the physical problem like earlier started with physical problem and then the converted into an equivalent problem. So, what is the physical problem over here? So, I have some current sources over here J and M and there radiating in some volume ok. So, let us take a hypothetical surface. The surface does not actually exist I have just mathematically drawn a box and let us say this is overall this is my the domain that I am interested in ok. So, we will we will call this region 1. So, let us call this V_1 let us call this V_2 ok. So, let us play a small game let us put one observer over here and there is another observer over here ok.

Now, what are the properties of these observers it is a strange game. This guy observer 2 can walk around anywhere in volume 2, but cannot walk into volume 1. And observer 1 over here vice versa can walk in volume 1, but cannot crossover into volume 2. So, this is we will call this a real problem right. Now, this dotted surface which I have shown you over here it is a hypothetical surface, it does not actually exist I have just drawn it right. So, what do you expect of the fields across the boundary? It would be the same because I have just drawn a line in air right the fields will be continuous across this hypothetical surface. So, when I look at if I ask observer 2 observer 2 has access to which of these quantities?

Student: E 2.

E_2 right E_1 the observer cannot cross over right. So, if I asked you observe what is $\hat{n} \times E_2$ for you? He will say $\hat{n} \times E_2$ for me will be equal to he will say hey observer 1 tell me what is the value of E_1 on your side he will write that over here. So, he will write $\hat{n} \times E_1$ this has come from the other observer and some neutral referees required to tell them what is the is there any magnetic current over there?

Because, there is magnetic current is on the is a pure surface current it does not it is not either here nor there is exactly on the boundary right. So, they are told there is there a magnetic current in the hypothetical situation will there be? No right because it is in thin air, I just made line in the air and said what is the current over here there is no current it is free space there is no current hanging out there how will they be right. I just drew a line in air. So, this will be written as plus 0.

So, observer 2 has access to this $\hat{n} \times E_2$ and observer 1 has access to this and they say fine they compare $\hat{n} \times E_2$ $\hat{n} \times E_1$ what do you expect the answers to be they will agree boundary conditions, even though I have created hypothetical surface boundary conditions should still be valid. So, they will agree no problem. Now, will play small will play yet another game. So, the game is to sort of make a fool of observer 2 in a way that observer 2 does not know. So, what we do is, let us take the same situation over here observer 2 remains over here cannot walk into volume 1 this is V_2 and this is V_1 V 1.

What I do is; so, here fields where E H and E H 22 11. Here what I do is I set the fields equal to 0 remember this is how equivalent problem it is not a physical problem. The physical problem was here this is a math problem. So, let us play along with this game the current sources are here actually we need not worry about the current sources over here, I am setting the fields over here E, H equal to 00 that is that is what observer 1 is going to report. Observer 1 is going to play little bit of a mischief, observer 1 is going to say fields are 0 inside.

So, observer 2 asks the same question he says value of my field over here $\hat{n} \times E_2$ E_2 he has access to. Now, he is asking the observer 1 hey what is the field inside ok. Observer 1 being a mischief says oh field inside a 0. So, the at the field inside a 0. To save the is there anywhere to save the situation and still be truthful to physics? Observer 1 has he is fixed I am going to report zero field is there any way to fix this situations such observer 2 does not know that anything different happened?

Student: Surface (Refer Time: 20:32) that produces (Refer Time: 20:33).

Exactly so, if I look at the boundary conditions over here, what is these two are being fixed by observer 1 and observer 2. So, where is the remaining gap? It is in this term over here. So, if I introduce a mathematical current which is equal to. So, if I introduce. So, this will be. So, M_s in this case was 0 write this was M_s I introduce a new M_s and what should I set as the value of this M_s?

Student: Minus h plus (Refer Time: 21:14).

Right so, supporting a set its value equal to $-\hat{n} \times E_1$ ok. So, observer 1 set the fields equal to 0 that is alright I save the situation by introducing a mathematical remember this is a mathematical current over here which is equal to this. So, now, this hypothetical surface has a current M s similarly to save the other boundary condition I will have to introduce the current Js. So, if I did this trick will observer to know any difference.

Observer 2 will not know any difference, observer 2 walks around everywhere inside this volume 2 and at the boundary it just verifies $\hat{n} \times E_2$ is equal to what it was earlier? Answer is going to be yes because I have put this hypothetical current over here $M_s J_s$ ok. So, the boundary conditions are satisfied right what was their earlier is there now. So, the fields inside this volume and this volume are the same by uniqueness theorem because e tan has been specified and is the same in both cases field will be same in both cases. So, observer 2 did not know what happened, but just see what is if you get the overall picture what have I done, I have punched out one volume and replaced it by a 0 fields and set of equivalent surface currents. So object is replaced ok.

So, again this is this can be useful if for example, I want to reduce the area of my computation I do not want to compute the fields everywhere in $V_1 + V_2$. If I can remove V_1 the small price I have to pay is the surface fields on this. So, they will become the new unknowns, but what have done is punctured out some volume over here. So, these are these this is the sort of reasoning behind equivalence principles, the volume equivalence we saw that the object was replaced by volume current J and Ms we saw. J equivalent M equivalent and in the surface equivalent principle I replaced the object by tangential surface currents ok.

So, again this will seem a little bit abstract right now that why are we doing this, but when we begin to take the first method in computational electromagnetic which is which are integral equation method, it will become very very useful having done this. So, one example can give you is object over here, it may have some very complicated epsilon r and I do not have to deal with it by doing something like this by setting by removing excluding this volume but.

Student: Also you removed j n h (Refer Time: 24:18).

I removed the field E and H inside the thing I do not have to worry about this currents over here because the role of this currents finally, is to produce the fields E and H.

Student: Yeah, that is (Refer Time: 24:27).

Yeah right. So, this sort of brings us to an end of the two main equivalence principles that we use. By the way this particular surface equivalence principle that I told you goes by the name of Love's equivalence theorem. There are other possible variations of this you can imagine that since this game that I am playing I have to keep en cross E 2.

So, what I did I put all the field 0 and everything was absorbed into the current, but I can think of some other contribution will come up with some other equivalence principle right and there are several in the literature. So, in the hands out I will give you can look at these other situations. What is the main objective? As I mentioned once again earlier we want to reduce pain in calculations. So, we will construct these equivalence principles such that the problem becomes easier to solve ok. So, that is the idea behind the surface equivalence principle ok.

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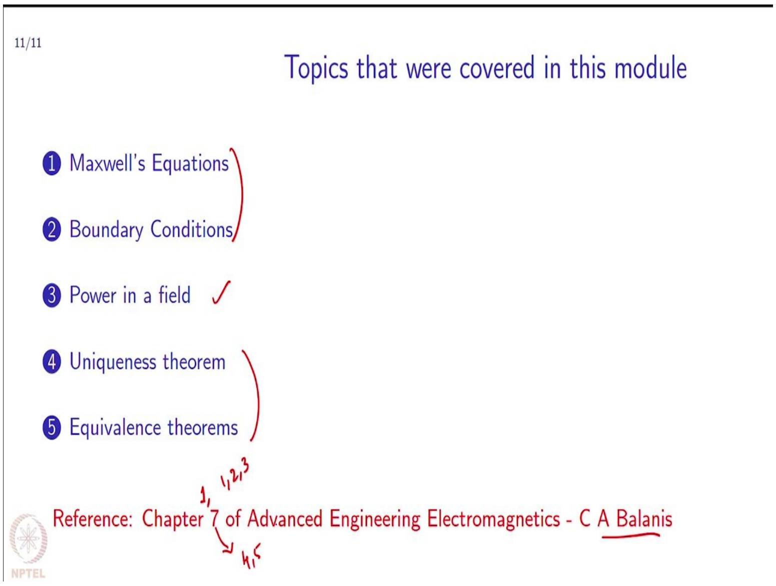

So, to sort of summarize this module we started with Maxwell's equations and boundary conditions, looked at the power and these two were sort of advance concepts in electromagnetic ok. So, you can look at; so, chapter 1 and 7. So, chapter 1 of Balanis will talk about topics 1, 2 and 3 and chapter 7 of Balanis will tell you about equivalence theorems and uniqueness theorems. It is a very nice readable book, please have a read through it.