Digital Signal Processing
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Lecture 09:
Elementary Signals (3) - Is $e^{j\omega_0 n}$ always periodic?

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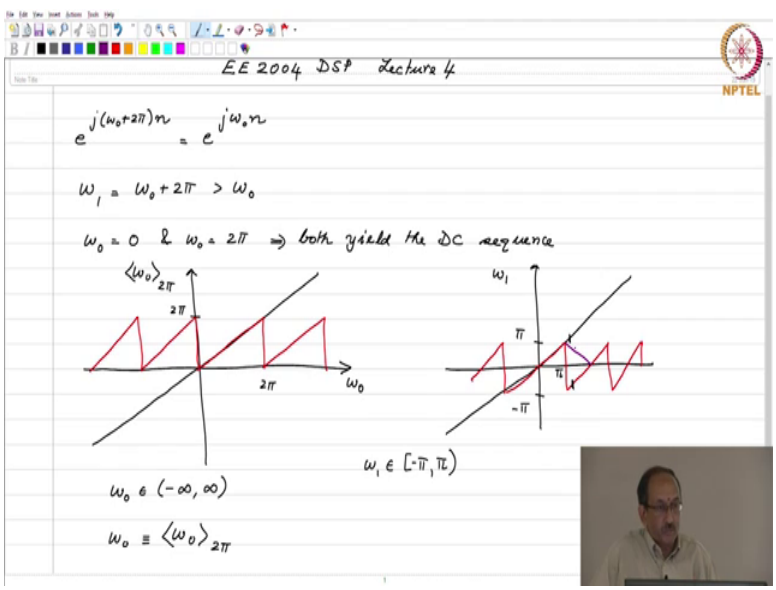

Let us continue from the last lecture. So, we were looking at discrete-time sinusoids and the difference that we saw was, $e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}$. Therefore, $\omega_1 = \omega_0 + 2\pi > \omega_0$ yields exactly the same signal. Even though the frequency is different as far as the signal is concerned, you get the identical signal back and we saw that $\omega_0 = 0$ and $\omega_0 = 2\pi$ both yield the DC sequence.

But as ω_0 increases, you expect the rapidity of the oscillations to increase, but looks like it is going to slow down at some point and come back to the DC sequence and this is easily seen in this example.

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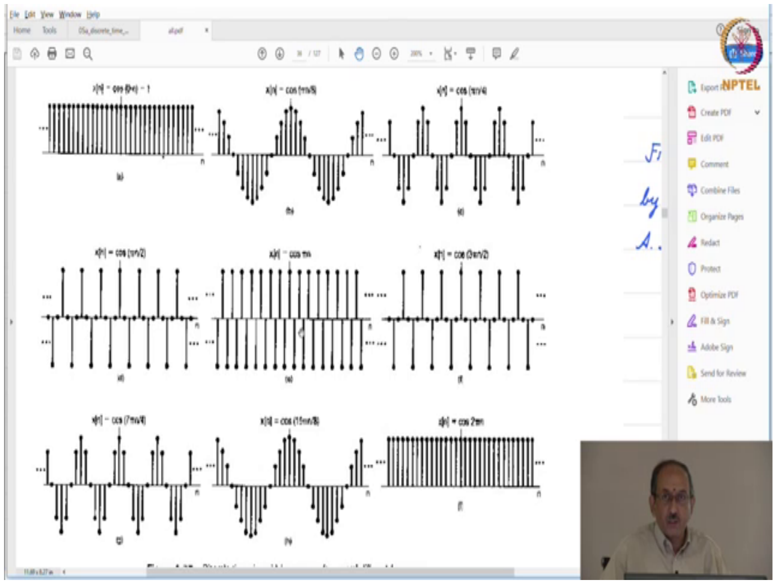

So, this is from Oppenheim's book. So, we are looking at the cosine signal here. So, this is ω_0 being 0 and then the frequency is increasing, as we go along. This is $\cos\left(\frac{\pi}{8}n\right)$, this is $\cos\left(\frac{\pi}{4}n\right)$, this is $\cos\left(\frac{\pi}{2}n\right)$ and so on and this is $\cos(\pi n)$, $\cos(\pi n)$ is $(-1)^n$. So, this is 1, -1, 1, -1 and so on.

And now, ω_0 keeps on increasing beyond π and this is $\cos\left(\frac{3}{2}\pi n\right)$ and this is further increase in ω_0 numerically. So, this is $\cos\left(\frac{7}{4}\pi n\right)$, this is $\cos\left(\frac{16}{8}\pi n\right)$. And clearly you see beyond π, the sequence, the rapidity of the oscillations slows down and when you hit 2π you are back to where you started off, namely ω_0 equal to 0.

So, this is a simple example that illustrates the fact that, as you increase ω_0 the rapidity does indeed increase, but unlike in the continuous-time case, looks like it reaches highest frequency of oscillation and then the signal starts to slow down and then comes back to the DC sequence on $\omega_0 = 2\pi$.

So, this can be kind of captured in this. So, this is ω_0; ω_0 belongs to the interval $-\infty$ to ∞; however, as far as the signal is concerned, because ω_0 and $\omega_0 + 2\pi$ are the same frequency in terms of the signal that is being produced, you can replace ω_0 by $\omega_0 + 2\pi$. And if you plot $\langle\omega_0\rangle_{2\pi}$,this is what you get.

So, this is $\langle\omega_0\rangle_{2\pi}$ and this is the behavior of the modulus function with respect to 2π. And moment you take mod, it is clear that you need to consider values of ω_0 only between 0 and 2π because any value outside this interval can be mapped to within this range.

The other way of looking at this is rather than restricting it from 0 to 2π, you can go between $-\pi$ and π. So, the other graph that shows this, is this. So, this point is pi, this also is π, this value is $-\pi$. As ω_0 becomes greater than π, you wrap it and then it maps a point here, maps to a point here.

Let me call this as ω_1 and ω_1 belongs to the interval $-\pi$ to π. And looking at ω in this range between $-\pi$ to π, get a feel for why the signal frequency slows down. If you know plot mod ω_1 then of course, from 0 to pi it is the same curve nothing changes because it is positive, but if you look at the value of ω_1 beyond π, then if you look at the modulus then the curve looks like this.

So, this also gives you a feel as to why the frequency decreases. So, from 0 to π, the rapidity of the oscillations increases. At π, you reach the highest frequency and then beyond π, the frequency actually decreases, as you can see from the modulus function that is taking the absolute value of ω_1.

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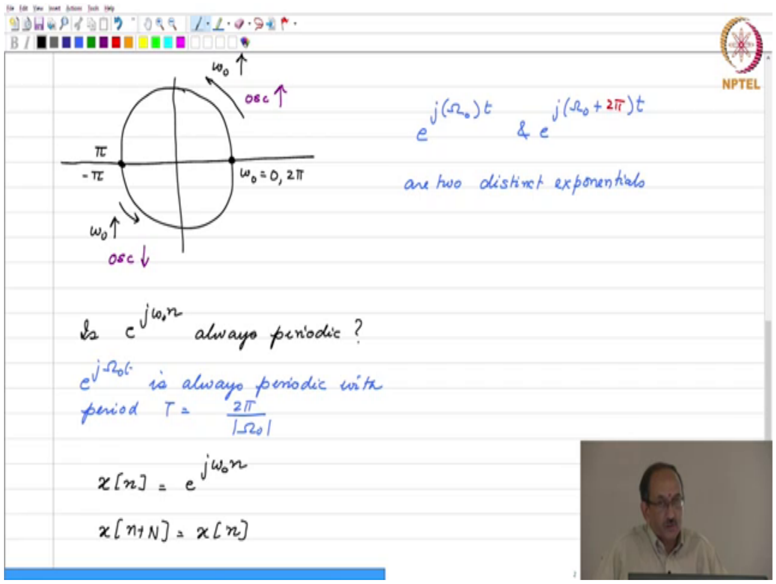

So, because ω is going to be restricted in the interval 0 to 2π, it is good to plot that on a circle rather than on a line. So, this is $\omega_0 = 0$. Here, ω_0 is increasing, the oscillations are also increasing.

So, this happens till ω_0 reaches π. So, this is $\omega_0 = \pi$, π is also the same point as $-\pi$. Now, beyond π, ω_0 is still increasing; however, the rapidity of the oscillations decreases and it keeps on decreasing till you hit 2π when you now reach the DC sequence.

So, this is another way of looking at how things are as ω_0 changes from the interval 0 to 2π and typically we will be focusing on ω_0 in the range $-\pi$ to π. So, this is one distinct difference between discrete-time and continuous-time. Recall that in continuous-time, $e^{j\Omega_0 t}$ and $e^{j(\Omega_0+2\pi)t}$; these two are two distinct exponentials, they are not the same.

So, this is the first difference. The other difference again, this is a review for you. Is $e^{j\omega_0 n}$ always periodic? Remember $e^{j\Omega_0 t}$ is always periodic with period $T = \dfrac{2\pi}{|\Omega_0|}$. Now, let us consider the discrete-time sequence, we want $x[n] = e^{j\omega_0 n}$, under what conditions is $x[n+N] = x[n]$?

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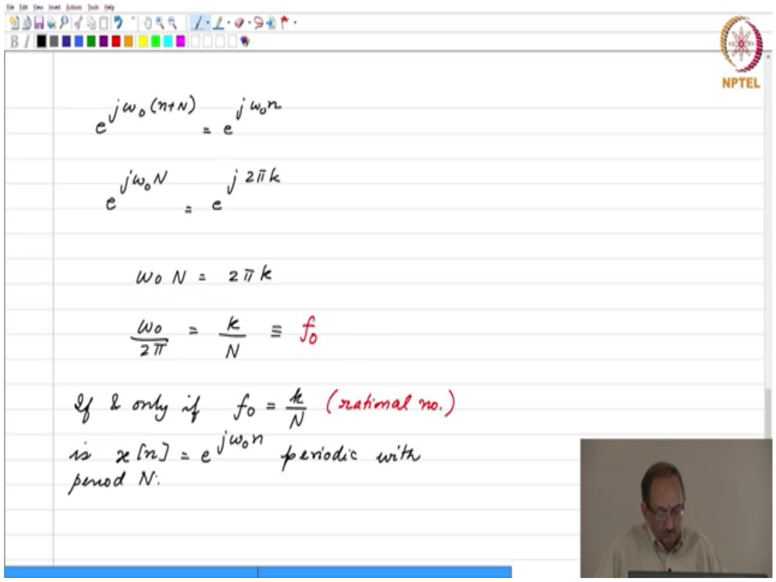

All you need to do is you need to see under what conditions, $e^{j\omega_0(n+N)} = e^{j\omega_0 n}$. So, clearly you require $e^{j\omega_0 N}$ should be $e^{j2\pi k}$. Therefore, you require ω_0 must be the same as $2\pi k$ or $\frac{\omega_0}{2\pi} = \frac{k}{N}$.

And $\frac{\omega_0}{2\pi}$, the commonly used notation is to denote this as f_0. So, if and only if $f_0 = \frac{k}{N}$. So, this means this is a rational number. Only in these conditions, $x[n] = e^{j\omega_0 n}$ is periodic with period N. Now, the further inferences can be drawn from this development.

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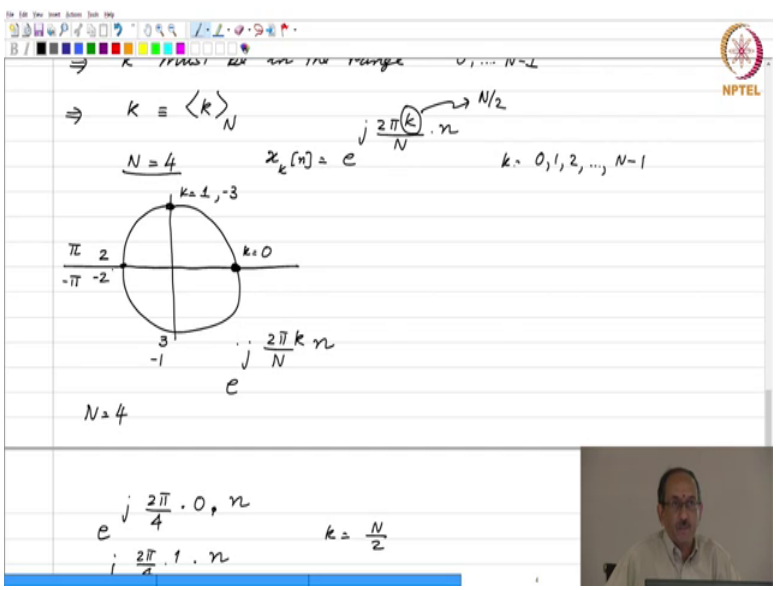

Recall that ω_0 is really $\langle \omega_0 \rangle_{2\pi}$, which means that f_0 which is nothing but $\frac{\omega_0}{2\pi}$ is really $\langle f \rangle$ what?

Student: 1.

mod 1, right; because, if ω_0 is mod 2π since $f_0 = \frac{\omega_0}{2\pi}$, f_0 has to be in the range 0 to 1 or between?

Student: $-\frac{1}{2}$.

$-\frac{1}{2}$ to $\frac{1}{2}$. So, this means that f_0 which is $\langle f_0 \rangle_1$, remember, f_0 for the exponential to be periodic has to be of the form $\frac{k}{N}$ and k takes on the values $0, \pm 1, \pm 2$, and so on, but because f_0 is the same as $\langle f_0 \rangle_1$, then you need to restrict yourself as far as k is concerned to the range, what is the value of k that you need to consider?

Student: (Refer Time: 13:22) $-\frac{N}{2}$ (Refer Time: 13:24).

$-\frac{N}{2}$ to $\frac{N}{2}$ that opens up for N odd, N even and so on. An easier way of stating that would be 0 to $N-1$. So, this must be in the range $0, 1, \ldots, N-1$ because, if k were equal to N then that is same as k taking the value 0. So, this implies that k is really $\langle k \rangle$, k mod?

Studemt: N.

N, all right. For example, if you take $N = 4$.

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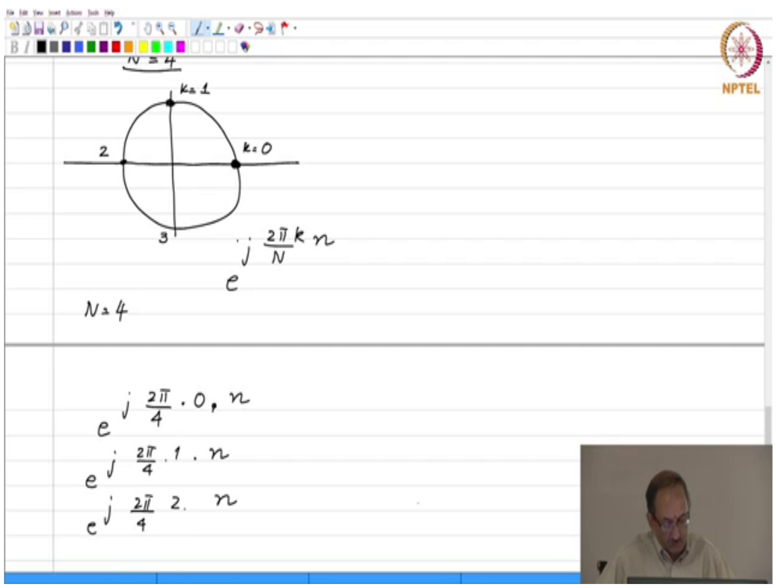

Remember, we are now looking at complex exponentials that are periodic. So, if you consider $N = 4$, I am now plotting the various frequencies that are possible assuming that the periodicity is 4. So, which means k needs to take the value $0, 1, 2$ and 3.

Therefore, if k takes on the value 0, you will have ω_0 here. So, this is $k = 0$ and this is $k = 1, 2$ and 3. Therefore, you have $e^{j\omega_0 n}$. Now this is, in this particular case, we have N to be 4. Therefore, let us look at all the possibilities. So, this can be written as $e^{j\frac{2\pi}{N}kn}$.

So, these are the various complex exponentials that are possible for period N and in this particular case, we are going to let $N = 4$. Therefore, we have $e^{j\frac{2\pi}{N}.0.n}$. So, this gives you the first complex exponential. Then, $e^{j\frac{2\pi}{N}.1.n}$, $e^{j\frac{2\pi}{N}.2.n}$, $e^{j\frac{2\pi}{N}.3.n}$.

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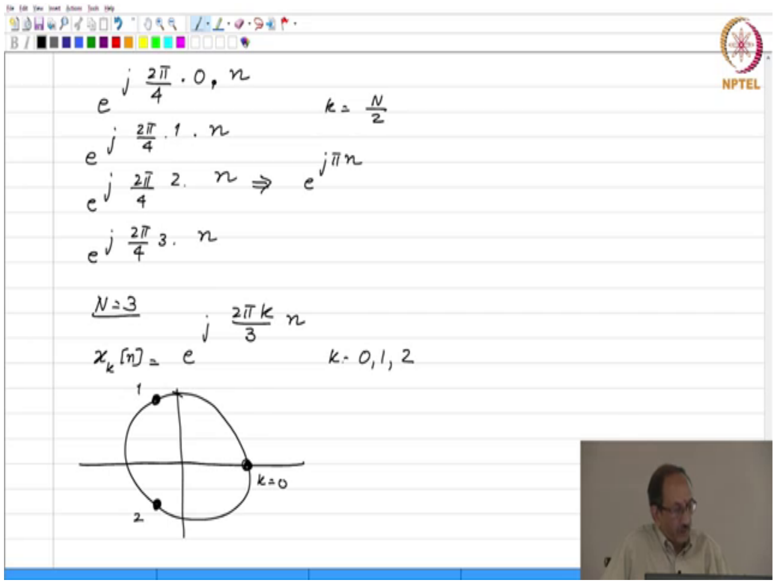

So, these are the four distinct exponentials that are possible here. Note that when $k = \dfrac{N}{2}$ when N is even, then the frequency corresponding to that particular value of k is; when k is $\dfrac{N}{2}$ remember, we are looking at the set of complex exponentials $e^{j\frac{2\pi}{N}.k.n}$.

So, let me call this as $x_k[n]$ and k itself will take on the value $0, 1, 2, \ldots N-1$. Here I am considering the case as a specific example, when N is 4. So, I get these four distinct exponentials, whatever distinct means we will make it more precise. So, when N is even, when $k = \dfrac{N}{2}$, what will be the signal that is produced?

Student: (Refer Time: 18:07).

So, in this particular case, you get $e^{j\pi n}$. Because, in this if you put $k = \dfrac{N}{2}$ where N is even, you will get $e^{j\pi n}$. So, remember this also corresponds to the highest frequency π which is also the same as $-\pi$. Because k is the same as $\langle k \rangle_N$, the index $k = 3$ also corresponds to the index k equal to, what is one other k that you can immediately think of in the place of 3?

Student: 7.

7.

Student: Minus 1.

7 also is correct. -1 also is the right answer. So, this also corresponds to $k = -1$, this corresponds to $k = 2, -2$ and this corresponds to $k = -3$. And these are called a set of harmonic sinusoids. These sinusoids are set of sinusoids that are harmonically related. And when N is even, when k hits $\dfrac{N}{2}$, you will reach the highest frequency, namely π.

On the other hand, suppose $N = 3$, then $x_k[n] = e^{j\frac{2\pi}{3}kn}$. And here, k takes on the values $0, 1, 2$ and if you plot these points on the circle $k = 0$ will correspond to this frequency, $k = 1$ and $k = 2$.

So, these are the three values of k and these are the values of $k.\omega_0$ that I have plotted on the circle here. Notice that in this particular case, you do not hit the highest value of π.

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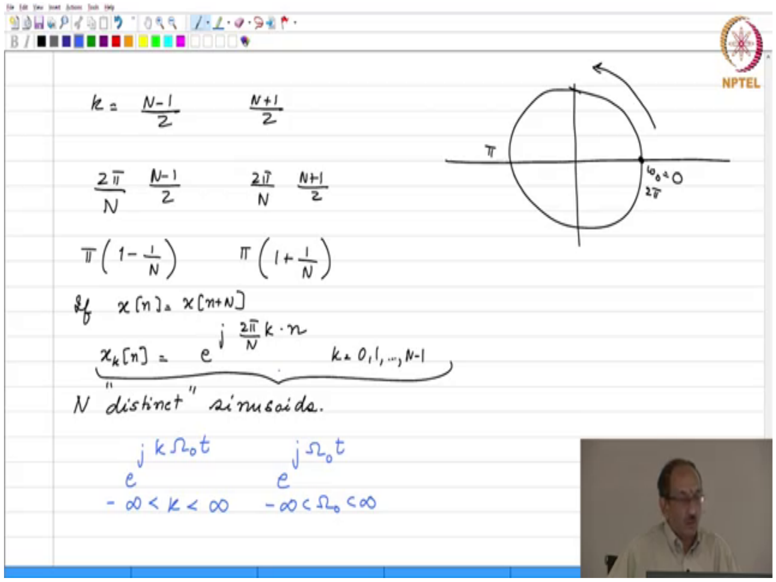

So, remember when N is odd, what is the closest integer that gets you to π? What value of k gets you a frequency that is closest to π?

Student: (Refer Time: 21:28)

$\frac{N-1}{2}$. So, $k = \frac{N-1}{2}$ produces a signal whose frequency is closest to π. The other value of k that will also produce something that this closest to π will be?

Student: $N+1$.

$\frac{N+1}{2}$ and what will be the actual frequency? So, $\frac{2\pi}{N}k = \frac{2\pi}{N}\left(\frac{N-1}{2}\right)$ and here you have $\frac{2\pi}{N}k = \frac{2\pi}{N}\left(\frac{N+1}{2}\right)$. So, the actual frequencies are $\pi\left(1-\frac{1}{N}\right)$ and $\pi\left(1+\frac{1}{N}\right)$. So, these are the two sinusoids whose frequencies are closest to π. You do not actually hit the π frequency because N is odd.

Therefore, if in general if $x[n] = x[n+N]$, then we produce sinusoids of the form $x_k[n] = e^{j\frac{2\pi}{N}kn}$ for $k = 0, 1 \ldots N-1$. And then, we have each of these sinusoids is periodic with period N and we have N distinct sinusoids.

Now, let us kind of understand this a little more. So, we need to restrict ourselves only in the range 0 to 2π. So, we start off with $\omega_0 = 0$ and then we keep on increasing ω_0. For every value of ω_0, you produce a sinusoid and ω_0 varies continuously from 0 to 2π.

So, as ω_0 varies continuously from 0 to 2π, you produce uncountably infinite different sinusoids. In this range, among these uncountably infinite number of sinusoids, there is one finite set of sinusoids, namely $e^{j\frac{2\pi}{N}kn}$, among these uncountably infinite set of sinusoids, this set of sinusoids is the only one that is periodic with period N.

Whereas in the continuous-time case, you have $e^{jk\Omega_0 t}$, this is the counterpart there and then you let k in the range $-\infty$ to ∞. So, these are harmonic sinusoids in the continuous-time case, you have a countably infinite set and all of them are periodic.

The general case is you have $e^{j\Omega_0 t}$ and now you can let Ω_0 in the range $-\infty$ to ∞, Ω_0 varies continuously in this range. Again you get an uncountably infinite number of sinusoids.

Here, every single sinusoid is periodic. In the continuous-time case, every single sinusoid is periodic. Whereas in the discrete-time case, as you let ω_0 vary from 0 to 2π continuously, you get an uncountably infinite number of sinusoids, but among them only this finite set of sinusoids, namely N in number are periodic. So, this kind of, reinforces the similarities and differences. We made the statement, we have N distinct sinusoids. So, this is a rather descriptive term we will make this term more precise and see what distinct really means.