Digital Signal Processing
Prof. C.S. Ramalingam
Department Electrical Engineering
Indian Institute of Technology, Madras

Lecture 72: The Discrete Fourier Transform (4) - Circular convolution

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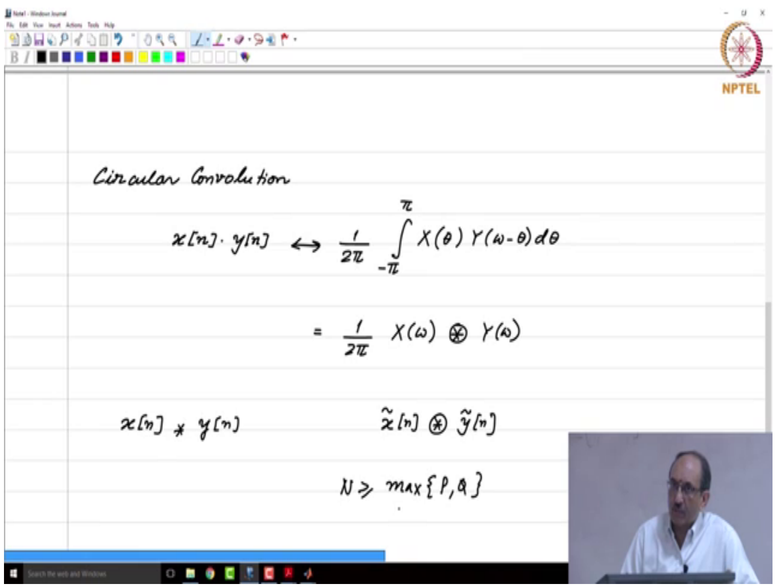

Now, let us look at Circular Convolution and if you recall, our first mention of circular convolution happened in the context of the properties of the DTFT. We made the remark that, if you have a sequence $x[n].y[n]$, then the transform was convolution. If you multiply in the time domain, you convolve in the frequency domain.

However, in this particular case, the frequency domain transforms are periodic with period 2π and hence recall, we use the $X(\omega)$ notation here. And, if you look at this definition, we made the remark that this is nothing but $\dfrac{1}{2\pi}X(\omega) \circledast Y(\omega)$ except that this convolution is circular convolution and not linear. And, we said that, we will talk more about circular convolution later and we are now going to get into more details of circular convolution.

And, circular convolution applies to periodic signals and the example as given here is circular convolution applied to signals whose independent variable is continuous. Because, this is 2π periodic $Y(\omega)$ also is also 2π periodic and this is the definition and this $1/2\pi$ comes because of this. So, what is captured within the integral sign, the integral as given above is denoted in shorter notation by this circular convolution.

And this circular convolution also applies to periodic sequences and in that case, instead of the integral, we will have summation. And, now we are going to see more details of circular convolution as applied to sequences. Same concept applies to signals whose independent variable is continuous.

And, then we will also examine whether there is if at all, any connection between circular and linear convolution.

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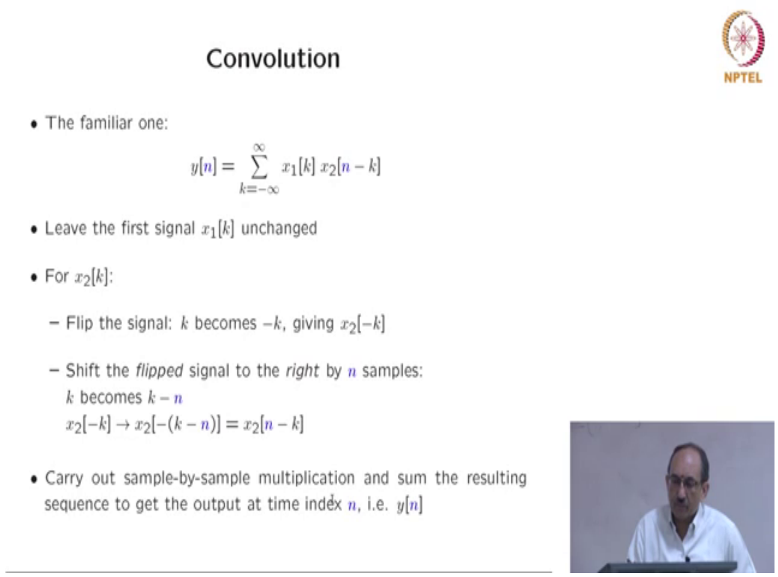

So, now let us look at this. So, we are familiar with linear convolution. So, $y[n] = \sum_{k=-\infty}^{\infty} x_1[kx_2[n-k]]$. And if you recall, we replace n by k. We keep one signal unchanged; take the other signal, flip it and shift it and then multiply sample by sample and then sum up over all k and then you will get a number, that number will be a function of the shift which is n.

And, hence you call the resulting number as $y[n]$. And then you evaluate this for all possible shifts, namely for all possible values of n. So, this is our linear convolution. Now, what we are going to do is, we are going to talk about sequences that are periodic.

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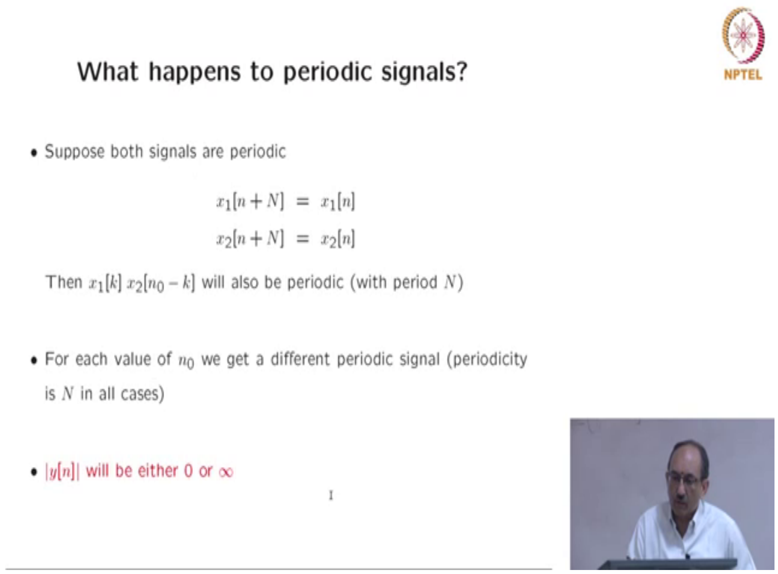

So, what happens in the periodic case? So, we have 2 sequences, x_1 and x_2. They are both periodic with the same period N. So, $x_1[n+N] = c_1[n]$; similarly for x_2. And hence, $x_1[k]x_2[n_0 - k]$ will also be periodic. Here, I am using n naught to denote the shift.

So, for a particular value of n_0, this also will be periodic with period N. Because, you are going to take one sequence keep it as it is, take the other sequence which is periodic, reflect it, it will continue to be periodic. Shift it, it will still be periodic and hence the product also will be periodic with the same period. And then, remember, you need to sum this up over all possible values of k and in the linear convolution, the sum went from k going from $-\infty$ to $+\infty$.

And remember, n_0 which we are denoting the shift value, for each n_0, we will get a different periodic signal. But, periodicity is N in all cases. And if you sum the product over all k from $-\infty$ to $+\infty$, you can think of this as being broken up into sum over each period and then summing this number. But, if you think of this in that terms, then it is clear that if you sum up over one period, we will get one number. And then you need to get the final answer, you need to sum up over all k. So, basically what you are doing is, you are adding this number to itself repeatedly.

Because, sum up over all k can be broken up into segments of sum over each period which will be the same and you are repeatedly adding this number to itself. So, if the value over one period has a certain value, then if you repeatedly add that value to itself, you will either get 0 or ∞; the absolute value of this sum. The sum over one period is 0. Here, we are assuming we are dealing with real valued sequences. If sum over one period is 0, then the entire sum over all k will be 0. If it is either positive or negative, then repeatedly you are going to add that number to itself infinite number of times. So, the absolute value will be infinity in that case.

So, this is linear convolution as separate to periodic signal or sequences is not useful, but the problem really comes in the last step. So, you are repeatedly adding the same number to itself over and over and again.

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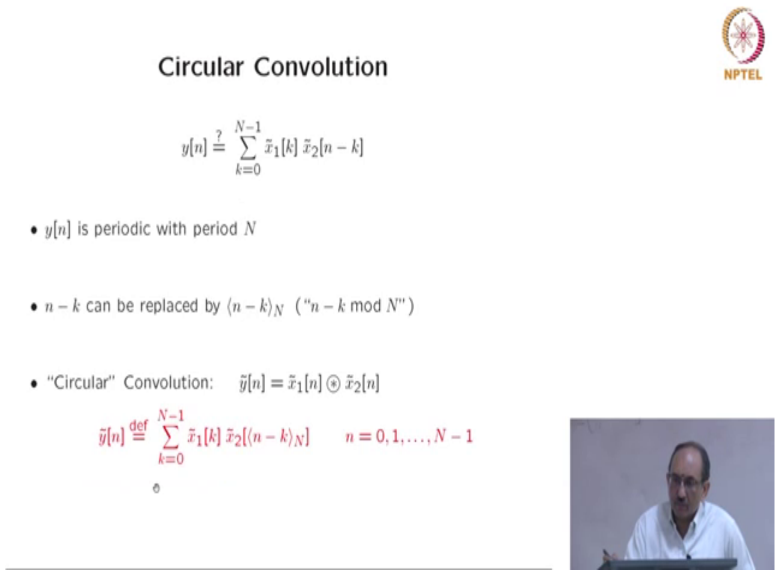

And hence, instead of letting the summation go from $-\infty$ to $+\infty$, if you restrict it to go from 0 to $N-1$, then you will not hit the problem that we hit earlier. Therefore, as a working definition, we have put $y[n] = \sum_{k=0}^{N-1} \tilde{x}_1[k]\ \tilde{x}_2[n-k]$ and we have put a question mark because we want to consider whether this will work. So, $y[n]$ is periodic with period N because x_1 and x_2 are periodic with period N.

And in this definition, $n-k$ can be replaced by $< n-k >_N$ because each sequence is periodic. And hence what we started off by thinking whether this will work, indeed does. So, the actual definition of circular convolution is $y[n] = \sum_{k=0}^{N-1} \tilde{x}_1[k]\ \tilde{x}_2[< n-k >_N]$. And you need to worry about values of shifts only in the range 0 to $N-1$, because you are going to keep one sequence fixed and then you are going to flip the other signal and shift. In the linear convolution, all possible shifts meant the shifts going from $-\infty$ to $+\infty$.

In this case, because of periodicity, the only shifts you need to consider are between 0to $N-1$, because any shift outside of this range will be the same as a shift to within this range. So, this is indeed the definition for circular convolution.

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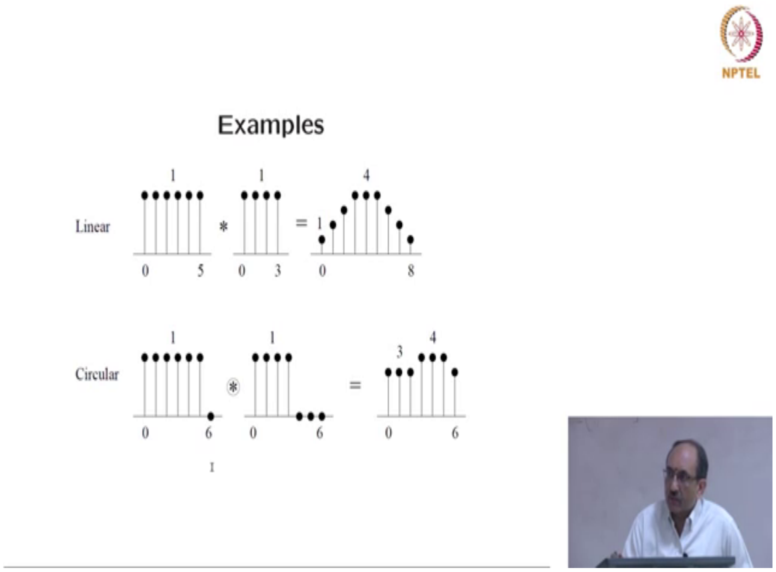

And here are some examples. So, what we have done is, here is the sequence there is all 1's. So, this is 6 length long, convolved with another sequence that is 4 long. Linear convolution is the familiar one. Now, when you want to talk about circular convolution for these sequences, if they are as they are, that is, a periodic circular convolution does not make sense because circular convolution is defined only for sequences that are periodic.

So, what we will do is, we will take this sequence and then make it periodic with period 6, just as an example. Similarly, we have taken the second sequence and also made it periodic with period 6. And to make it periodic with period 6, remember the original lengths are 6 and 4. So, rather, we are going to make it periodic with period 7 because this is going from 0 to $N-1$.

And the periodicity of $N = 7$ means that you need to zero-pad each of these sequences by as many zeros required to make the total length 7. And immediately, you can see if this is of length P and this is of length Q, the linear convolution will be of length $P + Q - 1$. And if you want to make these sequences periodic, if you want to look at their periodic counterparts, you need to zero-pad them. And the minimum length for periodicity has to be, if this is of length P and if this is of length Q and you want both of them to be periodic with period N, the minimum N that is possible would be?

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Very good; the minimum length N should be $\max(P, Q)$. So, $N \geq \max(P, Q)$. And so this is straightforward convolution and this happens to be circular convolution. If you apply the definition, this is what you will get.

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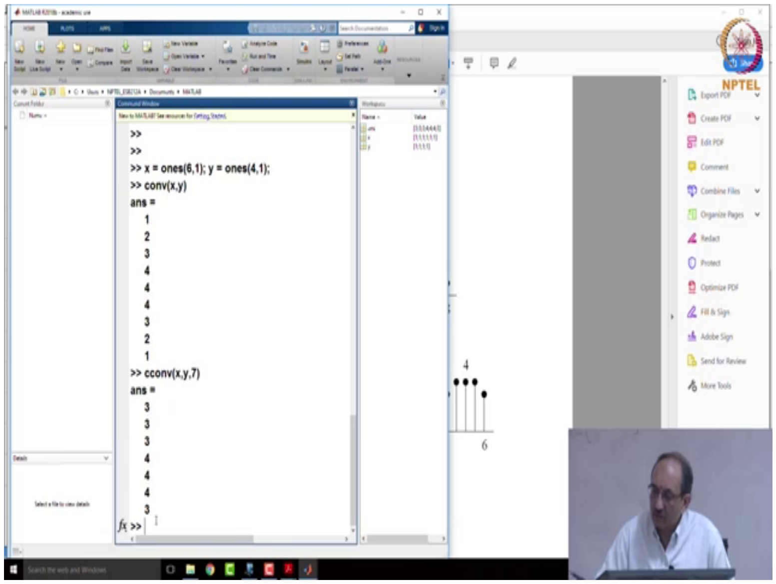

So, now, let us look at matlab, all right. So, x is, so this is the first sequence and y is the second sequence. And if I $conv(x, y)$ which is linear convolution, so this is what you will get; [1 2 3 4 4 4 3 2 1]. And this is of length $P + Q - 1$. So, this is expected and there is another command called $cconv$, which is the circular convolution counterpart. So $cconv(x, y,\ )$, I also need to specify the period.

So, in the figure that I had shown you, I had assumed the period to be 7. It went from 0 to 6 therefore, the period is 7. So, if $cconv(x, y, 7)$; this is the circular convolution, and this is [3 3 3 4 4 3]. So, now,this is exactly what was shown here [3 3 3 4 4 3], exactly what is given by matlab. Now, we have to ask the question whether there is any relationship if at all between circular convolution and linear convolution. So, we are trying to compare $x[n]$ convolved with $y[n]$; so this is linear convolution.

And then, $\tilde{x}[n]$ circularly convolved with $\tilde{y}[n]$ and this $\tilde{x}$ and $\tilde{y}$ are not arbitrary, but they are related to x and y and they are related to x and y through them being the periodic extension of x and y. So, you take x, make it periodic with period N; take y, make it periodic with period N to give you $\tilde{x}$ and $\tilde{y}$. And hence, you need to choose the period N that is at least this. So, you need to add as many zeros as needed to make x and y periodic with period N.

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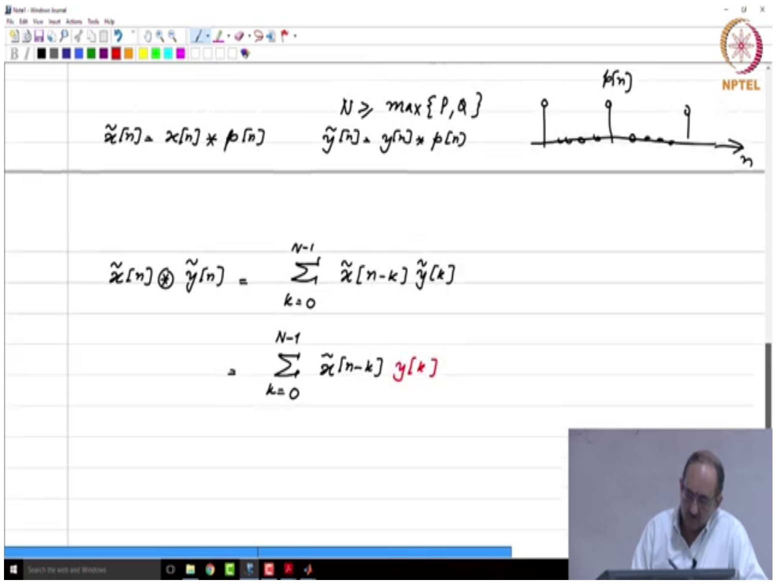

Therefore, $\tilde{x}[n] \circledast \tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{x}[n-k]\tilde{y}[k]$. So, this is just the straightforward definition. And remember, $\tilde{x}[n] = x[n] * p[n]$, where $p[n]$ is your impulse strain with period N.

Similarly, $\tilde{y}[n] = y[n] * p[n]$; $p[n]$ of course is the impulse strain. So, this is $p[n]$. So this is nothing but $\sum_{k=0}^{N-1} \tilde{x}[n-k]y[k]$ and you can replace $\tilde{y}[k]$ by $y[k]$ because over the interval 0 to $N-1$, $\tilde{y}[k]$ is the same as $y[k]$.

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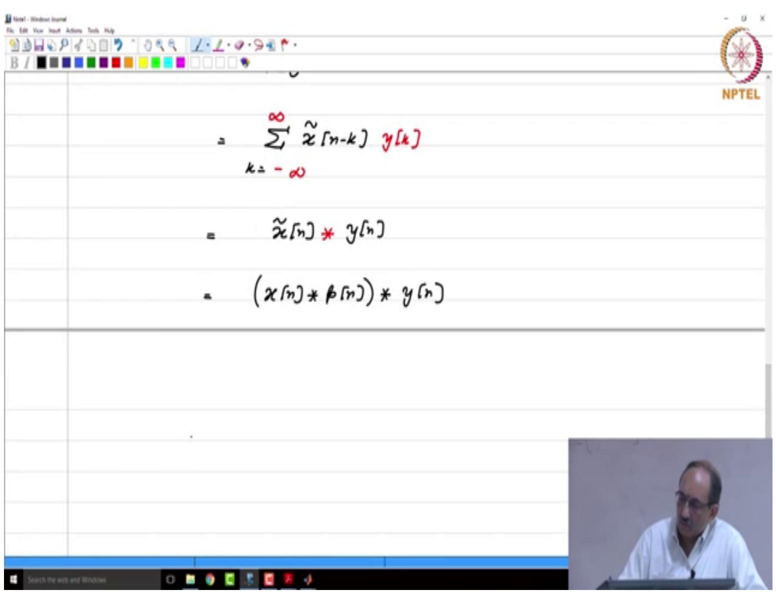

And, this is $\sum_{k=} \tilde{x}[n-k]y[k]$. Now k, you can let k go from $-\infty$ to $+\infty$ because outside 0 to $N-1$,

$y[k] = 0$. Therefore, this is perfectly fine and this has the form of linear convolution, except that this is the linear convolution of what?

Student: (Refer Time: 16:33) $\tilde{x}$.

$\tilde{x}[n] * y[n]$.

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Right, but $\tilde{x}[n]$ is really $x[n] * p[n]$, convolved with $y[n]$. And now you can use associativity and commutativity and hence this is really $(x[n] * y[n]) * p[n]$.

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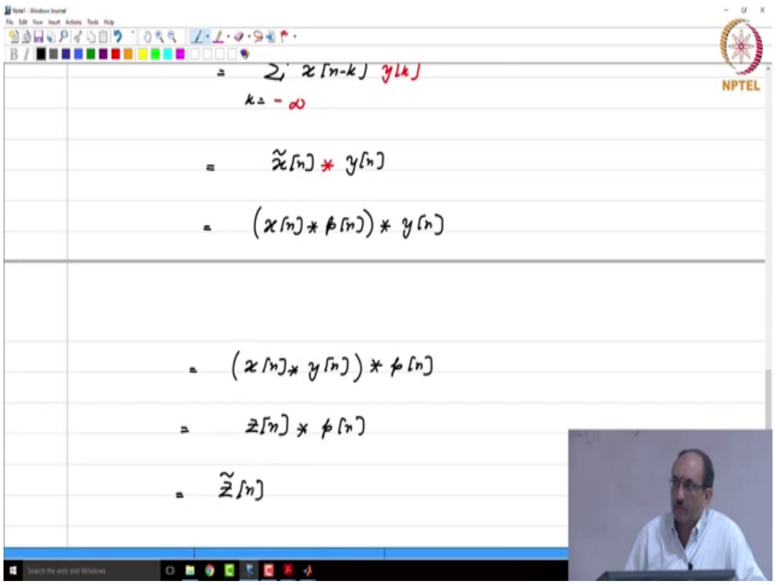

And $x[n] * y[n]$, we can call it as the $z[n]$, and $z[n]$ is the result of linear convolution with of x and y. And, hence is this and this is really $\tilde{z}[n]$. So, what this is saying is, you take the result of linear convolution and then repeat it periodically and that is the same as circular convolution. And, if you go back to this example, we will get the same result of circular convolution by taking linear convolution and repeating it periodically.

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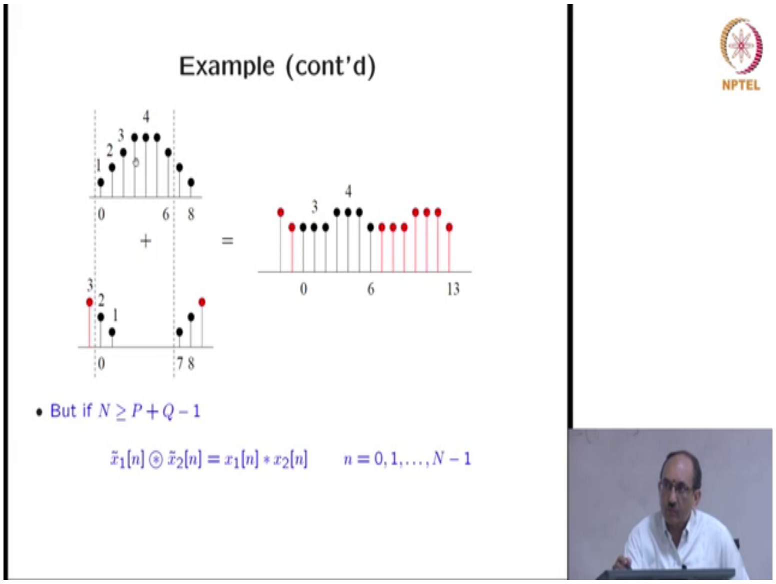

And to see that, so this is the result of linear convolution. Now, you need to make this periodic, the periodicity has to be 7. So, you take this and shift it to the right by 7 samples, which means you will go beyond this dotted line. And then take the same thing and shift it to the left by 7, you will get this. Note that, for periodic convolution you need to restrict yourself only in the range 0 to $N-1$; in this case, 0 to 6.

Therefore, the repetition to the right which starts from here, you need not worry about because this is outside 0 to $N-1$. What you need to worry about is, only the left shift. The left shift overlaps in the region 0 to $N-1$ with 2 samples in this particular case. And hence, from 0 to 6, if you add this and this, you will get this as, simple as that. So, this is the result of obtaining the, I mean, taking the result of linear convolution and then repeating it periodically. Wherever there is overlap, you add and over one period, if you look at the result, you will get this.

So, immediately this tells you, if you choose N such that N equals at least $P+Q-1$, then if you shift to the left and to the right, the repetitions will not overlap. If you choose N such that cap N is at least $P+Q-1$ or larger, shifting it to the left and to the right will cause no overlap. And hence, you can get the result of linear convolution via circular convolution provided you choose N to be at least as large as $P+Q-1$.

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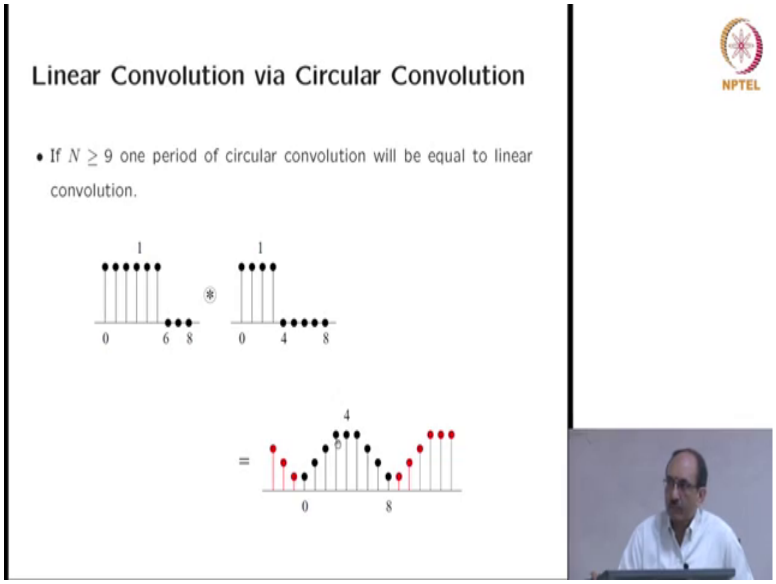

So, that is what is shown here. Remember, the one sequences of length 6, the other sequences of length 4. So, $P + Q - 1 = 9$ therefore, if you choose N to be 9 or greater, over one period circular and linear convolutions will match. And the minimum N is $P + Q - 1$ therefore, if you make the lengths to be 9 in both sequences and then if you do circular convolution, the repetitions will not overlap and from 0 to $N - 1$, the result of circular and linear convolutions match exactly.