Digital Signal Processing

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Lecture 76:
The Discrete Fourier Transform (2)
- Matrix-vector representation of the DFT
- DFT as the samples of the DTFT
- Effects of zero-padding
- Converting bin index true frequency

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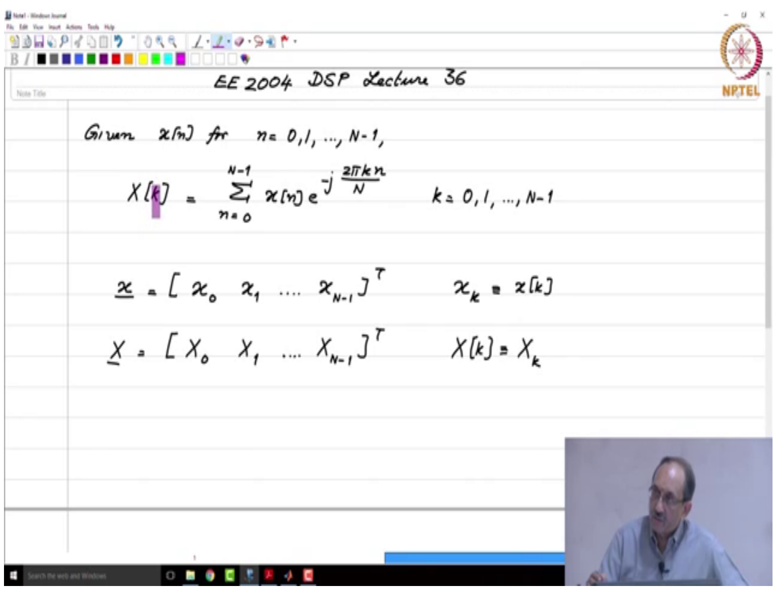

Let us continue our discussions on the Discrete Fourier Transform. We saw that, given $x[n]$ for $n = 0, 1, \ldots, N-1$, the DFT is defined like this. And we saw that, you need to consider only k for the indices 0 to $N-1$, because $X[k+N]$ is the same as $X[k]$. So, this is the DFT definition.

Now, this can be written in compact matrix vector notation. So, let us define the column vector $\underline{x}$, $\underline{x}$ which will be shown as boldface in typed or in printed work. So, in handwritten work we use an underscore here. We denote this as $[x_0, x_1, \ldots, x_{N-1}]^T$ and this is a column vector and we are using the notation x_k. So, x_k really stands for $x[k]$. So, to use this notation is more convenient in this context. Now let us express this in matrix vector notation.

Similarly, $\mathbf{X}$ or $\underline{X}$ is denoted as $[X_0, X_1, \ldots, X_{N-1}]^T$ and once again $X[k]$ is denoted as X_k. Remember, for the k^{th} bin index, the DFT is given by this and this can be thought of as an inner product of 2 vectors; one of the vectors is x as given here and the other vector is $e^{-j2\pi kn/N}$ and you can vary this over n going from 0 to $N-1$. You will generate a vector, that is, N long and this definition of $X[k]$ can be thought of as an inner product of 2 vectors.

And, this is an inner product of 2 vectors for a particular value of k and therefore, you can write all such inner products letting k go from 0 to $N-1$.

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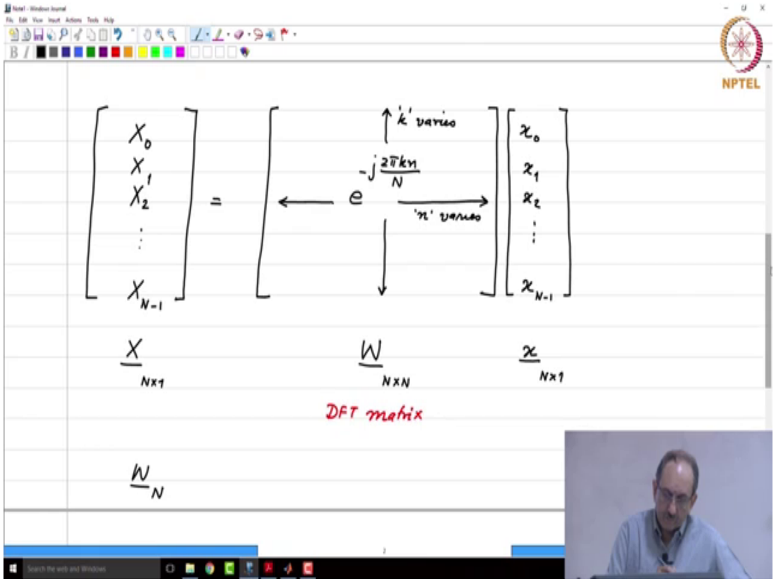

So, hence, we have $[X_0, X_1, \ldots, X_{N-1}]^T$. So, these are the DFT coefficients. For each value of k, you have an inner product. One of the vectors is fixed and that is $\underline{x}$ and that is given by $[x_0, x_1, \ldots, x_{N-1}]^T$. So, this is what is called as your data vector. The other vector is generated by $e^{-j2\pi kn/N}$ and then for each inner product you are going to vary n.

For a given k, you are going to vary n as given in this summation. To capture that, you can write this as $e^{-j2\pi kn/N}$. So, along this dimension, you are going to vary n. You are going to fix k and then you are going to vary n from 0 to $N-1$. You will generate an N dimensional vector and that when it is taken into an inner product with this vector, you will generate X_k. But what you need to do is, you need to do this for all values of k and hence you can think of each row as being a function of k and hence in this dimension, k varies.

So, the DFT equation can be represented in matrix vector notation. So, this can be written as $\underline{X}$ and this is of dimension $N \times 1$. This matrix is called the W matrix and this has dimension $N \times N$ and this is your data vector which is again a column vector and $\underline{x}$ is $N \times 1$ and this is also called as the DFT matrix.

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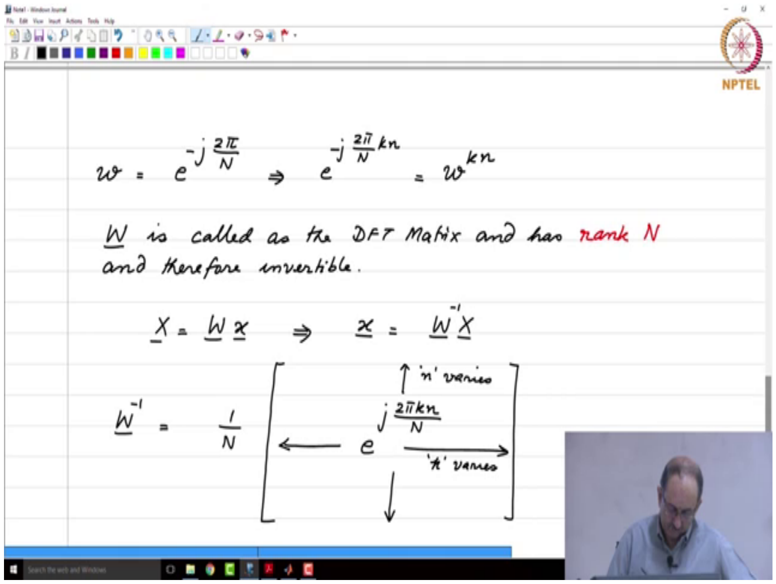

And, the notation $\underline{W}$ is used, because you can denote $e^{-j2\pi/N}$ as w. This is the notation that is used in this context and $e^{-j2\pi/N}$ is called as w. And hence, these elements are nothing but powers of w. Because, if you let $w = e^{-j2\pi/N}$, $e^{-j2\pi kn/N}$ is nothing, but w^{kn}. So, this implies $e^{-j2\pi kn/N} = w^{kn}$.

Sometimes, the DFT matrix is also denoted by $\underline{W}_N$ to make explicit its size. If the size is understood from the context, you can omit the subscript N. If you want to make the size explicit, you make the suffix explicit and show that it is of size $N \times N$. So, this is a very compact notation and one inference that can be made from this is that, remember, this is $N \times N$ matrix and all the columns are independent.

They are independent, because we have already seen that $e^{j\omega_1 n}$ and $e^{j\omega_2 n}$ are independent whenever $\omega_1 neq \omega_2$. So, from that, it follows that the columns of $\underline{W}$ are independent and this is an $N \times N$ matrix with N independent columns. And, if you recall from your linear algebra course, this has rank N which means this is non-singular and hence invertible. So, $\underline{W}$ is called as the DFT matrix and has rank N and therefore, invertible. So now, we have the case that $\underline{X} = \underline{W}\ \underline{x}$.

So, this immediately implies $\underline{x}$ is, pre multiply both sides by $\underline{W}^{-1}$, so this becomes $\underline{x} = \underline{W}^{-1}\underline{X}$. And, you can show that $\underline{W}^{-1}$ is this. So, that is also an $N \times N$ matrix, it is $e^{j2\pi kn/N}$. And in this dimension, k varies and in this dimension, n varies.

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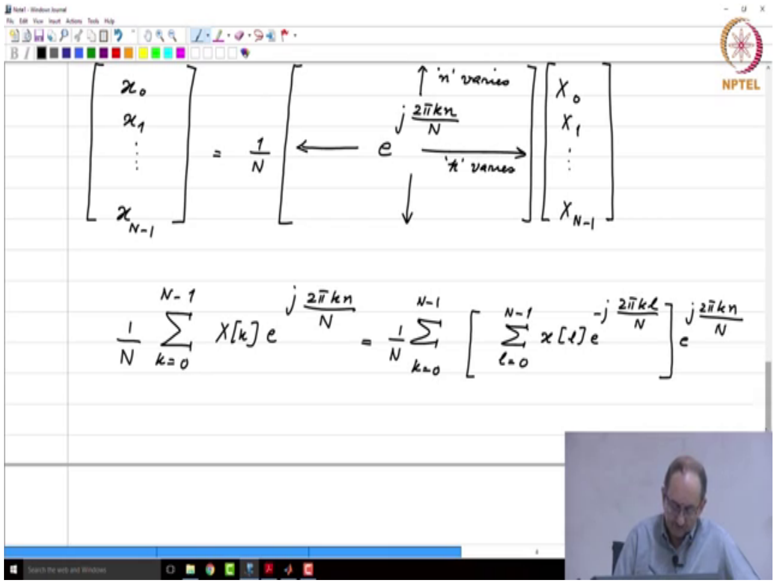

So, you can think of $\underline{x}$ which is nothing but $[x_0, x_1, \ldots, x_{N-1}]^T$. So, this is nothing but $\underline{W}^{-1}$ which is what you have seen just now. So, this is the $\underline{W}^{-1}$ and this multiplies the transform coefficients and you recover the data vector. So, you can go from data vector to the endpoint transform vector. From the endpoint transform vector, you can go back to the endpoint data vector via the DFT matrix.

And, you can easily verify that the inverse transform that was kind of given to you without any explanation, that really follows from the fact that the DFT matrix $\underline{W}$ has this inverse and hence, if you have $\dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$. This is the definition of the inverse Fourier transform, all we are trying to do now is verify that this is indeed true and this is almost trivially shown. So, this is $\dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1}(\;)$. Remember, we have to bring in $\underline{x}$ into the equation, because we are trying to show that this indeed gives you the original sequence that you started off with.

And therefore, you can replace $X[k]$ by it is definition and hence, this is $\displaystyle\sum_{l=0}^{N-1} x[l]e^{-j2\pi kl/N}$. And you want to use the dummy index l here, because n is already part of this you do not want to repeat that variable here because it will cause confusion, you will not get the right answer. This after all is a dummy index, that is why I am using l and now I have $e^{j2\pi kn/N}$.

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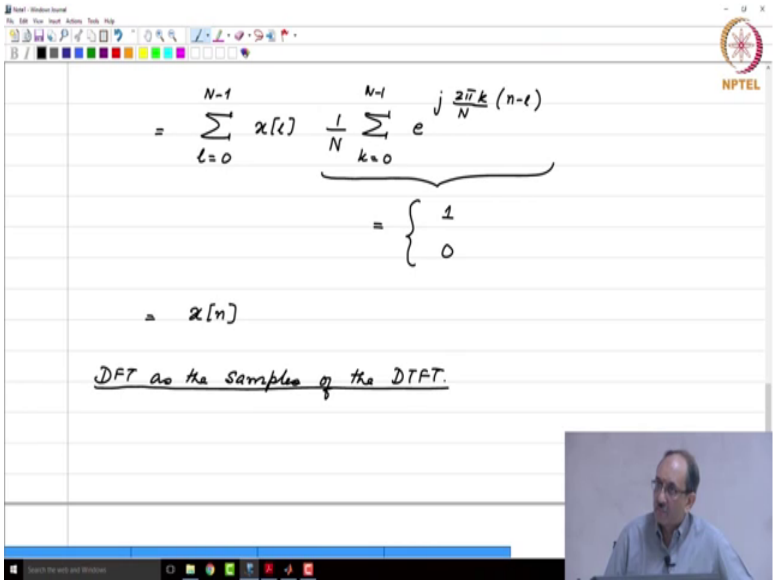

Now, you can interchange these 2 summations without any worries, because these are finite summations. And hence, $\sum_{l=0}^{N-1} x[l](\ )$ and now look at this. So, this is nothing but $\frac{1}{N}\sum_{k=0}^{N-1} e^{j2\pi k(n-l)/N}$. All I have done is, I have collected all the remaining 2 exponential terms and this is what this is. When $l = n$, what would be this summation?

Student: (Refer Time: 15:37).

Remember you also have to have this $1/N$ factor. So, this will be.

Student: 1.

1. When $l \neq n$, it will be 0. Therefore, the only term that will survive will be $x[n]$. So, this indeed shows that, you get back $x[n]$ starting from this. So, the inversion formula is indeed correct. Now this another viewpoint of the DFT, we can think of the DFT as being samples of the DTFT. When you started off with the DFT definition, we just gave it as, here is the formula take it, no explanation was given. But, now whatever we have seen so far has shown that this is consistent that, if you start off with this as the definition of the DFT, then you get this as the inverse and they are consistent.

The only thing was, even though you do not make any assumption about the data outside 0 to $N-1$, moment you put the data into the DFT framework, you are guaranteed that $x[n]$ will be periodic with period N which is also consistent with the fact that DFT is nothing but the DTFS written slightly differently. And the DTFT, rather the DTFS assumption is the signal sequence is periodic. So, everything is consistent.

Now, we can think of the DFT as the samples of the DTFT. For this, we make the assumption that the data are 0 outside 0 to $N-1$. So, here we assume that $x[n] = 0$ outside 0 to $N-1$ and now we will show that the periodicity of $x[n]$ will make sense as we go through this development.

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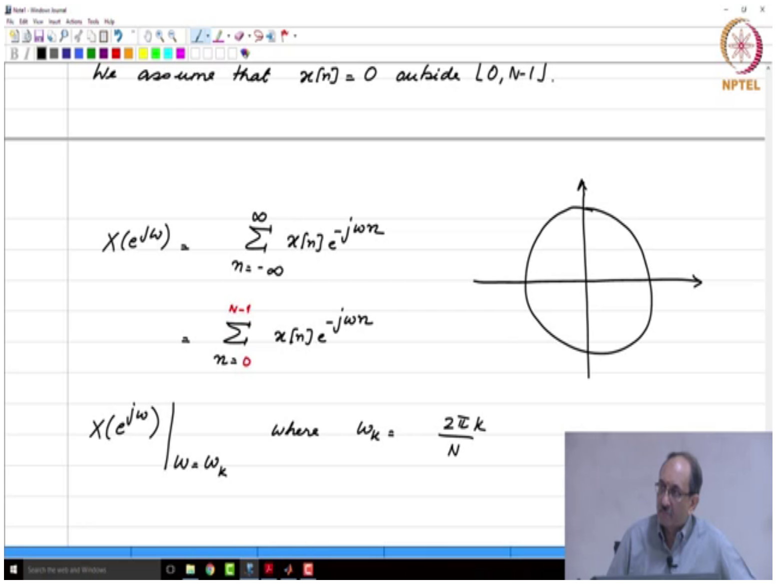

So, $X(e^{j\omega})$ by definition is, $\sum_{n==-\infty}^{\infty} x[n]e^{-j\omega n}$. And this is where we are going to make use of the assumption that the data are 0 outside 0 to $N-1$. So, now, the summation collapses to 0 to $N-1$, because $x[n] = 0$ outside this interval. So, you have $x[n]e^{-j\omega n}$ and we know the interpretation of the DTFT. You can think of the DTFT as evaluating the Z-transform around the unit circle.

So, we get a continuum values from 0 to 2π. Now what we are going to do is, we are going to take the DTFT which is $X(z)$ evaluated along a continuum namely 0 to 2π for as far as ω is concerned. Now we are going to sample this, we are not going to evaluate the DTFT continuously, but you are going to evaluate this only at N points that are uniformly spaced. And hence, we take $X(e^{j\omega})$ and then evaluate it at $\omega = \omega_k$, where $\omega_k = 2\pi k/N$.

This make sense because the interval is 2π. You have taken N uniform points, so the spacing between points is $2\pi/N$. And you are evaluating it at k uniformly space points and hence you can call these samples as $X[k]$.

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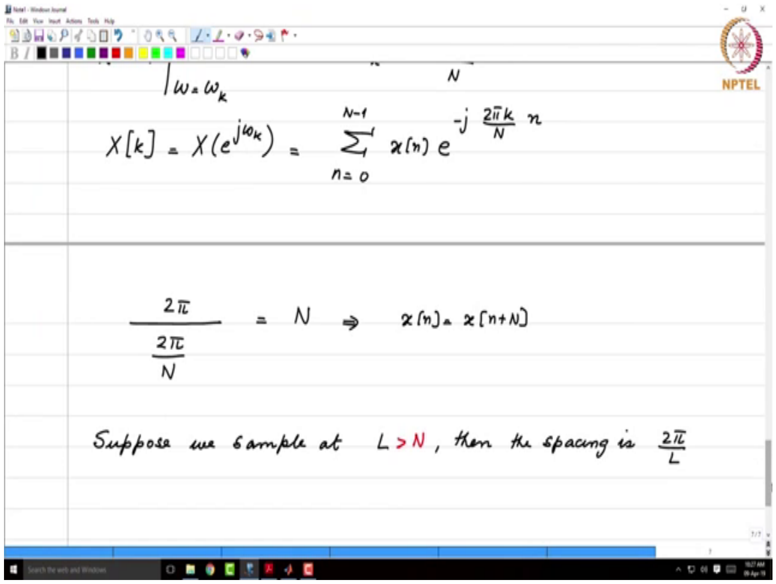

So, this is nothing but $X(e^{j\omega_k})$. So, this is nothing but $\sum_{n=0}^{N-1} x[n]e^{-j\omega_k n}$, ω_k is $2\pi k/N$. And hence, one interpretation of the DFT is taking the DTFT and evaluating it at a set of N uniformly space points in the interval 0 to 2π.

Now, if you sample in one domain, you periodically repeat in the other domain. So, when we sampled in the time domain, where the spacing was T apart. In the other domain, the repetition was $2\pi/T$, where T was a spacing. Now the spacing is $2\pi/N$ and hence in the other domain, it has to repeat by 2π by the period.

And hence, you see that the repetition in the other domain happens with a periodicity of N, which is what we should expect because just because you view the DFTS being samples of the DTFT, you cannot get rid of the periodicity. Therefore, even though we started off by saying $x[n]$ is 0 outside 0 to $N-1$ which is what we needed to restrict the summation in this range, moment you start sampling, immediately it introduces periodicity in the other domain. And that periodicity is exactly N. So, everything is consistent.

So, this implies $x[n+N] = x[n]$. Again, the ideas that we learned from sampling in the time domain are applicable here. There, you needed the signal to be band limited, so that when you sample, the repetitions did not overlap if you did not want to cause aliasing. So, now, here you are sampling in the frequency domain therefore, in the other domain which is time the signal has to be time limited.

So, everything is consistent. So, the signal is time limited. You sample, periodic repetitions occur with periodicity cap N and there is no aliasing. Question?

Student: (Refer Time: 24:18)

Student: Then what will happen in the (Refer Time: 24:24).

Actually, I was going to come to exactly that question here. So, you raised it at the right time. When you sample at twice the set of points, you are sampling at a finer spacing. If you sample at a finer spacing, remember, in the time domain, if you over sample, the repetitions will occur later. And hence, if you sample at say twice the number of points, here also the repetition will occur later. The periodicity will exactly be the number of points at which you sample. So, now, suppose we sample at L points which is greater than N, then the spacing is what? The spacing is $2\pi/L$.

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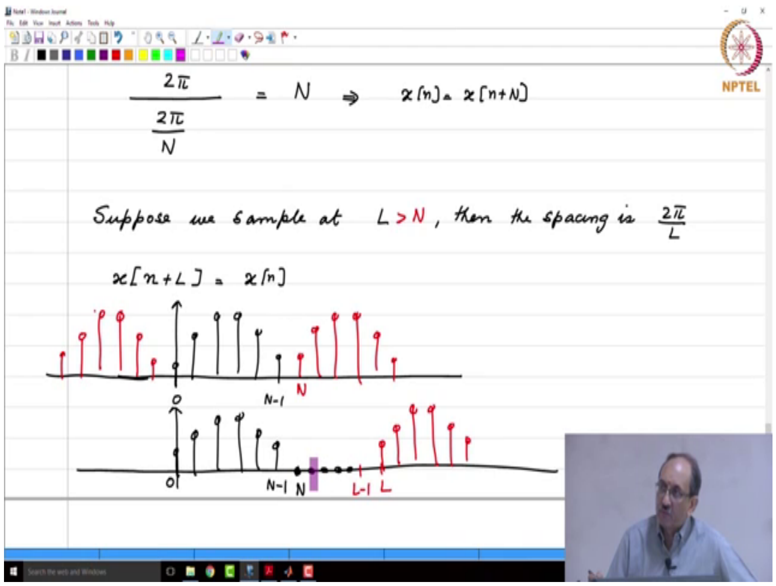

And hence, in the time domain, the repetition will be $2\pi/L$ and hence you are now left with the situation where $x[n + L] = x[n]$. And hence, if you have, say this as a sequence as an example and if this is 0 to $N - 1$, or rather let me draw a different sequence so that you see the difference. Suppose, I have this as my sequence, if I sample it at exactly N points, then what I will do is, I will repeat this and I will repeat this like this. And similarly, I will cause repetition on this side, because I am going to repeat periodically on both sides.

So, this is what my repetition will look. Now coming to the situation that you raised, now you are going to sample it at a larger number of points. And hence, so this is my original sequence, I now sampling it at L which is larger than N. So, my repetition will occur with a periodicity of period L. Therefore, what happens at 0 happened at N in the previous case. Whereas, what is going to happen now is, it is going to happen at index L. Therefore, the repetition will be like this which means from N to $L - 1$, you will have zeros.

Student: Sir.

Yes.

Student: They were there will samples from the (Refer Time: 28:16)

When you say time period, can you?

Student: Time (Refer Time: 28:20)

Now, remember, we are sampling in the frequency domain. In the frequency domain, your interval is always 0 to 2π. So, within 0 to 2π, what is your sampling period? You need $2\pi/N$ at a minimum. Because, the signal is one way of seeing this is, signal is time limited from 0 to $N - 1$, if you sample it at endpoints, the repetitions will occur with the periodicity of N which means that the repeating periods will not overlap with each other, that is the minimum.

So, which again correlates the fact that you need time limited signal for you to sample in the frequency domain, so that the repetitions do not overlap provided you sample adequately. Now, what you are going to do is, you are taking the same 0 to 2π interval and sampling it at L number of points. So, the spacing is now $2\pi/L$ which is finer than $2\pi/N$ and hence in the time domain, the repetition will occur with a periodicity of L.

So, which is what is being shown here. So, these, this repetition occurs later, consistent to the fact that you now sampling it at L. So, the sampling happens in the frequency domain. Does that answer the question? The other way of looking at this is, we have already seen what a finer sampling does, the same thing can be obtained by starting off from the time domain.

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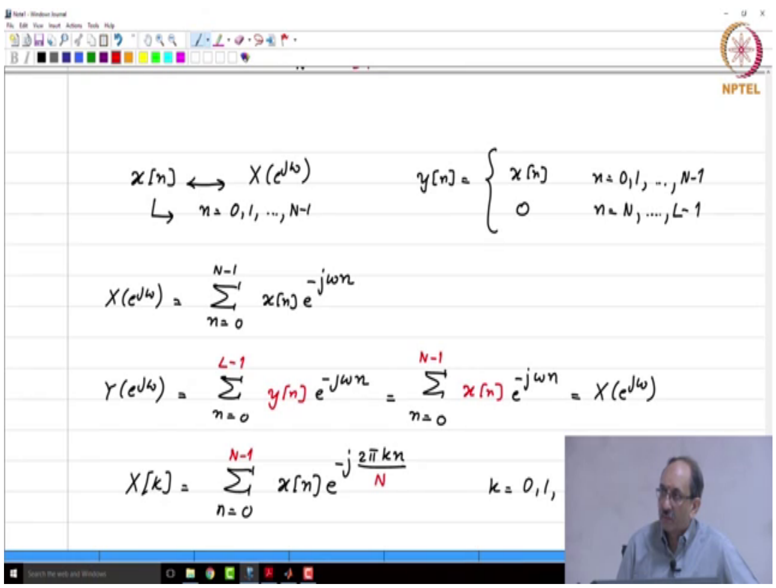

That is, suppose you have $x[n]$ which has DTFT $X(e^{j\omega})$. And $x[n]$ is defined for $n = 0, \ldots, N-1$ and it is 0 outside this interval. For this sequence, you have this $X(e^{j\omega})$ and hence $X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$. Now, suppose I form $y[n]$ which is same as $x[n]$ for $0, \ldots, N-1$ and then I add 0s for $n = N, \ldots, L-1$. So, this can be thought of taking a given sequence and zero-padding it. Now, what about $Y(e^{j\omega})$. So, this is n going from 0 to $L-1$. I now have $\sum_{n=0}^{L-1} y[n]e^{-j\omega n}$ and this is easily seen to be n going from 0 to $N-1$. Not only that, within this interval, $y[n] = x[n]$. So, this is times $\sum_{n=0}^{N-1} x[n]e^{-j\omega n}$ and this immediately tells you that $Y(e^{j\omega})$ and $X(e^{j\omega})$ are identical. Now, let us look at the corresponding DFT coefficients, $X[k]$ is $\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$.

And, k itself goes from $0, 1, \ldots, N-1$. So, the situation here is, we have an N-point sequence $x[n]$ and we have computed its N-point transform, so far so good.

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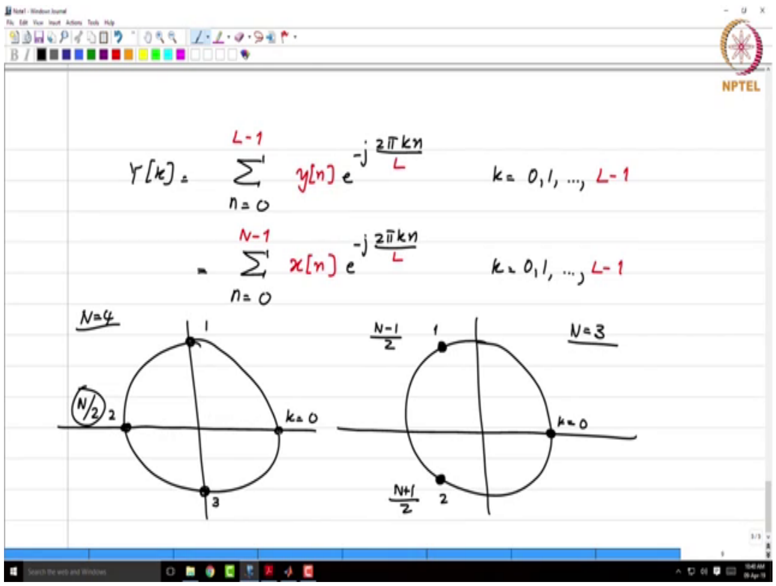

Now, let us look at $Y[k]$. $Y[k]$ is an L-point sequence. So, this is $\sum_{n=0}^{L-1} y[n]e^{-j2\pi kn/L}$. And, if you have an N-point sequence, you get an N-point transform. If you have an L point sequence, you will get an L point transform.

So, you will have L DFT coefficients therefore, $k = 0, 1, \ldots, L-1$. So, this is nothing but $\sum_{n=0}^{N-1}(\;)$ and $Y[k]$ is the same as, rather this should be of $y[n]$. So, this is $\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/L}$, this remains exactly the same. And therefore, this is $0, 1, \ldots, L-1$.

So, this is exactly the same idea, but looking at it from the other point of view, that is, we are starting from the time domain. So, earlier what we did was, we said let us take the DFT. rather the DTFT and sample it at L points where $L > N$. And then, we saw that the repetitions occur later and between the repetitions, zeros get added. The other way of looking at exactly the same concept is, take the given N-point sequence, zero-pad and compute the L-point DFT.

Again, what this gives you is, you get the samples of the DTFT. Remember, the DTFT is unchanged because the DTFT of both the original sequence and the zero-padded sequence are the same as it should be. But, what you have done now is, by zero-padding, you have evaluated the underlying DTFT at a finer set of points.

So, the DTFT remains unchanged, but when you zero-pad, you sample the DTFT at a larger set of points, that is the only thing that happens. So, either you can take this view point by starting off with a given sequence and zero-padding and then realizing that, you have sampled the DTFT at a finer set of points; that is one way of looking at it. The other way of looking at it is, what we saw earlier. You take the DTFT, sample it and then realize sampling it at a finer set of point means, the repetitions occur later in the time domain which is equivalent to zero-padding. So, these are two sides of the same coin.

And one thing to note in this context is, let us look at the case of the number of samples being odd or even. And if you take $N = 3$, then you sample the DTFT at a uniformly spaced points; here, the spacing is $2\pi/3$. So, this is 0, this is 1 and this is 2. So, this is $k = 0$, 1, then 2 and in general, this point will correspond to $(N-1)/2$ and this point will correspond to $(N+1)/2$.

If you had even set of points, this is the, this is how these samples will be distributed. So, this is again $k = 0, 1, 2, 3$. And this point will be corresponding to $N/2$. So, when you sample the DTFT at an even set of points, you will have a sampled at $\omega = \pi$. Whether N is odd or even, at $\omega = 0$, you will always have a sampled point. Whether you get a sampled point at pi or not depends on whether the number of points is even or odd. If it is even, you will hit the $\omega = \pi$ point, otherwise you will not hit that point.

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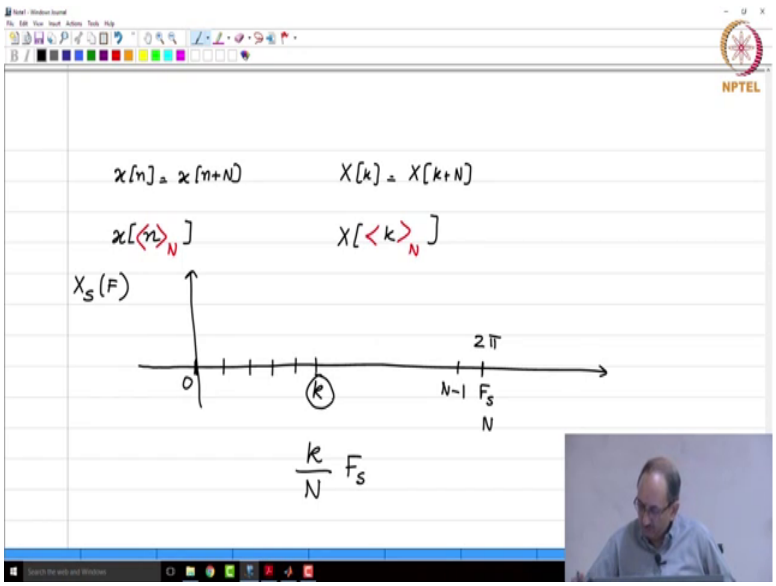

And in this context, it is very important to realize that $x[n] = x[n+N]$ and it is also the case that $X[k]$ which is the DFT coefficient, that also is periodic with period N. So, one immediate consequence of this, we have already seen this before when we are looking at discrete time periodic complex exponentials. Here, because of the periodicity, the index n is really index $<n>_N$.

Because, no matter what the index is, you can always map it to the interval 0 to $N-1$. Similarly, here, the index is really index mod N because both of them are periodic with period N. Therefore, both the sequence index and the transform index or index mod N. And hence, this index is also the same as -1. Similarly, this index also is the same as -1.

And moment you understand what is happening here, if you want to convert the DFT frequency to true frequency, first of all to do that you need the sampling frequency information. And when you sample in the time domain, you periodically repeat in the frequency domain. Therefore, we saw that as far as the spectrum goes, the spectrum is periodically repeating.

And hence, if you think of this in terms of Hertz, this will repeat periodically with a period of F_s, right. That is, I am talking about $X_s(F)$. I am showing, now this in F, earlier we had seen this as $X_s(\Omega)$ and be denoted instead of F_s, we denoted this point as Ω_s. All I have done now is, I have scaled it by 2π, so that the repetition is F_s rather than Ω_s.

We also know that, when this is mapped to the DTFT, this F_s will get mapped to 2π. Now, our goal is to convert the DTFT back to true Hertz, but in practice we will only be dealing with the DFT because that is the only thing that is machine implementable. Therefore, we cannot compute the DTFT in on a machine, you have to compute the DFT.

But what are we doing as far as the DFT goes? We are taking the interval 0 to 2π and dividing it into N points. Therefore, we are dividing this into N points, the penultimate the last point is really $N-1$. And if you let $k = N$, that is the same as k being 0. Therefore, k being N will correspond to the point 2π therefore, the general point on the DFT bin index is k.

Now given the k^{th} bin index, suppose you want this to correspond to true frequency, you need to divide by N and then what else do you need to do to get the true frequency in Hertz? You need to multiply by F_s. And hence, if you take the k^{th} DFT bin index, if you want to know what frequency it corresponds to, divide by N and multiply by F_s. This will give you that bin index's true frequency location. So, this is how you convert bin index to true frequency. So, everything is consistent with what we have seen so far.

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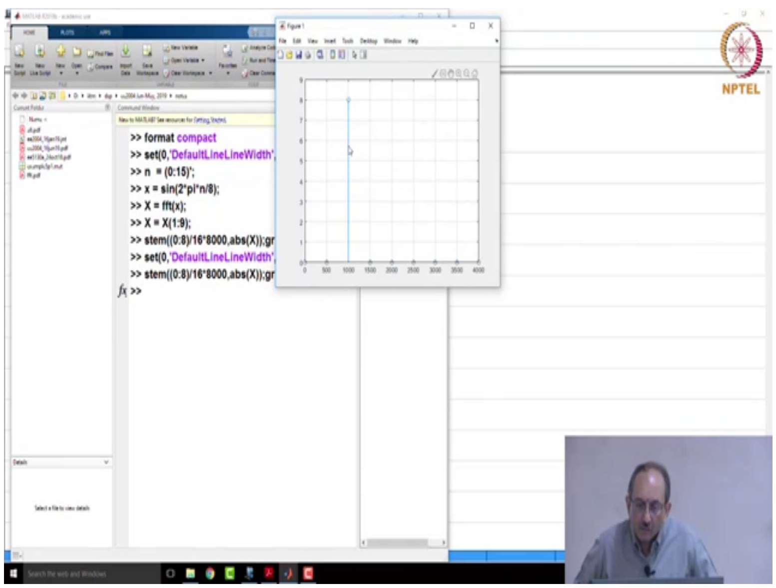

Let me see if I can quickly show this in matlab. Suppose, I have n going from 0 to 15, I have 16 points and then I am evaluating, rather forming as sinusoid with frequency of 1 kilo Hertz and then sampling frequency is 8 kilo Hertz.

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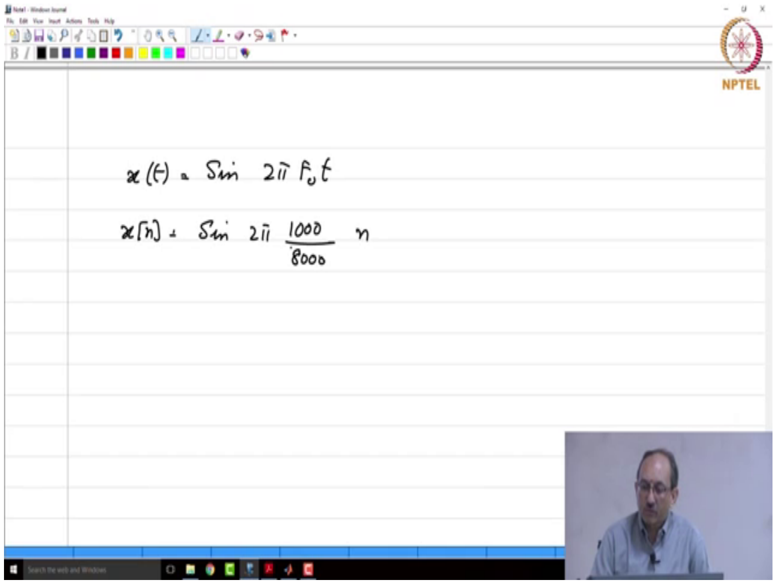

So, I have $x(t)$ which is $\sin(2\pi F_0 t)$ and I want to form $x[n]$ which is $\sin(2\pi F_o t)$, I am going to take it as 1000 Hertz and I am going to sample it at 8 kilo Hertz. So, I am going to replace t by nT, T is $1/F_s$ and this is what I have. So, $\sin(2\pi(1000/8000)n)$ is the sequence I want to generate. So, I have sin 2 times pi times n times, now I have the normalized frequency which is 1/8; 1000/8000. So, this is my sequence. Now I want to compute the DFT and the corresponding command in the in matlab is fft.

So, I have $fft(x)$. So, if you do not give any argument, we will compute the N-point DFT which is 16 points here. Now remember, this is a real valued sequence therefore, 0 to $N/2$ is all that I need because from $(N/2)+1$ to $N-1$ because of symmetry, it will be the complex conjugate version of the first upper half of this and hence I need to restrict x only from 0 to 8. Matlab is 1 based index therefore, I resting it restricting it to 1 to 9 and now I need to plot this.

So, I will use the stem command, right. Now I am going from 0 to $N/2$. So, I need to divide this, remember, I need to do k/N. So, this is $16 \times F_s$; F_s is this and then $abs(X)$. So, this should actually.

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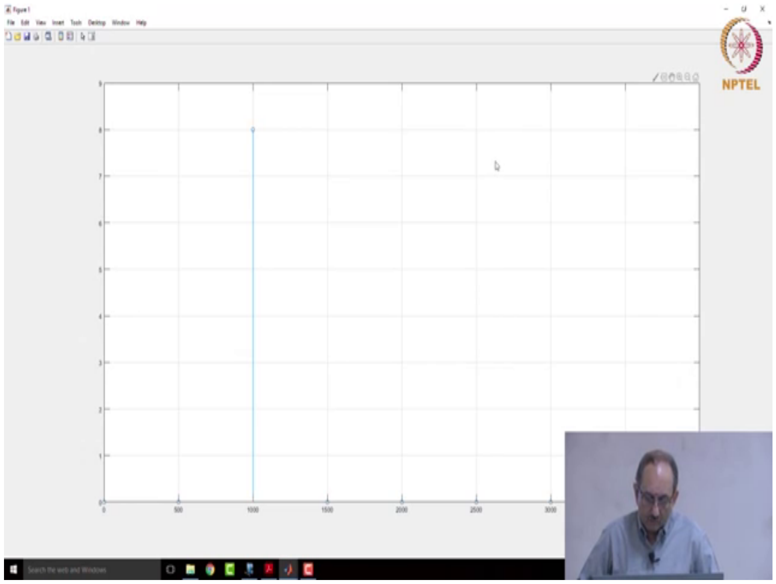

So, the x axis is frequency. So, this goes from 0 to $F_s/2$ because from $F_s/2$ to F_s, magnitude plot will be the same. And now, the frequency was 1 kilo Hertz and you see at this point, a value of 1. At every other bin index, the value is 0.

So, we will continue this exampled in the next class, we will see why the other values are 0 here and then what we will also do is, we will sample the DFT. Right now, we have sampled it only at 16 points, we had a 16 point sequence, we had a 16 point transform. Now we will zero-pad and then get the DFT by zero-padding and then overlay those plots on this and see the connection. And then, we will compare this with the DTFT and then we will convince ourselves that the DFT is indeed thee samples of the DTFT.